Isogeometric Residual Minimization Method for Time-Dependent Maxwell Problem

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National Science Centre, Poland grant 2017/26/M/ST1/00281

Time-Dependent problems

[1] M.Łoś, J.Munoz-Matute, I.Muga, M.Paszyński **2021** Isogeometric Residual Minimization (iGRM) for Non-Stationary Stokes and Navier–Stokes Problems *Computers&Mathematics with Applications*

[2] M.Łoś, J.Munoz-Matute, I.Muga, M.Paszyński **2019** Isogeometric Residual Minimization (iGRM) with Direction Splitting for Non-Stationary Advection-Diffusion Problems *Computers&Mathematics with Applications*

[3] M.Łoś,M.Woźniak,M.Paszyński,A.Lenharth,K.Pingali **2017** IGA-ADS: Isogeometric Analysis FEM using ADS *Computer&PhysicsCommunications*

Marcin Łoś, Ph.D.
AGH University, Kraków, Poland

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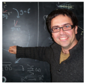


Ignacio Muga, Professor,
The Pontifical Catholic University of Valparaiso, Chile

Keshav Pingali, Professor,
Oden Institute, UT Austin USA

Overview

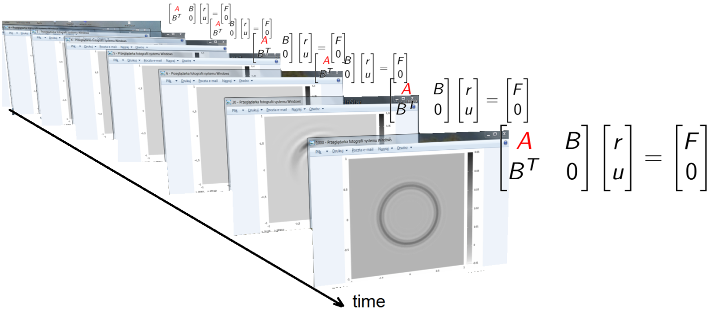


Figure: **Stabilization of time-dependent problems:**

+ finite difference discretization in time (time steps),

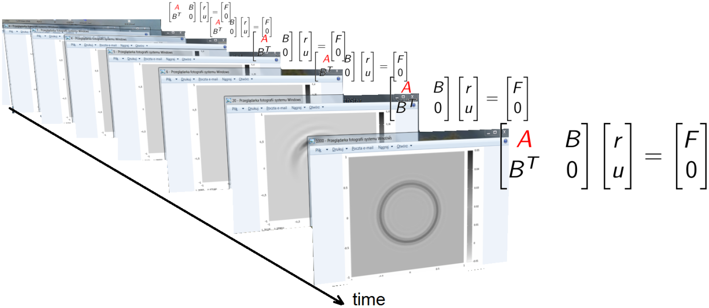
+ unconditionally stable second order time discretization scheme allowing for direction-splitting in time (stability in time)

+ higher order finite element discretization in space (separate problem in each time step, B-spline based discretization),

+ residual minimization method in every time step (stability in space),

+ Kronecker product structure of matrices (fast $\mathcal{O}(N)$ solver)

Overview



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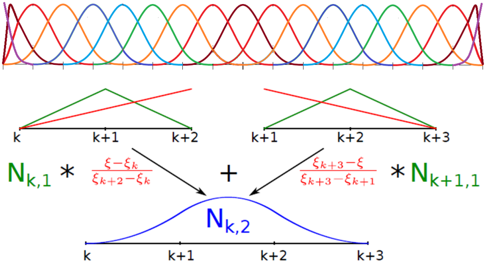
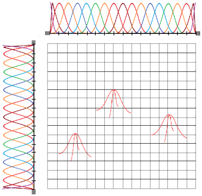
+ finite difference discretization in time (time steps),
+ unconditionally stable second order time discretization scheme allowing for direction-splitting in time (stability in time)
+ higher order finite element discretization in space (separate problem in each time step, B-spline based discretization),
+ residual minimization method in every time step (stability in space),
+ Kronecker product structure of matrices (fast $\mathcal{O}(N)$ solver),
+ Linear computational cost for arbitrary material data

Discretization with B-splines





$$N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \le \xi < \xi_{i+1}, \\ 0 & \text{otherwise} \end{cases}$$

