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Almost certain time-optimal positional control

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The present paper deals with a time-optimal control of positional objects whose dynamics are described by a differential inclusion with discontinuous right-hand side. The contents of this paper consist of a probabilistic concept of solving the above task. The existence and characteristics of an almost certain time-optimal control are shown. In compliance with the results obtained, the switching curve, well known from the (deterministic) classical case, has been 'blurred' by a random factor introduced here to the switching region. The material presented constitutes a suitable basis for the creation of technical suboptimal control structures, which can be applied successfully in engineering practice. Empirical experiments confirm numerous advantages of the systems designed, especially in the area of robustness.

1. Introduction

There is a broad class of industrial devices which realize their technological cycles mainly through a change of the position in particular mechanisms, e.g. machining attachments, reversing mills, and especially automata and robots (Tourassis 1988). Such systems are called positional. The dynamics of those devices are described by the differential inclusion

$$\ddot{y}(t) \in \mathcal{H}(\dot{y}(t), y(t), t) + u(t), \tag{1.1}$$

where u is a bounded control function, y denotes a positional parameter of the object, and the function $\mathcal{H}$ models the resistance to motion. If one omits this factor, i.e. with $\mathcal{H} \equiv \{0\}$, formula (1.1) simply expresses the second law of Newtonian mechanics for straight-line or rotary motion. In this paper, the time-optimal stabilization of system (1.1), or the task of reaching the equilibrium state $\dot{y}(t) = y(t) = 0$ in a minimal and finite time, will be considered.

The essential element of the above model is the bounded set-valued function $\mathcal{H}$ describing the resistance to motion. For the majority of cases in practice, this function can be expressed in the form

$$\mathcal{H}(\dot{y}(t), y(t), t) = v(\dot{y}(t), y(t), t)\, \mathcal{F}(\dot{y}(t)), \tag{1.2}$$

where v denotes a bounded real continuous function, and $\mathcal{F}$ is almost everywhere a singleton, representing a bounded real piecewise continuous function which can be multivalued at the points of discontinuity. For the sake of illustration, a simple form of such a function might be

$$\mathcal{F}(\dot{y}(t)) = \begin{cases} \{1\} & \text{if } \dot{y}(t) > 0, \\ [-s, s] & \text{if } \dot{y}(t) = 0, \\ \{-1\} & \text{if } \dot{y}(t) < 0, \end{cases} \tag{1.3}$$

where the parameter $s > 1$ is connected with static friction. Of course, in reality, the form of the function $\mathcal{F}$ can be much more complicated; but, in accordance with the properties of frictional phenomena, even the above trivial shape makes it possible to assume the univalence and continuity of the function v.

Naturally, if the forms of the functions $\mathcal{F}$ and especially v are more complex, then the process of synthesis of time-optimal control and analysis of the system obtained, and also the consecutive identification of its parameters, becomes disproportionately more difficult. In practice, unless the form of the function v is simple—e.g. a constant function (Hejmo & Kloch 1981) or a function in only the third (time) argument (Kulczycki 1995a)—the synthesis of the time-optimal control system turns out to be impossible to realize in a deterministic way.

In this paper a probabilistic concept of solving the time-optimal positional control problem will be proposed. In the model of resistance to motion adopted here, it is assumed that the function v introduced in equation (1.2) is the realization of a given stochastic process V with almost all the realizations being continuous and jointly bounded. Therefore, the dependence of the function v on $\dot{y}(t)$, $y(t)$, and t, is replaced by the dependence on a random factor. Moreover, such a model of resistance to motion also accommodates its probabilistic dependence on various factors—not only $\dot{y}(t)$, $y(t)$, and t, but also those which are usually omitted in the deterministic approach to simplify the model, e.g. temperature. The probabilistic concept also admits the perturbations and noise occurring in the system.

The random factor introduced to the system by the stochastic process V implies that the dynamical system will be described by a random differential inclusion. Because the distribution of the probabilistic measure connected with this process can be concentrated at only one realization, the random problem thus formulated is a generalization of a certain deterministic task in which the controlled system is discontinuous and non-autonomous (Kulczycki 1995a). If, further, the above realization is constant and the function $\mathcal{F}$ is univalent (i.e. singleton-valued), then the problem can be reduced to the basic task of the time-optimal positional control (Hejmo & Kloch 1981). Moreover, when this constant realization has the value zero, the classical case of time-optimal transfer of a mass (Athans & Falb 1966: §7.2) is obtained.

2. Mathematical background

Let $\mathcal{T}$ be an interval with nonempty interior. First, consider a deterministic differential inclusion:

$$\dot{\boldsymbol{x}}(t) \in \mathcal{G}(\boldsymbol{x}(t), t), \tag{2.1}$$

where $\mathcal{G} : \mathbb{R}^n \times \mathcal{T} \to \mathcal{P}(\mathbb{R}^n)$, $\boldsymbol{x} : \mathcal{T} \to \mathbb{R}^n$, and $\mathcal{P}(\mathcal{A})$ denotes the set of subsets of $\mathcal{A}$. The solutions of differential inclusions in three different senses, which are usually used in the analysis of discontinuous dynamical systems, are described below.

Definition 1 The function $\boldsymbol{x}$, absolutely continuous on every compact subinterval of the set $\mathcal{T}$, is a solution of differential inclusion (2.1)

- in the *Caratheodory* sense (C solution), if it satisfies inclusion (2.1) almost everywhere in $\mathcal{T}$,

- in the *Filippov* sense (F solution), if

$$\dot{x}(t) \in \mathsf{F}\mathcal{G}(x(t), t) \quad \text{almost everywhere in } \mathcal{T}, \tag{2.2}$$

- in the *Krasovski* sense (K solution), if

$$\dot{x}(t) \in \mathsf{K}\mathcal{G}(x(t), t) \quad \text{almost everywhere in } \mathcal{T}, \tag{2.3}$$

where the operators **F** and **K** are defined by

$$\mathsf{F}\mathcal{G}(x(t), t) = \bigcap_{e>0} \bigcap_{\mathcal{Z}\subset\mathbb{R}^n : \mathrm{m}(\mathcal{Z})=0} \operatorname{conv} \mathcal{G}([x(t) + e\mathrm{B}_n] \setminus \mathcal{Z}, t), \tag{2.4}$$

$$\mathsf{K}\mathcal{G}(x(t), t) = \bigcap_{e>0} \operatorname{conv} \mathcal{G}(x(t) + e\mathrm{B}_n, t), \tag{2.5}$$

B_n denotes the open unit ball in the space $\mathbb{R}^n$, $\mathrm{m}(\mathcal{Z})$ is the Lebesgue measurement of the set $\mathcal{Z}$, and $\operatorname{conv} \mathcal{C}$ means the convex closed hull of the set $\mathcal{C}$. □

Suppose also that $t_0 \in \mathcal{T}$ and $x_0 \in \mathbb{R}^n$.

