Evolutionary Strategy for Shape Design of Rotating Variable-Thickness Annular Elastic Disc

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Abstract

In the paper the optimisation of rotating variable-thickness annular elastic discs based on a simulation model by means of evolutionary strategies is discussed. The goal of the design is to find such a shape of the disc that would ensure maximal elastic carrying capacity. In the presented model the profile of the disc is defined by spline curves. After defining physical and mathematical models of the analysed problem, obtained experimental results are presented.

Keywords: evolutionary strategies, optimal shape design, elastic carrying capacity

1. Introduction

For very complex engineering problems mathematical models often prove so difficult that their analysis becomes a non-trivial task for most traditional methods. In such cases simulation experiments may be useful to verify the correctness of the proposed solutions, which may be further improved (optimised) by the human designer. However the globally optimal solution may be hardly found this way. It seems that this task may be successfully handled by some (meta)heuristic computational techniques such as *evolutionary algorithms*. This term covers a wide range of search and optimisation methods, based on analogies to phenomena of natural evolution. Particularly evolutionary strategies are the most often used for continuous optimisation problems [1, 4]. They are distinguished by deterministic selection and Gaussian mutation as main variation operator.

The paper discusses the optimisation of rotating variable-thickness annular elastic discs based on the proposed simulation model by means of evolutionary strategies. The goal of the design is to find such a shape of the disc that would ensure maximal elastic carrying capacity. While in many papers the shape is modelled with the use of some fixed class of functions (e.g. hyperbolic, n-th order polynomial), in the model presented the profile of the disc is defined by spline curves. This introduces many more "degrees of freedom" and makes it possible to find a more suitable shape. Selected experimental results with (μ, λ)-evolutionary strategy conclude the work.

2. The optimisation problem

Most important assumptions of the physical model are as follows:

1. We consider an annular elastic disc of variable thickness $h = h(r)$ rotating with constant angular velocity ω and subject to uniform traction p_b at the outer radius b. The disc is clamped at the inner radius a.
2. The classical theory of thin discs with small gradient dh/dr is assumed and hence the stresses τ_{rz} and σ_z are neglected.
3. The material is linear-elastic with Young's modulus E, Poisson's ratio ν and subject to the Huber-Mises-Hencky (H-M-H) yield condition.
4. The small-strain theory is adopted.

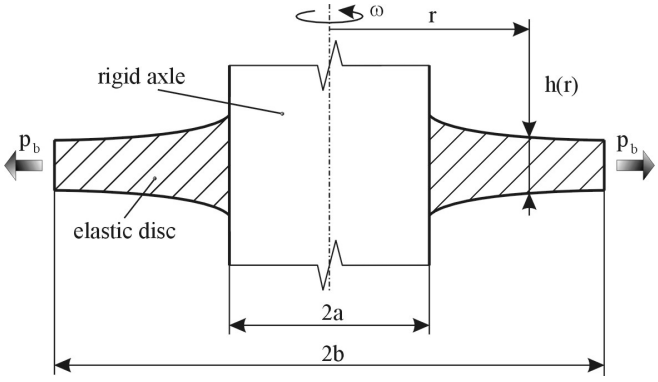

Fig. 1. Annular disc under consideration

2.1. Basic equations

The condition of internal equilibrium in polar coordinates may be expressed as follows:

$$\frac{1}{h}\frac{d}{dr}(h\sigma_r)+\frac{\sigma_r-\sigma_\theta}{r}+\rho_m\omega^2 r=0 \qquad (1)$$

where ρ_m stands for mass density and h for the disc thickness.

Constitutive equations (described by the Hooke's law) take the following form:

$$\sigma_r=\frac{E}{1-\nu^2}\left(\frac{du}{dr}+\nu\frac{u}{r}\right),\quad \sigma_\theta=\frac{E}{1-\nu^2}\left(\frac{u}{r}+\nu\frac{du}{dr}\right) \qquad (2)$$

and combine radial displacement u with radial σ_r and circumferential σ_θ stresses.