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi)$$

Figure: Recursive formulae for B-splines. Tensor products on 2D grid

J.A. Cottrel, T.J.R. Hughes, Y. Bazilevs, *Isogeometric Analysis. Toward Integration of CAD and FEA*, Wiley, (2009)

Mass matrices over 2D domain $\Omega = \Omega_x \times \Omega_y$

Here we will derive linear $\mathcal{O}(N)$ cost solver for tensor product grids. We discretize with tensor product B-spline basis functions B_{ij}

$$\mathcal{M} = (B_{ij}, B_{kl})_{L^2} = \int_\Omega B_{ij} B_{kl} \, d\Omega =$$

$$\int_\Omega B_i^x(x) B_j^y(y) B_k^x(x) B_l^y(y) \, d\Omega = \int_\Omega (B_i B_k)(x) \, (B_j B_l)(y) \, d\Omega$$
$$= \left(\int_{\Omega_x} B_i B_k \, dx \right) \left(\int_{\Omega_y} B_j B_l \, dy \right) = \mathcal{M}^x \otimes \mathcal{M}^y$$

A matrix with Kronecker product structure can be LU factorized in a linear O(N) computational cost.

Can we apply it to the iGRM case?

Stiffness matrices over 2D domain $\Omega = \Omega_x \times \Omega_y$

We discretize with tensor product B-spline basis functions

$$\mathcal{S} = \int_{\Omega} \frac{\partial B_i^x(x)}{\partial x} B_j^y(y) \frac{\partial B_k^x(x)}{\partial x} B_l^y(y) + B_i^x(x) \frac{\partial B_j^y(y)}{\partial y} B_k^x(x) \frac{\partial B_l^y(y))}{\partial y} \, \mathrm{d}\Omega$$
$$= \int_{\Omega_x} \frac{\partial B_i}{\partial x} \frac{\partial B_k}{\partial x} \, \mathrm{d}x \int_{\Omega_y} B_j B_l \, \mathrm{d}y + \int_{\Omega_x} B_i B_k \, \mathrm{d}x \int_{\Omega_y} \frac{\partial B_j}{\partial y} \frac{\partial B_l}{\partial y} \, \mathrm{d}y$$

$$= \mathcal{S}^x \otimes \mathcal{M}^y + \mathcal{M}^x \otimes \mathcal{S}^y$$

The Kronecker product structure and linear cost solver is preserved for **“half” of the matrix ONLY**

$$\mathcal{S}^x \otimes \mathcal{M}^y u^{k+1/2} = \mathcal{M}^x \otimes \mathcal{S}^y u^k$$

$$\mathcal{M}^x \otimes \mathcal{S}^y u^{k+1} = \mathcal{S}^x \otimes \mathcal{M}^y u^{k+1/2}$$

Time-dependent Maxwell problem

Let us consider the time-dependent Maxwell equations on domain $\Omega = [0,1]^3$:

$$\frac{\partial \mathbf{E}}{\partial t}(t) = \frac{1}{\epsilon}\nabla \times \mathbf{H}(t); \qquad \frac{\partial \mathbf{H}}{\partial t}(t) = -\frac{1}{\mu}\nabla \times \mathbf{E}(t) \quad t \in \mathcal{R}, x \in \Omega$$

$$\mathrm{div}\epsilon\mathbf{E}(t) = 0 \quad t \in \mathcal{R}, x \in \Omega; \qquad \mathrm{div}\mu\mathbf{H}(t) = 0 \quad t \in \mathcal{R}, x \in \Omega$$

$$\mathbf{E}(t) \times \mathbf{n} = 0 \quad t \in \mathcal{R}, x \in \partial\Omega; \qquad \mathbf{H}(t) \cdot \mathbf{n} = 0 \quad t \in \mathcal{R}, x \in \partial\Omega$$

$$\mathbf{E}(x,0) = \mathbf{E_0}(x) \quad x \in \Omega; \qquad \mathbf{H}(x,0) = \mathbf{H_0}(x) \quad x \in \Omega$$

where $\mathbf{E}(x,t)$ is an electric field, $\mathbf{H}(x,t)$ is the magnetic field, $\mathbf{E_0} \in L^2(\Omega)^3$ and $\mathbf{H_0} \in L^2(\Omega)^3$ are initial states, permittivity $\epsilon \in L^\infty(\Omega)$, and permeability $\mu \in L^\infty(\Omega)$ are given functions assumed to be constant in time, and they fullfil $\epsilon(x) \geq \delta > 0$, and $\mu(x) \geq \delta > 0$.