Definition 2 The C, F or K solutions of the deterministic differential inclusion (2.1) with an initial condition

$$x(t_0) = x_0 \tag{2.6}$$

are *unique* if all C, F, or K solutions, respectively, are identically equal functions. □

Any C solution is a K solution and any F solution is a K solution, but there is no general relation between C and F of solutions; however, K solutions comprise a very large class. The result is that differential inclusions having a discontinuous right-hand side present a considerable difficulty because of the lack of a universal concept of a solution for them. Of course, if the existence of unique and equal C, F, and K solutions can be shown, then further analysis is simplified.

The above concepts of solutions will be generalized below to random differential inclusions. Such a generalization, however, is not unique. In this paper, the concept of almost certain solutions (with probability 1, first type) will be applied, because of its obvious interpretation.

Let (Ω, Σ, P) be a probability space. From a practical point of view, its completeness can be assumed without any loss in generality (Rudin, 1974: §1.36). Consider the random differential inclusion

$$\dot{X}(\omega, t) \in \mathcal{G}(\omega, X(\omega, t), t), \tag{2.7}$$

where $\mathcal{G} : \Omega \times \mathbb{R}^n \times \mathcal{T} \to \mathcal{P}(\mathbb{R}^n)$ and X denotes an n-dimensional stochastic process defined on (Ω, Σ, P) and $\mathcal{T}$.

Definition 3 A stochastic process X is an *almost certain C, F,* or *K solution* of random differential inclusion (2.7) if almost all its realizations are C, F, or K solutions, respectively, of the corresponding deterministic differential inclusions obtained by fixing some $\omega \in \Omega$. □

Assume also that X_0 is an n-dimensional random variable defined on (Ω, Σ, P).

DEFINITION 4 The almost certain C, F, or K solution of the random differential inclusion (2.7), with an initial condition

$$X(\omega, t_0) = X_0(\omega) \quad \text{for almost all } \omega \in \Omega, \tag{2.8}$$

is *unique* if all almost certain C, F or K solutions, respectively, are an equivalent stochastic process (i.e. $P[\{\omega \in \Omega : X^{\sim}(\omega, t) = X^{\approx}(\omega, t)\}] = 1$ for every $t \in \mathcal{T}$). □

The generalization of the concept of time-optimal control to random systems is not unique, either. From a practical point of view, it would be most useful to define a control that is only a function of time and the state in closed-loop systems, realizing a minimum of the expected value of the time to reach the target set. Unfortunately, such a formulation of the problem does not provide hope for its solution.

In what follows, a different definition of the time-optimal control for random systems is formulated. This control, by analogy to the almost certain solution, will be called an almost certain time-optimal control.

DEFINITION 5 Let $\mathcal{G} : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{T} \to \mathcal{P}(\mathbb{R}^n)$ and suppose that the differential inclusion

$$\dot{X}(\omega, t) \in \mathcal{G}\big(\omega, X(\omega, t), U(\omega, t), t\big), \tag{2.9}$$

with initial condition

$$X(\omega, t_0) = X_0(\omega) \quad \text{for almost all } \omega \in \Omega, \tag{2.10}$$

describes the dynamics of a random system submitted to the control $\mathcal{U}$. Then, an m-dimensional stochastic process U° defined on (Ω, Σ, P) and $\mathcal{T}$ will be called an *almost certain time-optimal control* if almost all its realizations are time-optimal controls for proper deterministic systems obtained at a fixed $\omega \in \Omega$. □

The almost certain time-optimal control ensures realization of the minimum expected value of the time to reach the target set; however, it depends additionally on the random factor. The result of this dependence is that the above control is difficult to apply directly, but it constitutes a useful basis for the creation of technical constructions of suboptimal structures in which the direct dependency of the control function on the random factor is eliminated. This concept will be considered with great care in Section 4.

3. Main results

The following theorem provides a mathematical base for the probabilistic concept of solving the time-optimal control problem investigated in this paper.

THEOREM 1 Assume the following conditions.

(1) $t_0 \in \mathbb{R}$, $\mathcal{T} = [t_0, \infty)$, $x_0 \in \mathbb{R}^2$, and $v_-, v_+ \in \mathbb{R}$ such that $-1 < v_- \leqslant v_+ < 1$;
(2) the origin of coordinates constitutes a target set;
(3) $\mathcal{U}_a = \{u : \mathcal{T} \to [-1, 1]\}$ represents a set of admissible controls;

(4) $\mathcal{F} : \mathbb{R} \to \mathcal{P}([-1,1])$ denotes a real piecewise continuous function, except at a finite number of points at most where it can be multi-valued, and this function is locally Lipschitz except at points of discontinuity and multivalence, and such that $z\,\mathcal{F}(z)$ has only nonnegative elements for every $z \in \mathbb{R}$;
(5) (Ω, Σ, P) is a complete probability space;
(6) V is a real stochastic process defined on (Ω, Σ, P) and $\mathcal{T}$, with almost all realizations being continuous, satisfying the boundary condition $V(\omega, t) \in [v_-, v_+]$ for $t \in \mathcal{T}$;
(7) a random differential inclusion

$$\dot{X}_1(\omega, t) = X_2(\omega, t), \tag{3.1}$$

$$\dot{X}_2(\omega, t) \in U(\omega, t) - V(\omega, t)\mathcal{F}\big(X_2(\omega, t)\big), \tag{3.2}$$

with initial condition

$$\big(X_1(\omega, t_0), X_2(\omega, t_0)\big) = \boldsymbol{x}_0 \quad \text{for almost all } \omega \in \Omega, \tag{3.3}$$

describes the dynamics of the system submitted to the control U.