Taking advantage of constitutive equations (2) after simple calculations the condition of internal equilibrium has the form (3):

$$\frac{d^2u}{dr^2}+\frac{1}{r}\left(1+\frac{r}{h}\frac{dh}{dr}\right)\frac{du}{dr}-\frac{1}{r^2}\left(1-\nu\frac{r}{h}\frac{dh}{dr}\right)u=-\rho_m\omega^2\frac{1-\nu^2}{E}r\,.$$

Boundary conditions:

$$u(a)=0\,,\ \frac{E}{1-\nu^2}\left[\frac{du(b)}{dr}+\nu\frac{u(b)}{r}\right]=p_b \tag{4}$$

allow to find a numerical solution depending on external loadings (angular velocity ω and uniform traction p_b).

The stress intensity calculated according to Huber-Mises-Hencky hypothesis:

$$\sigma_i^2=\sigma_r^2+\sigma_\theta^2-\sigma_r\sigma_\theta \tag{5}$$

takes its maximal value at the boundary of the disc (usually at the inner radius) or at a certain point inside the disc. It obviously depends on the shape of the disc.

When the maximum value of the stress intensity reaches the value of yield stress σ_0:

$$\left|\sigma_i(r_0)\right|_{MAX}=\sigma_0\,, \tag{6}$$

the elastic carrying capacity is exhausted. Therefore (6) allows to find the external loadings value (a combination of angular velocity ω and uniform traction p_b), which we call the elastic carrying capacity of the disc.

2.2. Shape representation

As it was mentioned above, the profile of the disc is represented by the 3rd order spline built on equidistant nodes (with δ being the distance between nodes). Coefficients of the spline may be found from the set of linear equations:

$$\begin{bmatrix}2&1&0&\dots&0\\1&4&1&\dots&0\\0&1&4&\dots&0\\\vdots&&\ddots&&\vdots\\0&&1&4&1\\0&\dots&0&1&2\end{bmatrix}\cdot\begin{bmatrix}M_0\\M_1\\M_2\\\vdots\\M_{n-1}\\M_n\end{bmatrix}=\begin{bmatrix}d_0\\d_1\\d_2\\\vdots\\d_{n-1}\\d_n\end{bmatrix} \tag{7}$$

where:

$$d_j=\frac{6}{\delta^2}\cdot\left(y_{j-1}-2y_j+y_{j+1}\right),\quad j=1,\dots,n-1\,, \tag{8}$$

$$\begin{cases}d_0=\dfrac{6}{\delta^2}\cdot(y_1-y_0)\\[2mm] d_n=\dfrac{6}{\delta^2}\cdot(y_n-y_{n-1})\end{cases}. \tag{9}$$

After some calculations one may find the value of interpolating function as well as its derivative at any point from the range $r\in\langle a,b\rangle$ (or, after introducing dimensionless radius $x=r/a$, $x\in\langle 1,b/a\rangle$):

$$s(x)=\frac{1}{6\delta}\cdot\left[M_{j-1}\cdot(x_j-x)^3+M_j\cdot(x-x_{j-1})^3\right]+\\+\left[\frac{y_j-y_{j-1}}{\delta}-\frac{\delta}{6}\cdot(M_j-M_{j-1})\right]\cdot(x-x_{j-1})+\\+y_{j-1}-M_{j-1}\frac{\delta^2}{6} \tag{10}$$

$$\frac{ds(x)}{dx}=\frac{1}{2\delta}\cdot\left[-M_{j-1}\cdot(x_j-x)^2+M_j\cdot(x-x_{j-1})^2\right]+\\+\frac{y_j-y_{j-1}}{\delta}-\frac{\delta}{6}\cdot(M_j-M_{j-1}) \tag{11}$$

where $x\in\langle x_{j-1};x_j\rangle$ and $j=1,\dots,n$.