We have

$$\nabla\times = \begin{bmatrix} 0 & -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & -\frac{\partial}{\partial x_1} \\ -\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_1} & 0 \end{bmatrix} - \begin{bmatrix} 0 & \frac{\partial}{\partial x_3} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} & 0 & 0 \end{bmatrix} = C_1 - C_2$$

Time-dependent Maxwell problem

Following

M. Hochbruck, T. Jahnke, R. Schnaubelt, **2015**
Convergence of an ADI splitting for Maxwell's equations,
Numerishe Mathematik

we use the following time integration scheme

$$\mathbf{E}^{n+\frac{1}{2}} = \mathbf{E}^{n} - \frac{\tau}{2\epsilon} C_2 \mathbf{H}^{n} + \frac{\tau}{2\epsilon} C_1 \mathbf{H}^{n+\frac{1}{2}};$$

$$\mathbf{H}^{n+\frac{1}{2}} = \mathbf{H}^{n} - \frac{\tau}{2\mu} C_1 \mathbf{E}^{n} + \frac{\tau}{2\mu} C_2 \mathbf{E}^{n+\frac{1}{2}}$$

$$\mathbf{E}^{n+1} = \mathbf{E}^{n+\frac{1}{2}} + \frac{\tau}{2\epsilon} C_1 \mathbf{H}^{n+\frac{1}{2}} - \frac{\tau}{2\epsilon} C_2 \mathbf{H}^{n+1};$$

$$\mathbf{H}^{n+1} = \mathbf{H}^{n+\frac{1}{2}} + \frac{\tau}{2\mu} C_2 \mathbf{E}^{n+\frac{1}{2}} - \frac{\tau}{2\mu} C_1 \mathbf{E}^{n+1}$$

Substituting the second equation in the first one in both substeps lead to

$$\left(I - \frac{\tau^2}{4\epsilon} C_1 \mu^{-1} C_2\right) \mathbf{E}^{n+\frac{1}{2}} = \mathbf{E}^n + \frac{\tau}{2\epsilon}(C_1 - C_2)\mathbf{H}^n - \frac{\tau^2}{4\epsilon} C_1 \mu^{-1} C_1 \mathbf{E}^n$$

$$\mathbf{H}^{n+\frac{1}{2}} = \mathbf{H}^n - \frac{\tau}{2\mu} C_1 \mathbf{E}^n + \frac{\tau}{2\mu} C_2 \mathbf{E}^{n+\frac{1}{2}}$$

and

$$\left(I - \frac{\tau^2}{4\epsilon} C_2 \mu^{-1} C_1\right) \mathbf{E}^{n+1} = \mathbf{E}^{n+\frac{1}{2}} + \frac{\tau}{2\epsilon}(C_1 - C_2)\mathbf{H}^{n+\frac{1}{2}} - \frac{\tau^2}{4\epsilon} C_2 \mu^{-1} C_2 \mathbf{E}^{n+\frac{1}{2}}$$

$$\mathbf{H}^{n+1} = \mathbf{H}^{n+\frac{1}{2}} + \frac{\tau}{2\mu} C_2 \mathbf{E}^{n+\frac{1}{2}} - \frac{\tau}{2\mu} C_1 \mathbf{E}^{n+1}$$

Time-dependent Maxwell problem

$$C_1\mu^{-1}C_2 = \begin{bmatrix} \frac{\partial}{\partial x_2}\mu^{-1}\frac{\partial}{\partial x_2} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_3}\mu^{-1}\frac{\partial}{\partial x_3} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_1}\mu^{-1}\frac{\partial}{\partial x_1} \end{bmatrix},$$

$$C_2\mu^{-1}C_1 = \begin{bmatrix} \frac{\partial}{\partial x_3}\mu^{-1}\frac{\partial}{\partial x_3} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_1}\mu^{-1}\frac{\partial}{\partial x_1} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_2}\mu^{-1}\frac{\partial}{\partial x_2} \end{bmatrix}$$

as well as

$$C_1\mu^{-1}C_1 = \begin{bmatrix} 0 & \frac{\partial}{\partial x_2}\mu^{-1}\frac{\partial}{\partial x_1} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_3}\mu^{-1}\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1}\mu^{-1}\frac{\partial}{\partial x_3} & 0 & 0 \end{bmatrix},$$

$$C_2\mu^{-1}C_2 = \begin{bmatrix} 0 & 0 & \frac{\partial}{\partial x_3}\mu^{-1}\frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_1}\mu^{-1}\frac{\partial}{\partial x_2} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_2}\mu^{-1}\frac{\partial}{\partial x_3} & 0 \end{bmatrix}$$