Then, there exists an almost certain time-optimal control U°, whose realizations take on the values 1 and -1 and have at most one point of discontinuity. This control generates a unique almost certain C solution, which is also a unique almost certain F solution and a unique almost certain K solution.

Proof. Section 3.1 presents the lemma whose thesis determines the truth of the above theorem with a random factor fixed. Theorem 1 itself will be proved in Section 3.2. Some comments concerning the assumptions made are given in Section 3.3.

3.1 Lemma

Lemma 1 Assume that t_0, $\mathcal{T}$, $\boldsymbol{x}_0$, v_-, v_+, target set $\mathcal{U}_{\mathrm{a}}$, and $\mathcal{F}$ satisfy assumptions (1)–(4) of Theorem 1. Suppose also that:

(5) $v : \mathcal{T} \to [v_-, v_+]$ is a continuous function;
(6) a deterministic differential inclusion

$$\dot{x}_1(t) = x_2(t), \tag{3.4}$$

$$\dot{x}_2(t) \in u(t) - v(t)\,\mathcal{F}\big(x_2(t)\big), \tag{3.5}$$

with initial condition

$$\big(x_1(t_0), x_2(t_0)\big) = \boldsymbol{x}_0, \tag{3.6}$$

describes the dynamics of the system submitted to the control u.

Then there exists a time-optimal control $u^{\mathrm{o}} \in \mathcal{U}_{\mathrm{a}}$, which takes on the values ± 1, has at most one discontinuity point, and generates a unique C solution which is also a unique F solution and unique K solution.

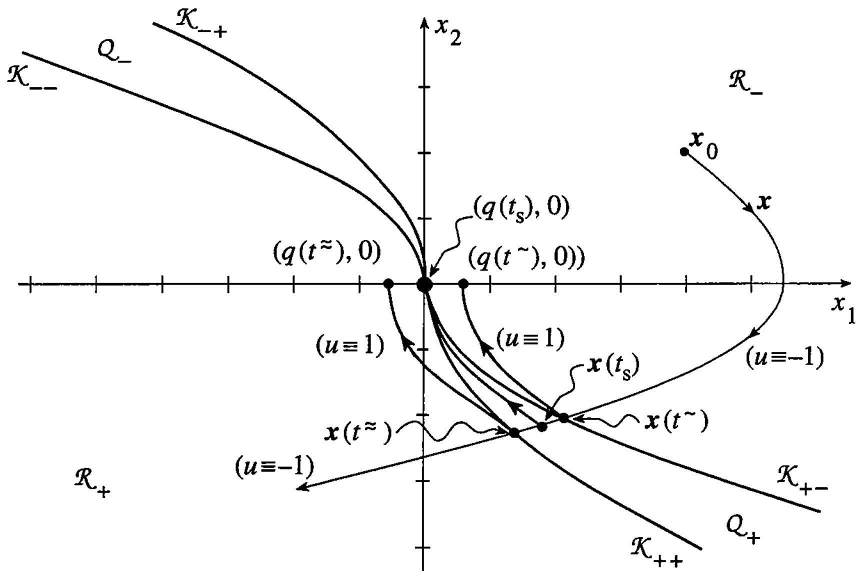

Fig. 1. Subdivision of state space into sets $\{(0,0)\}$, $\mathcal{Q}_+$, $\mathcal{Q}_-$, $\mathcal{R}_-$, $\mathcal{R}_-$, and illustration of proof in the case of $x_0 \in \mathcal{R}_-$.

Proof. A brief description of the proof, including the facts used subsequently in this paper, will be given below. See Kulczyski (1995a) for a full proof.

Denote by x_{--} and x_{-+} the unique C solutions of the system (3.4)–(3.5) with the terminal condition $x(0) = (0,0)$, defined on the interval $(-\infty, 0]$, and generated by the control $u \equiv -1$, when $v \equiv v_-$ or $v \equiv v_+$, respectively. Let

$$\mathcal{K}_{--} = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : (\xi_1, \xi_2) = x_{--}(t) \text{ for some } t \in (-\infty, 0)\}, \tag{3.7}$$

$$\mathcal{K}_{-+} = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : (\xi_1, \xi_2) = x_{--}(t) \text{ for some } t \in (-\infty, 0)\}. \tag{3.8}$$

Thus these are the sets of all states which can be brought to the origin by the control $u \equiv -1$, when $v \equiv v_-$ or $v \equiv v_+$, respectively (Fig. 1). Similarly, denote by x_{+-} and x_{++} unique C solutions of that system, generated by $u \equiv 1$, when $v \equiv v_-$ or $v \equiv v_-$, respectively. Define the sets $\mathcal{K}_{+-}$ and $\mathcal{K}_{++}$ analogously to the above ones. Moreover, consider the following sets:

$$\mathcal{Q}_+ = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \text{there exist } (\xi_1', \xi_2) \in \mathcal{K}_{++} \text{ and } (\xi_1'', \xi_2) \in \mathcal{K}_{+-} \text{ with } \xi_1' \leqslant \xi_1 \leqslant \xi_1''\}, \tag{3.9}$$

$$\mathcal{Q}_- = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \text{there exist } (\xi_1', \xi_2) \in \mathcal{K}_{--} \text{ and } (\xi_1'', \xi_2) \in \mathcal{K}_{-+} \text{ with } \xi_1' \leqslant \xi_1 \leqslant \xi_1''\}, \tag{3.10}$$

$$\mathcal{R}_+ = \{(\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \mathcal{Q} : \text{there exists } (\xi_1', \xi_2) \in \mathcal{Q} \text{ with } \xi_1 < \xi_1'\}, \tag{3.11}$$

$$\mathcal{R}_- = \{(\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \mathcal{Q} : \text{there exists } (\xi_1', \xi_2) \in \mathcal{Q} \text{ with } \xi_1' < \xi_1\}, \tag{3.12}$$

where $\mathcal{Q} = \mathcal{Q}_+ \cup \{(0,0)\} \cup \mathcal{Q}_-$. By virtue of this, the state space has been subdivided into disjoint and non-empty sets: $\{(0,0)\}$, $\mathcal{Q}_+$, $\mathcal{Q}_-$, $\mathcal{R}_+$, $\mathcal{R}_-$ (Fig. 1). Now let v be the given function occurring in assumption 5.