For numerical calculations we introduce the following dimensionless quantities:

$$\beta=\frac{b}{a},\quad x=\frac{r}{a},\quad y=\frac{h}{a},\quad w=\frac{u}{a}\,. \tag{12}$$

The above group of parameters is connected only with the geometry of the disc while the next one describes the material of the disc and its external loadings:

$$S=\frac{\sigma_0}{E},\ \Omega=\frac{\sqrt{3}\rho_m\omega^2a^2}{2\sigma_0},\ p=\frac{p_b}{\sigma_0} \tag{13}$$

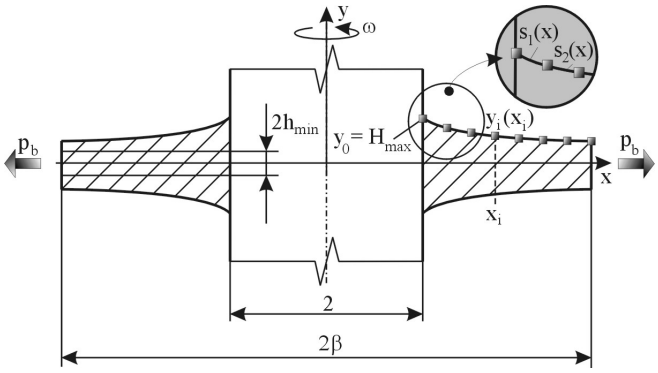

Fig. 2. Profile representation and constrains of the shape

2.3. The model

After introducing basic equations (the physical model) one may formulate the optimisation problem by defining a decision variables vector, a feasible region and an objective function.

The decision variables vector

$$\underline{Y}=(y_0,y_2,\dots,y_n)\in M\subset IR^{n+1} \tag{14}$$

represents the shape of the disc in $n+1$ equidistant points. The feasible region

$$M = \left\{ \underline{Y} \in IR^{n+1} \;\middle|\; \begin{array}{l} y_0 = H_{\max} \wedge \\ \wedge \left(k_d \cdot h_{\min} \le y_j \le k_g \cdot H_{\max} \;\; \forall j = 1,\ldots,n \right) \end{array} \right\} \qquad (15)$$

assumes that the disc is clamped at the inner radius having there fixed thickness ($y_0 = H_{\max}$) and that the disc can be neither too thin (not thinner than $k_d \cdot h_{\min}$) nor too thick (not thicker than $k_g \cdot H_{\max}$).

Objective function is described by the following formula

$$\Phi = \left[c \left(\frac{1}{\beta - 1} \int_1^{\beta} \sigma_i(x) dx \right) + (1-c) \sqrt{p^2 + \Omega^2} \right] \to \max \qquad (16)$$

where $0 \le c \le 1$ makes it possible to set the importance of each of the two criteria taken into account. The first of them (with the multiplier c) is connected with the equalization of the stress intensity and the second one with the external loadings (it is worth noting that if $c = 0$ this criterion becomes a simple maximization of elastic carrying capacity). Such a generalisation is very helpful in estimating the limit carrying capacity or decohesive carrying capacity.

3. Evolutionary algorithm approach

The search for the optimal solution to the problem defined above was based on the simulation model and (μ, λ)-evolutionary strategy ((μ, λ)-ES). The chromosome consisted of decision variables (14) and standard deviations of mutation. All genes were represented as floating-point values without any coding. It is worth noting that genetic algorithms approach usually requires some kind of coding which may be sometimes complicated and also may disturb the algorithm [2].