Time-dependent Maxwell problem

$$\mathbf{E}^{n+\frac{1}{2}} - \frac{\tau^2}{4\epsilon}\begin{bmatrix} \frac{\partial}{\partial x_2}\mu^{-1}\frac{\partial}{\partial x_2} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_3}\mu^{-1}\frac{\partial}{\partial x_3} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_1}\mu^{-1}\frac{\partial}{\partial x_1} \end{bmatrix}\mathbf{E}^{n+\frac{1}{2}} =$$

$$\mathbf{E}^{n} + \frac{\tau}{2\epsilon}\begin{bmatrix} 0 & -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & -\frac{\partial}{\partial x_1} \\ -\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{bmatrix}\mathbf{H}^{n}$$

$$-\frac{\tau^2}{4\epsilon}\begin{bmatrix} 0 & \frac{\partial}{\partial x_2}\mu^{-1}\frac{\partial}{\partial x_1} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_3}\mu^{-1}\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1}\mu^{-1}\frac{\partial}{\partial x_3} & 0 & 0 \end{bmatrix}\mathbf{E}^{n}$$

$$\mathbf{H}^{n+\frac{1}{2}} = \mathbf{H}^{n} - \frac{\tau}{2\mu}\begin{bmatrix} 0 & 0 & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_1} & 0 \end{bmatrix}\mathbf{E}^{n} + \frac{\tau}{2\mu}\begin{bmatrix} 0 & \frac{\partial}{\partial x_3} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} & 0 & 0 \end{bmatrix}\mathbf{E}^{n+\frac{1}{2}}$$

Time-dependent Maxwell problem

$$
\begin{aligned}
\mathbf{E}^{n+1} - \frac{\tau^2}{4\epsilon}\begin{bmatrix} \frac{\partial}{\partial x_3}\mu^{-1}\frac{\partial}{\partial x_3} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_1}\mu^{-1}\frac{\partial}{\partial x_1} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_2}\mu^{-1}\frac{\partial}{\partial x_2} \end{bmatrix}\mathbf{E}^{n+1} = \\
\mathbf{E}^{n+\frac{1}{2}} + \frac{\tau}{2\epsilon}\begin{bmatrix} 0 & -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & -\frac{\partial}{\partial x_1} \\ -\frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \end{bmatrix}\mathbf{H}^{n+\frac{1}{2}} \\
-\frac{\tau^2}{4\epsilon}\begin{bmatrix} 0 & 0 & \frac{\partial}{\partial x_3}\mu^{-1}\frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_1}\mu^{-1}\frac{\partial}{\partial x_2} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_2}\mu^{-1}\frac{\partial}{\partial x_3} & 0 \end{bmatrix}\mathbf{E}^{n+\frac{1}{2}}
\end{aligned}
$$

$$
\mathbf{H}^{n+1} = \mathbf{H}^{n+\frac{1}{2}} + \frac{\tau}{2\mu}\begin{bmatrix} 0 & \frac{\partial}{\partial x_3} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} & 0 & 0 \end{bmatrix}\mathbf{E}^{n+\frac{1}{2}} - \frac{\tau}{2\mu}\begin{bmatrix} 0 & 0 & \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_1} & 0 \end{bmatrix}\mathbf{E}^{n+1}
$$

We will introduce now a variational formulation.
We denote by $(\cdot,\cdot)$ both the usual inner products in $L^2(\Omega)$ and $L^2(\Omega)^3$, i.,e. $(\mathbf{H},\mathbf{V}) = (H_1,V_1)+(H_2,V_2)+(H_3,V_3)$
We consider for the moment that μ and ϵ are constant.
We multiply the equations by suitable test functions $\mathbf{V}$, integrate in space and integrate by parts the second order terms