The case $\boldsymbol{x}_0 \in \mathcal{R}_-$ will be considered first (Fig. 1). The C solution $\boldsymbol{x} = (x_1, x_2)$ of system (3.4)–(3.5), fulfilling initial condition (3.6), and generated by the control $u \equiv -1$, crosses the set $\mathcal{K}_{+-}$ in a finite time $t^{\sim}$ and then the set $\mathcal{K}_{++}$ also in a finite time $t^{\approx}$. If at the moment $t^{\sim}$ the value of the control is changed to 1, the second component of the above C solution reaches zero in a finite time $t^{\wedge}$, with $0 \leqslant x_1(t^{\wedge})$. A similar situation is encountered when the change occurs at the moment $t^{\approx}$, but then $x_1(t^{\wedge}) \leqslant 0$. And now, the function q which assigns the coordinate $x_1(t^{\wedge})$ of the point of the axis $x_2 = 0$ crossed by this C solution, to the time τ of changing the control value, namely $q : \tau \mapsto x_1(t^{\wedge})$, is continuous by the form of equation (3.4) and the continuity of an integral with parameter. Because a continuous function q defined on the connected set $[t^{\sim}, t^{\approx}]$ takes on all the intermediate values in $[q(t^{\sim}), q(t^{\approx})]$, there exists $t_s \in [t^{\sim}, t^{\approx}]$ such that $x_1(t^{\wedge}) = 0$, and so $\boldsymbol{x}(t^{\wedge}) = (0,0)$; then $t_f = t^{\wedge}$ is the finite time for the above C solution to reach the origin. To summarize, if $\boldsymbol{x}_0 \in \mathcal{R}_-$, there exists t_s such that the C solution generated by the control

$$u^{\circ}(t) = \begin{cases} -1 & \text{for } t \in [t_0, t_s), \\ 1 & \text{for } t \in [t_s, \infty), \end{cases} \tag{3.13}$$

reaches the origin in the finite time t_f, with $t_0 < t_s < t_f$ and $\boldsymbol{x}(t_s) \in \mathcal{Q}_+$ (Fig. 1).

Analogously, if $\boldsymbol{x}_0 \in \mathcal{R}_+$, there exists t_s such that the C solution generated by the control

$$u^{\circ}(t) = \begin{cases} 1 & \text{for } t \in [t_0, t_s), \\ -1 & \text{for } t \in [t_s, \infty), \end{cases} \tag{3.14}$$

reaches the origin in the finite time t_f, with $t_0 < t_s < t_f$ and $x(t_s) \in \mathcal{Q}_-$.

Consider now the case $\boldsymbol{x}_0 \in \mathcal{Q}_+$. The C solution $\boldsymbol{x} = (x_1, x_2)$ generated by the control $u \equiv 1$ reaches the axis $x_2 = 0$ in a finite time $t^{\wedge}$. If $x_1(t^{\wedge}) = 0$, so that $\boldsymbol{x}(t^{\wedge}) = (0,0)$, the control sought is

$$u^{\circ}(t) = 1 \quad \text{for } t \in [t_0, \infty), \tag{3.15}$$

and then $t_f = t^{\wedge}$ is the finite time for this C solution to reach the origin. However, if $0 < x_1(t^{\wedge})$, a consideration analogous to that of the case $\boldsymbol{x}_0 \in \mathcal{R}_-$ establishes the existence of the control (3.13), where the role of the point $\boldsymbol{x}(t^{\sim})$ is taken by $\boldsymbol{x}_0$. Finally, if $x_1(t^{\wedge}) < 0$, then $\boldsymbol{x}(t^{\wedge}) \in \mathcal{R}_-$; so, by prolongation of the positive control value, the consideration for the case $\boldsymbol{x}_0 \in \mathcal{R}_+$ can be continued for $t \geqslant t^{\wedge}$, giving an adequate control of the form (3.14).

The case $\boldsymbol{x}_0 \in \mathcal{Q}_-$ can be considered analogously. In this case, the counterpart of formula (3.15) is

$$u^{\circ}(t) = -1 \quad \text{for } t \in [t_0, \infty). \tag{3.16}$$

By combining the above four cases, a control u^0 of the form (3.13), (3.14), (3.15), or (3.16) has been assigned to every initial state $\boldsymbol{x}_0 \in \mathbb{R}^2 \setminus \{(0,0)\}$.

The proof of the optimality of such a control is based on a certain differential inequality (Hejmo & Kloch 1981). Namely it can be shown that, if z is any absolutely continuous function and y denotes a C solution of the differential inclusion with the right-hand-side mapping $\mathcal{G} : \mathbb{R} \times \mathcal{T} \to \mathcal{P}(\mathbb{R})$ satisfying certain assumptions which hold for the above formulated conditions, then the differential inequality

$$z(t_0) \leqslant y(t_0), \tag{3.17}$$

$$\dot{z}(t) \leqslant G\big(z(t), t\big) \quad \text{almost everywhere in } \mathcal{T}, \tag{3.18}$$

where $G(\zeta, t)$ is an arbitrary element of $\mathcal{G}(\zeta, t)$, implies the following inequality:

$$z(t) \leqslant y(t) \quad \text{for } t \in \mathcal{T}. \tag{3.19}$$

First, consider the initial state $\boldsymbol{x}_0 = (x_{01}, x_{02})$ to which the control (3.15) has been assigned. The optimality of this control will be proved by contradiction. So, let there exist a control $u^* \in \mathcal{U}_{\mathrm{a}}$ which brings the initial state under consideration along the C solution $\boldsymbol{x}^* = (x_1^*, x_2^*)$ to the origin in the time t_{f}^* such that $t_{\mathrm{f}}^* < t_{\mathrm{f}}$. It follows from formula (3.5) that, for the control (3.15), the function $\dot{x}_2$ is positive, and therefore the absolutely continuous function x_2 is strongly increasing; in particular,