In the algorithm the mutation operator is realised in two stages: first we calculate new standard deviations:

$$\begin{array}{l} \sigma_j^{\langle new \rangle} := \sigma_j^{\langle old \rangle} \cdot \exp\left(\tau_1 \cdot \xi_{N(0,1)} + \tau_2 \cdot \xi_{N(0,1),j} \right) \\ \forall j = 1,\ldots,n \end{array} \qquad (17)$$

and next new values of decision variables:

$$\begin{array}{l} y_j^{\langle new \rangle} := y_j^{\langle old \rangle} + \sigma_j^{\langle new \rangle} \cdot \xi_{N(0,1),j} \\ \forall j = 1,\ldots,n \end{array} \qquad (18)$$

where τ_1, τ_2 are algorithm parameters and $\xi_{N(0,1)}$ is the random number drawn from the one-dimensional Gaussian distribution. Evolution of standard deviations of mutation together with decision variables allows for auto-adaptation of computing process [3].

The second genetic operator applied in ES is a recombination operator. In the algorithm it is realised in the following way (arithmetic recombination; $m \in \langle 0,1 \rangle$ is the random number drawn from the uniform distribution):

$$\begin{cases} \underline{Y}^{\langle new \rangle,1} := m\underline{Y}^{\langle old \rangle,1} + (1-m)\underline{Y}^{\langle old \rangle,2} \\ \underline{Y}^{\langle new \rangle,2} := m\underline{Y}^{\langle old \rangle,2} + (1-m)\underline{Y}^{\langle old \rangle,1} \end{cases} \qquad (19)$$

and the same for the standard deviation:

$$\begin{cases} \underline{\sigma}^{\langle new \rangle,1} := m\underline{\sigma}^{\langle old \rangle,1} + (1-m)\underline{\sigma}^{\langle old \rangle,2} \\ \underline{\sigma}^{\langle new \rangle,2} := m\underline{\sigma}^{\langle old \rangle,2} + (1-m)\underline{\sigma}^{\langle old \rangle,1} \end{cases} . \qquad (20)$$

4. The optimisation results

Below optimal shapes – in the meaning of criterion (16) – connected with different ratio Ω / p are presented. All the results were obtained for the "typical" material of the disc for which:

$$\nu = 0.3;\; \sigma_0 / E = 0.001 . \qquad (21)$$

As regards geometrical parameters all calculations were connected with the following values:

$$\beta = b/a = 2,\; h_{\min} = 1,\; H_{\max} = 3,\; k_d = 0.9,\; k_g = 1.1 . \qquad (22)$$

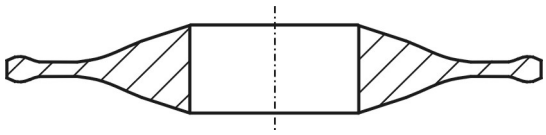

Fig. 3. Optimal disc for purely inertial loadings; $c = 0.4$

Fig. 3 presents the shape of optimal disc when the external pressure is equal to zero. There are two interesting things in the shape: first it is the flat part near the centre of the disc and the second one is the swelling near the outer radius. The flat segment results from the constraint connected with the minimal thickness which becomes active near the centre. The swelling results from the character of objective function – this swelling is the outcome of more equal distribution of the stress intensity. The elastic carrying capacity (ECC) is described by the pair: $\Omega = 1.21;\; p = 0$.

The optimal shape of the disc when the dimensionless angular velocity is twice greater than the dimensionless external pressure is shown in Fig. 4.

Fig. 4. Optimal disc for $\Omega / p = 2$; $c = 0.4$

In this case the shape is more regular but a small swelling near the outer radius still remains. A more regular shape can be explained by the fact that a part of inertia forces in this case is replaced by the external pressure action. In this case the ECC is described by the following pair: $\Omega = 0.94;\; p = 0.47$.

In Fig. 5 the optimal shape of the disc for dimensionless external pressure and rotation velocity is shown.

Fig. 5. Optimal disc for $\Omega/p = 1$; $c = 0.4$

It is interesting that for such a ratio Ω/p the shape is very smooth which may be, in a way, optimal for the work of the disc.

5. The ES analysis

Figures 6 and 7 show the minimum and maximum quality of the chromosomes and average standard deviation in subsequent steps of the algorithm.