$$\begin{aligned}
&(E_1^{n+\frac{1}{2}},V_1)+(E_2^{n+\frac{1}{2}},V_2)+(E_3^{n+\frac{1}{2}},V_3)\\
&+\frac{\tau^2}{4\epsilon\mu}\left[\left(\frac{\partial}{\partial x_2}E_1^{n+\frac{1}{2}},\frac{\partial}{\partial x_2}V_1\right)+\left(\frac{\partial}{\partial x_3}E_2^{n+\frac{1}{2}},\frac{\partial}{\partial x_3}V_2\right)+\left(\frac{\partial}{\partial x_1}E_3^{n+\frac{1}{2}},\frac{\partial}{\partial x_1}V_3\right)\right]\\
&=(E_1^n,V_1)+(E_2^n,V_2)+(E_3^n,V_3)\\
&+\frac{\tau}{2\epsilon}\left[\left(-\frac{\partial}{\partial x_3}H_2^n+\frac{\partial}{\partial x_2}H_3^n,V_1\right)+\left(\frac{\partial}{\partial x_3}H_1^n-\frac{\partial}{\partial x_1}H_3^n,V_2\right)\right.\\
&\left.+\left(-\frac{\partial}{\partial x_2}H_1^n+\frac{\partial}{\partial x_1}H_2^n,V_3\right)\right]\\
&+\frac{\tau^2}{4\epsilon\mu}\left[\left(\frac{\partial}{\partial x_1}E_2^n,\frac{\partial}{\partial x_2}V_1\right)+\left(\frac{\partial}{\partial x_2}E_3^n,\frac{\partial}{\partial x_3}V_2\right)+\left(\frac{\partial}{\partial x_3}E_1^n,\frac{\partial}{\partial x_1}V_3\right)\right]
\end{aligned}$$

Time-dependent Maxwell problem (weak form) (2/4)

$$
\begin{aligned}
(H_1^{n+\frac{1}{2}}, V_1) + (H_2^{n+\frac{1}{2}}, V_2) + (H_3^{n+\frac{1}{2}}, V_3) &= (H_1^n, V_1) + (H_2^n, V_2) + (H_3^n, V_3) \\
&\quad - \frac{\tau}{2\mu}\left[\left(\frac{\partial}{\partial x_2} E_3^n, V_1\right) + \left(\frac{\partial}{\partial x_3} E_1^n, V_2\right) + \left(\frac{\partial}{\partial x_1} E_2^n, V_3\right)\right] \\
&\quad + \frac{\tau}{2\mu}\left[\left(\frac{\partial}{\partial x_3} E_2^{n+\frac{1}{2}}, V_1\right) + \left(\frac{\partial}{\partial x_1} E_3^{n+\frac{1}{2}}, V_2\right) + \left(\frac{\partial}{\partial x_2} E_1^{n+\frac{1}{2}}, V_3\right)\right]
\end{aligned}
$$

Time-dependent Maxwell problem (weak form) (3/4)

$$
\begin{aligned}
&(E_1^{n+1}, V_1) + (E_2^{n+1}, V_2) + (E_3^{n+1}, V_3) \\
&+\frac{\tau^2}{4\epsilon\mu}\left[\left(\frac{\partial}{\partial x_3}E_1^{n+1}, \frac{\partial}{\partial x_3}V_1\right) + \left(\frac{\partial}{\partial x_1}E_2^{n+1}, \frac{\partial}{\partial x_1}V_2\right) + \left(\frac{\partial}{\partial x_2}E_3^{n+1}, \frac{\partial}{\partial x_2}V_3\right)\right] \\
&= (E_1^{n+\frac{1}{2}}, V_1) + (E_2^{n+\frac{1}{2}}, V_2) + (E_3^{n+\frac{1}{2}}, V_3) \\
&+\frac{\tau}{2\epsilon}\left[\left(-\frac{\partial}{\partial x_3}H_2^{n+\frac{1}{2}} + \frac{\partial}{\partial x_2}H_3^{n+\frac{1}{2}}, V_1\right) + \left(\frac{\partial}{\partial x_3}H_1^{n+\frac{1}{2}} - \frac{\partial}{\partial x_1}H_3^{n+\frac{1}{2}}, V_2\right)\right. \\
&\left. + \left(-\frac{\partial}{\partial x_2}H_1^{n+\frac{1}{2}} + \frac{\partial}{\partial x_1}H_2^{n+\frac{1}{2}}, V_3\right)\right] \\
&+\frac{\tau^2}{4\epsilon\mu}\left[\left(\frac{\partial}{\partial x_1}E_3^{n+\frac{1}{2}}, \frac{\partial}{\partial x_3}V_1\right) + \left(\frac{\partial}{\partial x_2}E_1^{n+\frac{1}{2}}, \frac{\partial}{\partial x_1}V_2\right) + \left(\frac{\partial}{\partial x_3}E_2^{n+\frac{1}{2}}, \frac{\partial}{\partial x_2}V_3\right)\right]
\end{aligned}
$$