$$x_2(t_{\mathrm{f}}^*) < x_2(t_{\mathrm{f}}) = 0. \tag{3.20}$$

The form of the set $\mathcal{U}_{\mathrm{a}}$ implies that

$$\begin{aligned} \dot{x}_2^*(t) &= u^*(t) - F\big(x_2^*(t)\big)\, v(t) \\ &\leqslant u^{\mathrm{o}}(t) - F\big(x_2^*(t)\big) v(t) \quad \text{almost everywhere in } [t_0, \infty), \end{aligned} \tag{3.21}$$

where $F(\xi)$ is an arbitrary element of $\mathcal{F}(\xi)$. From this and from the dependence $x_2^*(t_0) = x_{02} = x_2(t_0)$, on the basis of inequality (3.19),

$$x_2^*(t) \leqslant x_2(t) \quad \text{for } t \in [t_0, \infty), \tag{3.22}$$

or especially

$$0 = x_2^*(t_{\mathrm{f}}^*) \leqslant x_2(t_{\mathrm{f}}^*). \tag{3.23}$$

Inequalities (3.20) and (3.23) constitute a contradiction, which proves the optimality of the control (3.15) for the initial state under consideration.

The initial state to which the control (3.16) has been assigned can be considered analogously. A similar situation occurs in the cases when the control is of the form (3.13) or (3.14), where the proof breaks into two stages, each requiring the use of inequality (3.19).

Thanks to the results of Hajek (1979), it is readily shown that the control u^{o} generates unique and equal to each other C, F, and K solutions of system (3.4)–(3.5) with initial condition (3.6). This concludes the brief description of the proof of Lemma 1, given in detail by Kulczycki (1995a).

3.2. *Proof of Theorem 1*

Denote by $\Omega^{\sim}$ the set of those $\omega \in \Omega$ for which the assumptions of Lemma 1 are fulfilled. Let $\Omega^{\approx} = \Omega \setminus \Omega^{\sim}$; of course, $P(\Omega^{\approx}) = 0$. It will be proved that the function $U^{\mathrm{o}} : \Omega \times \mathcal{T} \to \{-1, 1\}$ defined by

$$U^{\mathrm{o}}(\omega, \cdot) \equiv \begin{cases} u^{\mathrm{o}} : \mathcal{T} \to \{-1,1\} & \text{assigned to } \boldsymbol{x}_0 \text{ in Lemma 1 for } v \equiv V(\omega, \cdot) \\ & \text{if } \omega \in \Omega^{\sim}, \\ u : \mathcal{T} \to \{1\} & \text{if } \omega \in \Omega^{\approx}, \end{cases} \tag{3.24}$$

is a stochastic process. Because the probability space is complete, the zero-measure set $\Omega^{\approx}$ does not influence the measurability, and will be omitted in this part of the proof.

First, consider the case $\boldsymbol{x}_0 \in \mathcal{R}_-$. Let $\mathcal{J} = [t_0, t'']$, where $t'' = \sup_{\omega \in \Omega} t'(\omega) < \infty$ and $t'(\omega)$ is a time for the set $\mathcal{K}_{++}$ to be crossed by the C solution $\boldsymbol{x}$ of system (3.4)–(3.5), fulfilling initial condition (3.6), and generated by the control $u^{\mathrm{o}} \equiv -1$, when $v \equiv V(\omega, \cdot)$. Suppose that the function $\boldsymbol{p} : \Omega \times \mathcal{J} \to \mathbb{R}^2$ is such that

$$\boldsymbol{p}(\omega, t) = \boldsymbol{x}(t). \tag{3.25}$$

With a fixed $t \in \mathcal{J}$, the function $\boldsymbol{p}(\cdot, t)$ may be expressed as the composition $\boldsymbol{p}_3 \circ \boldsymbol{p}_2 \circ p_1$ of the mappings

$$p_1 : \Omega \to \mathrm{C}(\mathcal{J}, \mathbb{R}) \quad : \omega \mapsto V(\omega, \cdot) \restriction \mathcal{J}, \tag{3.26}$$

$$\boldsymbol{p}_2 : p_1(\Omega) \to \mathrm{C}(\mathcal{J}, \mathbb{R}^2) : V(\omega, \cdot) \restriction \mathcal{J} \mapsto \boldsymbol{x} \restriction \mathcal{J}, \tag{3.27}$$

$$\boldsymbol{p}_3 : \boldsymbol{p}_2(p_1(\Omega)) \to \mathbb{R}^2 \quad : \boldsymbol{x} \restriction \mathcal{J} \mapsto \boldsymbol{x}(t). \tag{3.28}$$

For any open set $\mathcal{D} \subseteq \mathrm{C}(\mathcal{J}, \mathbb{R})$, there exists a sequence of polynomial functions $v_i : \mathcal{J} \to \mathbb{R}$ $(i = 1, 2, \ldots)$ with rational coefficients, together with a sequence of real numbers d_i $(i = 1, 2, \ldots)$, such that

$$\begin{aligned} \mathcal{D} &= \bigcup_{i=1}^{\infty} \mathcal{B}(v_i, d_i) = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \bar{\mathcal{B}}(v_i, e_{i,j}) \\ &= \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \{v \in \mathrm{C}(\mathcal{J}, \mathbb{R}) : \max_{t \in \mathcal{J}}(|v(t) - v_i(t)|) \leqslant e_{i,j}\} \\ &= \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \{v \in \mathrm{C}(\mathcal{J}, \mathbb{R}) : \max_{t \in \mathcal{J} \cap \mathbb{Q}}(|v(t) - v_i(t)|) \leqslant e_{i,j}\} \\ &= \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcap_{t \in \mathcal{J} \cap \mathbb{Q}} \{v \in \mathrm{C}(\mathcal{J}, \mathbb{R}) : |v(t) - v_i(t)| \leqslant e_{i,j}\}, \end{aligned} \tag{3.29}$$

where $\mathcal{B}(\xi, r)$ is the open ball of centre ξ and radius r, with $\bar{\mathcal{B}}(\xi, r)$ the corresponding closed ball, in $\mathrm{C}(\mathcal{J}, \mathbb{R})$, and, for each $i \in \mathbb{N} \setminus \{0\}$, $(e_{i,j} : j \in \mathbb{N} \setminus \{0\})$ is an increasing sequence of real numbers convergent to d_i. As a result:

$$p_1^{-1}(\mathcal{D}) = \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcap_{t \in \mathcal{J} \cap \mathbb{Q}} p_1^{-1}\big(\{v \in \mathrm{C}(\mathcal{J}, \mathbb{R}) : |v(t) - v_i(t)| \leqslant e_{i,j}\}\big). \tag{3.30}$$