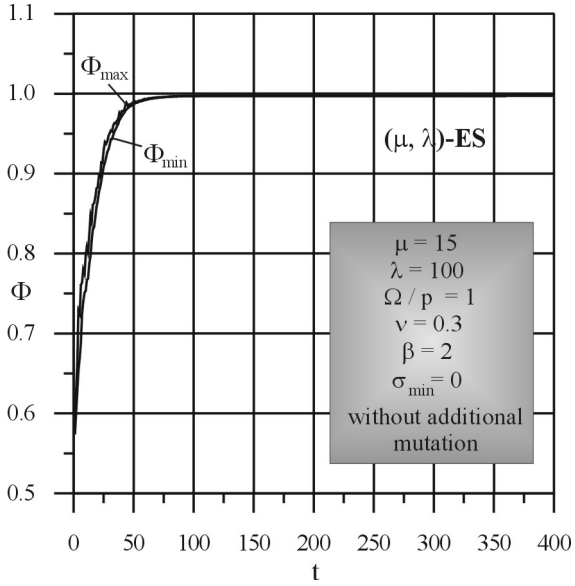

Fig. 6. Minimum and maximum value of objective function with respect to algorithm step number

The average of standard deviations for the population of parents was calculated as a Gaussian norm:

$$\sigma = \sigma(t) = \frac{1}{\mu}\sum_{i=1}^{\mu}\left\{\sum_{j=1}^{n}\left[\sigma_{ij}(t)\right]^2\right\}^{\frac{1}{2}} \qquad (23)$$

One should mention the fast convergence of classic (μ,λ)-ES, which may be considered an advantage. Yet it may be dangerous for the evolution process which can stop too early (premature convergence). After about the hundredth step of the algorithm obtained solutions do not improve so much, and in fact the computation may be stopped.

The parameter $\sigma_{\min}$ in all the presented graphs stands for minimum value of standard deviation (at a gene level).

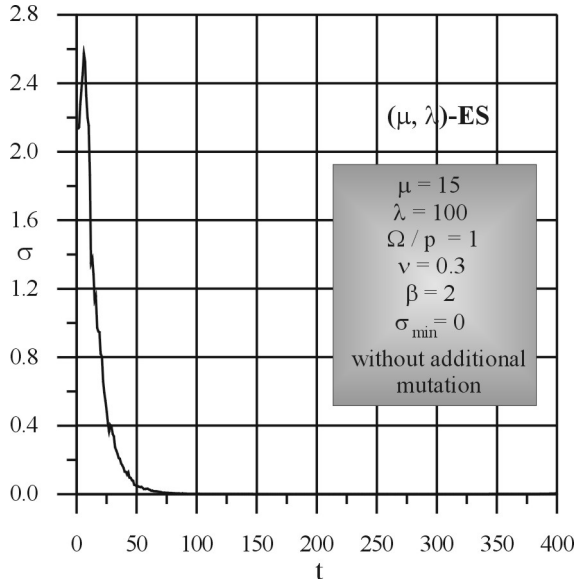

Fig. 7. Average standard deviation of the population with respect to algorithm step number

6. Concluding remarks

In many global optimisation problems that appear in engineering design, experiments (or their simulation) seem to be the only way to evaluate the quality of particular solutions. In these cases one must apply optimisation methods that do not require strict mathematical models of a problem like evolutionary algorithms or simulated annealing.

The paper describes the possible application of evolutionary strategies to the design of rotating variable-thickness annular elastic disc. Proposed representation of the shape (the use of 3rd order splines) turn out to be very useful. First of all it makes possible to describe the shape of the disc more precisely (giving many "degrees of freedom" to the optimisation problem). The results of simulation experiments show that the proposed approach allows to increase the disc carrying capacity (even up to 80% in comparison to the flat shape), and to even out effective stresses in the whole disc.

Further research should concentrate on different variants of evolutionary strategy used for optimisation including specialized operators and parallel (agent-based) implementation of the algorithm.

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