Time-dependent Maxwell problem (weak form) (4/4)

$$
\begin{aligned}
(H_1^{n+1}, V_1) + (H_2^{n+1}, V_2) + (H_3^{n+1}, V_3) &= (H_1^{n+\frac{1}{2}}, V_1) + (H_2^{n+\frac{1}{2}}, V_2) + (H_3^{n+\frac{1}{2}}, V_3) \\
&+ \frac{\tau}{2\mu}\left[\left(\frac{\partial}{\partial x_3} E_2^{n+\frac{1}{2}}, V_1\right) + \left(\frac{\partial}{\partial x_1} E_3^{n+\frac{1}{2}}, V_2\right) + \left(\frac{\partial}{\partial x_2} E_1^{n+\frac{1}{2}}, V_3\right)\right] \\
&- \frac{\tau}{2\mu}\left[\left(\frac{\partial}{\partial x_2} E_3^{n+1}, V_1\right) + \left(\frac{\partial}{\partial x_3} E_1^{n+1}, V_2\right) + \left(\frac{\partial}{\partial x_1} E_2^{n+1}, V_3\right)\right]
\end{aligned}
$$

Time-dependent Maxwell problem (spatial discretization)

Assuming the discretization with B-splines on tensor product grids,

$$\mathbf{M} = \int_\Omega B_{ijk} B_{abc} dx_1 dx_2 dx_3 =$$

$$\int_{\Omega_{x_1}} B_i B_a dx_1 \int_{\Omega_{x_2}} B_j B_b dx_2 \int_{\Omega_{x_3}} B_k B_c dx_3 = M_{x_1} \otimes M_{x_2} \otimes M_{x_3}$$

$$\left(\mathbf{M} + \frac{\tau^2}{4\epsilon\mu}\mathbf{K}_1\right)\mathbf{E}^{n+\frac{1}{2}} = \mathbf{M}\mathbf{E}^n + \frac{\tau}{2\epsilon}\mathbf{C}\mathbf{H}^n + \frac{\tau^2}{4\epsilon\mu}\mathbf{R}_1\mathbf{E}^n$$

$$\mathbf{M}\mathbf{H}^{n+\frac{1}{2}} = \mathbf{M}\mathbf{H}^n - \frac{\tau}{2\mu}\mathbf{C}_1\mathbf{E}^n + \frac{\tau}{2\mu}\mathbf{C}_2\mathbf{E}^{n+\frac{1}{2}}$$

$$\left(\mathbf{M} + \frac{\tau^2}{4\epsilon\mu}\mathbf{K}_2\right)\mathbf{E}^{n+1} = \mathbf{M}\mathbf{E}^{n+\frac{1}{2}} + \frac{\tau}{2\epsilon}\mathbf{C}\mathbf{H}^{n+\frac{1}{2}} + \frac{\tau^2}{4\epsilon\mu}\mathbf{R}_2\mathbf{E}^{n+\frac{1}{2}}$$

$$\mathbf{M}\mathbf{H}^{n+1} = \mathbf{M}\mathbf{H}^{n+\frac{1}{2}} + \frac{\tau}{2\mu}\mathbf{C}_2\mathbf{E}^{n+\frac{1}{2}} + \frac{\tau}{2\mu}\mathbf{C}_1\mathbf{E}^{n+1}$$

Time-dependent Maxwell problem (spatial discretization)

$$\mathbf{M} = \begin{bmatrix} M_{x_1} \otimes M_{x_2} \otimes M_{x_3} & 0 & 0 \\ 0 & M_{x_1} \otimes M_{x_2} \otimes M_{x_3} & 0 \\ 0 & 0 & M_{x_1} \otimes M_{x_2} \otimes M_{x_3} \end{bmatrix}$$

$$\mathbf{K}_1 = \begin{bmatrix} M_{x_1} \otimes S_{x_2} \otimes M_{x_3} & 0 & 0 \\ 0 & M_{x_1} \otimes M_{x_2} \otimes S_{x_3} & 0 \\ 0 & 0 & S_{x_1} \otimes M_{x_2} \otimes M_{x_3} \end{bmatrix}$$