Because V is a stochastic process, for fixed i, j, and t, the set $p_1^{-1}(\{v \in C(\mathcal{J}, \mathbb{R}) : |v(t) - v_i(t)| \leqslant e_{i,j}\})$ is measurable; so, thanks to the above equality, the set $p_1^{-1}(\mathcal{D})$ is measurable, too. The mapping p_1 is therefore measurable.

The continuity of the mapping $\boldsymbol{p}_2$ results from the continuous dependence of a C solution on a right-hand side of a differential inclusion. Also, the continuity of the mapping $\boldsymbol{p}_3$ is trivial. Thus, the properties of the mappings p_1, $\boldsymbol{p}_2$, and $\boldsymbol{p}_3$ shown above implies the measurability of the function $\boldsymbol{p}(\cdot, t)$ for any fixed $t \in \mathcal{J}$. Further, the continuity of the function $\boldsymbol{p}(\omega, \cdot)$ for any $\omega \in \Omega$ results from the definition of a C solution. This implies the measurability of the function $\boldsymbol{p}$ with respect to the product sigma-algebra in $\Omega \times \mathcal{J}$.

Let $\mathcal{E} = \{(\omega, t) \in \Omega \times \mathcal{J} : \boldsymbol{x}(t) \in \mathcal{Q}_{\pm}\}$, or $\mathcal{E} = \boldsymbol{p}^{-1}(\mathcal{Q}_+)$. Now

$$\mathcal{Q}_+ = (\operatorname{int} \mathcal{Q}_{\mp}) \cup [\mathcal{K}_{++} \cup \{(0,0)\}] \cup [\mathcal{K}_{\pm -} \cup \{(0,0)\}] \setminus \{(0,0)\} \tag{3.31}$$

and the set $\operatorname{int} \mathcal{Q}_{\pm}$ is open; also the sets $\mathcal{K}_{++} \cup \{(0,0)\}$, $\mathcal{K}_{+-} \cup \{(0,0)\}$, and $\{(0,0)\}$ are closed. Hence the set $\mathcal{E}$ is measurable.

Suppose that (ω, τ) is a given element of the set $\mathcal{E}$. Then, let $\hat{\boldsymbol{x}} = (\hat{x}_1, \hat{x}_2)$ be the C solution of system (3.4)–(3.5) for $t \geqslant \tau$, with the initial condition $\hat{\boldsymbol{x}}(\tau) = \boldsymbol{x}(\tau)$, generated by $u \equiv 1$, with $v(t) = V(\omega, t)$ for $t \geqslant \tau$.

Consider the mapping $\boldsymbol{r}^* : \mathcal{E} \to C([0, t^*], \mathbb{R}^2)$ such that

$$\boldsymbol{r}^*(\omega, \tau)(s) = \hat{\boldsymbol{x}}(s + \tau) \quad \text{for } s \in [0, t^*], \tag{3.32}$$

where $t^* = \sup_{\omega \in \Omega} t_* < \infty$ and $t_* + \tau$ is the time of crossing the axis $x_2 = 0$ by the C solution $\hat{\boldsymbol{x}}$. It has been proved that the domain of this mapping is measurable. The remaining condition for the measurability of the mapping $\boldsymbol{r}^*$ can be shown analogously to the above by the composition of the mappings r_1 and $\boldsymbol{r}_2$, which are the counterparts of p_1 and $\boldsymbol{p}_2$. Further, define the mapping $r_3 : \boldsymbol{r}^*(\mathcal{E}) \to \mathbb{R}$ by

$$r_3(\hat{\boldsymbol{x}}) = \hat{x}_1(t_* + \tau). \tag{3.33}$$

Of course r_3 is continuous; therefore, the mapping $r : \mathcal{E} \to \mathbb{R}$ defined hereby as the composition $r_3 \circ \boldsymbol{r}^*$, is measurable.

Denote by $\mathcal{E}_+ \subseteq \mathcal{E}$ and $\mathcal{E}_- \subseteq \mathcal{E}$ the sets of those elements (ω, τ) for which the C solutions $\hat{\boldsymbol{x}}$ that are their images under the mapping $\boldsymbol{r}^*$ cross the positive and negative part of the x_1 axis respectively. From the equality

$$\mathcal{E} \setminus (\mathcal{E}_+ \cup \mathcal{E}_-) = r^{-1}(0) \tag{3.34}$$

and the measurability of the mapping r, it follows that the set $\mathcal{E} \setminus (\mathcal{E}_+ \cup \mathcal{E}_-)$ is measurable in $\Omega \times \mathcal{J}$.

Now let $\sigma : \Omega \to \mathbb{R}$ be a function defined by

$$\sigma(\omega) = t_s, \tag{3.35}$$

where t_s is the time of change of value of the function u^o of the form (3.13), assigned in Lemma 1 to the initial state under consideration, when $v \equiv V(\omega, \cdot)$. The graph of the function σ is the set $\mathcal{E} \setminus (\mathcal{E}_+ \cup \mathcal{E}_-)$, measurable in $\Omega \times \mathcal{J}$; therefore, σ is measurable.