$$\mathbf{K}_2 = \begin{bmatrix} M_{x_1} \otimes M_{x_2} \otimes S_{x_3} & 0 & 0 \\ 0 & S_{x_1} \otimes M_{x_2} \otimes M_{x_3} & 0 \\ 0 & 0 & M_{x_1} \otimes S_{x_2} \otimes M_{x_3} \end{bmatrix}$$

$$\mathbf{R}_1 = \begin{bmatrix} 0 & A_{x_1} \otimes B_{x_2} \otimes M_{x_3} & 0 \\ 0 & 0 & M_{x_1} \otimes A_{x_2} \otimes B_{x_3} \\ B_{x_1} \otimes M_{x_2} \otimes A_{x_3} & 0 & 0 \end{bmatrix}$$

$$\mathbf{R}_2 = \begin{bmatrix} 0 & 0 & A_{x_1} \otimes M_{x_2} \otimes B_{x_3} \\ B_{x_1} \otimes A_{x_2} \otimes M_{x_3} & 0 & 0 \\ 0 & M_{x_1} \otimes B_{x_2} \otimes A_{x_3} & 0 \end{bmatrix}$$

Time-dependent Maxwell problem (spatial discretization)

$$\mathbf{C}_1 = \begin{bmatrix} 0 & 0 & M_{x_1} \otimes A_{x_2} \otimes M_{x_3} \\ M_{x_1} \otimes M_{x_2} \otimes A_{x_3} & 0 & 0 \\ 0 & A_{x_1} \otimes M_{x_2} \otimes M_{x_3} & 0 \end{bmatrix}$$

$$\mathbf{C}_2 = \begin{bmatrix} 0 & M_{x_1} \otimes M_{x_2} \otimes A_{x_3} & 0 \\ 0 & 0 & A_{x_1} \otimes M_{x_2} \otimes M_{x_3} \\ M_{x_1} \otimes M_{x_2} \otimes A_{x_3} & 0 & 0 \end{bmatrix}$$

$$\mathbf{C} = \mathbf{C}_1 - \mathbf{C}_2$$

where $M_{x_1}, M_{x_2}, M_{x_3}$ are 1D mass matrices, $M_{x_1} = \int_{\Omega_{x_1}} B_i B_a dx_1$
$S_{x_1}, S_{x_2}, S_{x_3}$ are 1D stiffness matrices, $S_{x_1} = \int_{\Omega_{x_1}} \partial_{x_1} B_i \partial_{x_1} B_a dx_1$
$A_{x_1}, A_{x_2}, A_{x_3}$ are 1D advection matrices with the derivatives in the trial functions, $A_{x_1} = \int_{\Omega_{x_1}} \partial_{x_1} B_i B_a dx_1$
$B_{x_1}, B_{x_2}, B_{x_3}$ are 1D advection matrices with the derivatives in the test functions, $B_{x_1} = \int_{\Omega_{x_1}} B_i \partial_{x_1} B_a dx_1$

Application of the RM method for non-stationary Maxwell

We choose the inner product in the $\mathbf{H}(\mathrm{curl}, \Omega)$ space

$$(\mathbf{H}, \mathbf{V})_{\mathbf{H}(curl,\Omega)} = (\mathbf{H}, \mathbf{V}) + (\nabla \times \mathbf{H}, \nabla \times \mathbf{V})$$

However, to preserve the Kronecker structure in every sub-step we introduce different inner products for every substep.
For updates of the magnetic fields, we employ $L^2(\Omega)^3$ inner product.
For updates of the electric field, we consider inner products

$$\begin{aligned}(\mathbf{E}, \mathbf{V})_A = {} & (E_1, V_1) + (E_2, V_2) + (E_3, V_3) \\ & + \left[\left(\frac{\partial}{\partial x_2} E_1, \frac{\partial}{\partial x_2} V_1 \right) + \left(\frac{\partial}{\partial x_3} E_2, \frac{\partial}{\partial x_3} V_2 \right) + \left(\frac{\partial}{\partial x_1} E_3, \frac{\partial}{\partial x_1} V_3 \right) \right]\end{aligned}$$

$$\begin{aligned}(\mathbf{E}, \mathbf{V})_B = {} & (E_1, V_1) + (E_2, V_2) + (E_3, V_3) \\ & + \left[\left(\frac{\partial}{\partial x_3} E_1, \frac{\partial}{\partial x_3} V_1 \right) + \left(\frac{\partial}{\partial x_1} E_2, \frac{\partial}{\partial x_1} V_2 \right) + \left(\frac{\partial}{\partial x_2} E_3, \frac{\partial}{\partial x_2} V_3 \right) \right]\end{aligned}$$