Next, suppose that $t \in \mathcal{T}$ is fixed. The function $U^o(\cdot, t)$ takes on only the values

± 1; so, in order to show its measurability, it is enough to prove the measurability of the set $U^\circ(\cdot, t)^{-1}(\{-1\})$. The form of the control defined by formula (3.13) then yields

$$U^\circ(\cdot, t)^{-1}(\{-1\}) = \sigma^{-1}\big((t, \infty)\big); \tag{3.36}$$

thus, the measurability of the function σ clearly implies the measurability of the above set. Finally, for $x_0 \in \mathcal{R}_-$, the measurability of the function $U^\circ(\cdot, t)$ for any $t \in \mathcal{T}$, has been shown.

In the case of $x_0 \in \mathcal{R}_+$, the proof is analogous.

Now let $x_0 \in \mathcal{Q}_+$. The proof of the measurability of the function $U^\circ(\cdot, t)$ for any fixed $t \in \mathcal{T}$ is similar; however, one should take into consideration separately the subsets of those $\omega \in \Omega$ for which the C solutions of systems (3.4)–(3.5), with (3.6), $u \equiv 1$, and $v \equiv V(\omega, \cdot)$, cross the positive and negative parts of the x_1 axis (which are connected here with a different form of the control: (3.13) and (3.14), respectively).

If $x_0 \in \mathcal{Q}_-$, the proof is analogous to the above one.

So, it has been shown that the function $U^\circ(\cdot, t)$ is measurable for any $t \in \mathcal{T}$; therefore, U° is a stochastic process. Thus the existence of the almost certain time-optimal control has been proved.

Now define the function $X : \Omega \times \mathcal{T} \to \mathbb{R}^2$ as follows:

- if $\omega \in \Omega^\sim$, then $X(\omega, \cdot) \equiv x : \mathcal{T} \to \mathbb{R}^2$ given as the C solution of system (3.5)–(3.6) with (3.7), $u^\circ \equiv U^\circ(\omega, \cdot)$, $v \equiv V(\omega, \cdot)$;
- if $\omega \in \Omega^\approx$, then $X(\omega, \cdot) \equiv x : \mathcal{T} \to \{(0, 0)\}$.

That a so-defined function is a stochastic process can be proved similarly. The only difference consists of the fact that, contrary to $V(\omega, \cdot)$, the functions $U^\circ(\omega, \cdot)$ are only piecewise continuous. But it is possible to approximate them by sequences of continuous functions and profit from a continuous dependence of C solutions on a right-hand side of a differential inclusion.

So, the function X is the sought almost certain C solution generated in system (3.1)–(3.2) by the control U° defined by formula (3.24). The existence of almost certain F or K solutions can be shown identically.

The uniqueness of almost certain C, F, and K solutions of differential inclusion (3.1)–(3.2) with initial condition (3.3) clearly results from the uniqueness of the deterministic solutions at a fixed $\omega \in \Omega^\sim$, implied by the thesis of Lemma 1. Of course, from the same lemma, they are also equal to each other.

Thus, the thesis of Theorem 1 has finally been proved.

3.3. Some comments on the assumptions made

The assumption that $z\,\mathcal{F}(z)$ can be only nonnegative has been formulated in Theorem 1 merely for the sake of clarity of notation. Anyway this inequality is physically justified, because positive values of the stochastic process V make it consistent with the property of energy dissipation.

The condition $[v_-, v_+] \subset (-1, 1)$ ensures controllability of the system. The assumption that the function $\mathcal{F}$ fulfills a Lipschitz condition where it is univalent and continuous has been introduced to guarantee the uniqueness and equality of C, F, and K solutions.

In assumption (6) of Theorem 1, the stochastic process V has almost all realizations continuous and jointly bounded by the inequality $v_- \leqslant V(\omega, t) \leqslant v_+$ for $t \in \mathcal{T}$. On the basis of an analytical description of a given stochastic process, it is easy to determine whether it satisfies the above conditions. However, such a description is hardly available in practice. Usually, using the formulated hypotheses (e.g. stationarity and Markov properties), empirical distributions induced by the tested process on the spaces $\mathbb{R}^n$ for $n = 1, 2, ...,$ or the so-called finite-dimensional distributions, are calculated. Due to identification errors, the question occurs of whether there even exists a stochastic process with the measured distributions—and, if so, whether it satisfies the assumed conditions of continuity and boundedness. If certain conditions, which are so insignificantly weak that they can be treated as a test for the correctness of measurement of the finite-dimensional distributions, are satisfied, then the Kolmogorov theorem ensures the existence of a suitable separable stochastic process. The assumption of boundedness of almost all its realizations can be examined by one-dimensional distributions; and another theorem, also originating from Kolmogorov, formulates a sufficient condition for the continuity of almost all realizations of a stochastic process on the basis of properties of two-dimensional distributions. To summarize, the assumptions formulated in Theorem 1 with respect to the stochastic process V are thus mutually independent and identifiable on the basis of finite-dimensional distributions (Wong 1971: Ch. 2).

It is worthwhile noticing that the thesis of Lemma 1 is also true if the function v is only piecewise continuous (Kulczycki 1995a). But the analogous generalization of Theorem 1 has no practical meaning, because a stochastic process with almost all realizations being piecewise continuous and jointly bounded is not identifiable using finite-dimensional distributions.

4. Conclusions

The subject of this paper has been a probabilistic way of solving the problem of time-optimal control of discontinuous positional systems. The existence and characteristics of the so-called almost certain time-optimal control are shown in Theorem 1 with Lemma 1. The state space has been subdivided here into the sets $\mathcal{R}_-, \mathcal{R}_+, \mathcal{Q}_-, \mathcal{Q}_-$, with the origin being a target (Fig. 1). The borderlines are the sets $\mathcal{K}_{--}, \mathcal{K}_{-+}, \mathcal{K}_{+-}, \mathcal{K}_{-+}$, defined constructively in the proof of Lemma 1. Thus, if $x_0 \in \mathcal{R}_-$, then almost all realizations of that control have the form of the control sequence $(-1, 1)$, where the change of the value, i.e. switching of the control, occurs when the system state belongs to the set $\mathcal{Q}_+$. Similarly if $x_0 \in \mathcal{R}_+$, such a sequence takes on the form $(1, -1)$ and the switching exists when the state is included in the set $\mathcal{Q}_-$. In the cases $x_0 \in \mathcal{Q}_-$ and $x_0 \in \mathcal{Q}_+$, both the above controls are possible, but can be simply -1 or 1, respectively. The precise shape of a control sequence, especially the rigorous time of switching, is dependent on the random factor. Because the switching of the control can appear only when the system state belongs to the closed region $\mathcal{Q} = \mathcal{Q}_+ \cup \{(0, 0)\} \cup \mathcal{Q}_-$, this set will be called a *switching region*.

In compliance with the above results, the switching curve γ, well known from the classical (deterministic) case of time-optimal transfer of a mass (Athans & Falb 1966: § 7.2), has been generalized here to the switching region $\mathcal{Q}$. It should be stated that $\gamma \subset \mathcal{Q}$ only if $0 \in [v_-, v_+]$—which, however, is never true in practice. Moreover, if the distribution connected with the stochastic process V reduces to the deterministic case $v_- = v_+$, then $\mathcal{K}_{+-} = \mathcal{K}_{++}$ and $\mathcal{K}_{--} = \mathcal{K}_{-+}$; in this case, the switching area $\mathcal{Q}$ simplifies to the switching curve, which was considered by Hejmo & Kloch (1981). Of course, an artificial additional condition, such as $v_- = v_+ = 0$ implies that $\mathcal{Q} = \gamma$. These facts were mentioned in the Introduction.

In many technical problems, the form of the applied model of resistance to motion has great influence on the complexity or even the feasibility of a successful analysis. In this paper the function $\mathcal{H}$, defined in the Introduction and representing the model of motion resistances, has been decomposed into two factors: $\mathcal{F}(\dot{y}(t))$ and $V(\omega, t)$. The former, being deterministic, has no great influences on the complexity of a theoretical analysis, and so makes it possible to incorporate the properties of discontinuity and multivalence of friction phenomena. The latter, thanks to its probabilistic nature, includes (among others) approximations and identification errors (of the first factor, too), the dependence of motion resistances on position, time, and temperature, as well as perturbations and noise naturally occurring in real systems. The switching curve which is implied by the first—deterministic—factor, has been 'blurred' by the second—random—one to a switching region.

Finally, some practical suggestions of application of the results presented here will be given below. Besides specific cases, the direct realization of the system generating the almost certain time-optimal control U° encounters difficulties. The exact value of this control is in fact dependent on a random factor, which is unknown *a priori*. However, the presented material constitutes a suitable basis for the creation of technical constructions of suboptimal control structures in which such a dependence is eliminated.

For example, in the case of open-loop systems, the expectation of the stochastic process U° can be used in the construction of the suboptimal control. If the limitations of the actuator allow only extreme values of the set of admissible controls, then it is possible to apply the control sequences $(-1, 1)$ or $(1, -1)$, where the time of switching is the expectation of the sign changes in particular realizations of the stochastic process U°. In both cases, C, F, and K solutions exist in the system.

Naturally, from a practical point of view, closed-loop structures of controllers are preferable. Similarly to the classical case, the time-optimal control considered in this paper can be defined as a feedback controller by

$$U^{\circ}\big(X(\omega, t)\big) = \begin{cases} 1 & \text{if } X(\omega, t) \in \mathcal{R}_+, \\ -1 & \text{if } X(\omega, t) \in \mathcal{R}_-, \end{cases} \tag{4.1}$$

and then, for $X(\omega, t) \in \mathcal{Q}_- \cup \mathcal{Q}_+$, this function can be additionally defined—without direction dependence on a random factor, but only in a suboptimal way—e.g.

$$U^{s}\big(X(\omega, t)\big) = \begin{cases} d & \text{if } X(\omega, t) \in \mathcal{Q}_+, \\ -d & \text{if } X(\omega, t) \in \mathcal{Q}_-, \end{cases} \tag{4.2}$$

where $0 < d \leqslant 1$. In practice the value of the parameter d can be obtained heuristically. Usually this value should be close to 1, but it can also vary in the area $\mathcal{Q}$, taking on the value d_* such that

$$d_* \leqslant 1 - (v_+ - v_-) \tag{4.3}$$

on the sets $\mathcal{K}_{+-}$ and $\mathcal{K}_{--}$, and increasing continuously up to the value 1 on the sets $\mathcal{K}_{++}$ and $\mathcal{K}_{-+}$. This makes it possible to achieve a result similar to the bicycle-racing track or bobsleigh track, which are horizontal on the interior part, and become more vertical the farther they go to the outside edge. From a theoretical point of view, the proposed idea constitutes a transposition of the so-called 'admissible adaptive law' (Polycarpou & Ioannou 1993) for random systems. Inequality (4.3) and the above postulated continuity of variation of the parameter d have the goal of allowing the existence of C solutions. However, assigning the value of the parameter d can be conducted similarly to the 'admissible adaptation', but additionally taking into account a random factor. In particular, the value of the parameter d should be equal to 1 even in the neighbourhood of the sets K_{++} and K_{-+}, to neutralize the most unfavourable realizations of that factor.

Analogously to the first pair of examples, if constraints of an actuator limit the control to the extreme values of the admissible set, the results of Theorem 1 may be modified according to the physical observation that the influence of motion resistance in both periods of time—before and after the switching—can be averaged. Thus, after performing a detailed analysis of the sensitivity of the control system to the values of motion resistances, one can use elements of statistical decision theory, where a loss function is connected with extending the time of reaching the target if the control switching has been too late or too early. Kulczycki (1995b) provides a detailed description of such a concept of a feedback controller. It is worth noticing that, in the general case, there are no C solutions in the system obtained, whereas F and K solutions are nonunique.

The probabilistic concept of the control systems designed in the present paper have been successfully empirically verified (Kulczycki 1993a). During the real-time control in industrial process, a convenient method, based on neural networks and investigated by Kulczycki & Schiøler (1993), can be profitably used. It should be underlined that the control system constructed turned out to be only slightly sensitive to the inaccuracy resulting from identification and perturbations; this is a very valuable property of random control systems (Kulczycki 1993b).

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