For updates of the electric field the matrix is

$$\begin{bmatrix} \mathbf{G}_A & -\left(\mathbf{M} + \frac{\tau^2}{4\epsilon\mu}\mathbf{K}_1\right) \\ (\mathbf{M} + \frac{\tau^2}{4\epsilon\mu}\mathbf{K}_1)^T & 0 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{G}_B & -\left(\mathbf{M} + \frac{\tau^2}{4\epsilon\mu}\mathbf{K}_2\right) \\ (\mathbf{M} + \frac{\tau^2}{4\epsilon\mu}\mathbf{K}_2)^T & 0 \end{bmatrix}$$

where $\mathbf{G}_A = \mathbf{M} + \mathbf{K}_1$ and $\mathbf{G}_B = \mathbf{M} + \mathbf{K}_2$.
For updates of the magentic field the matrix is

$$\begin{bmatrix} \mathbf{M} & -\mathbf{M} \\ \mathbf{M}^T & 0 \end{bmatrix}$$

Numerical results: manufactured solution

For $\Omega = [0,1]^3$, for $\epsilon = 1$ and $\mu = 1$, $\kappa, \lambda \in \mathcal{N}$, $\kappa, \lambda \neq 0$ we have solutions

$$u^1_{\kappa,\lambda}(x,t) = \begin{bmatrix} \sin(\kappa\pi x_2)\sin(\lambda\pi x_3)\cos(\sqrt{\kappa^2+\lambda^2}\pi t) \\ 0 \\ 0 \\ 0 \\ -\frac{\lambda}{\sqrt{\kappa^2+\lambda^2}}\sin(\kappa\pi x_2)\cos(\lambda\pi x_3)\sin(\sqrt{\kappa^2+\lambda^2}\pi t) \\ \frac{\kappa}{\sqrt{\kappa^2+\lambda^2}}\cos(\kappa\pi x_2)\sin(\lambda\pi x_3)\sin(\sqrt{\kappa^2+\lambda^2}\pi t) \end{bmatrix}$$

$$u^2_{\kappa,\lambda}(x,t) = \begin{bmatrix} 0 \\ \sin(\kappa\pi x_1)\sin(\lambda\pi x_3)\cos(\sqrt{\kappa^2+\lambda^2}\pi t) \\ 0 \\ -\frac{\lambda}{\sqrt{\kappa^2+\lambda^2}}\sin(\kappa\pi x_1)\cos(\lambda\pi x_3)\sin(\sqrt{\kappa^2+\lambda^2}\pi t) \\ 0 \\ \frac{\kappa}{\sqrt{\kappa^2+\lambda^2}}\cos(\kappa\pi x_1)\sin(\lambda\pi x_3)\sin(\sqrt{\kappa^2+\lambda^2}\pi t) \end{bmatrix}$$

$$u^3_{\kappa,\lambda}(x,t) = \begin{bmatrix} 0 \\ 0 \\ \sin(\kappa\pi x_1)\sin(\lambda\pi x_2)\cos(\sqrt{\kappa^2+\lambda^2}\pi t) \\ -\frac{\lambda}{\sqrt{\kappa^2+\lambda^2}}\sin(\kappa\pi x_1)\cos(\lambda\pi x_2)\sin(\sqrt{\kappa^2+\lambda^2}\pi t) \\ \frac{\kappa}{\sqrt{\kappa^2+\lambda^2}}\cos(\kappa\pi x_1)\sin(\lambda\pi x_2)\sin(\sqrt{\kappa^2+\lambda^2}\pi t) \\ 0 \end{bmatrix}$$

Numerical results: manufactured solution

More general solutions can be obtained by

$$\mathbf{u}(x,t) = \sum_{\kappa=0,\ldots,\kappa_{max}} \sum_{\lambda=0,\ldots,\lambda_{max}} \left(a^1_{\kappa\lambda} u^1_{\kappa\lambda}(x,t) + a^2_{\kappa\lambda} u^2_{\kappa\lambda}(x,t) + a^3_{\kappa\lambda} u^3_{\kappa\lambda}(x,t) \right)$$

The first manufactured solution function is

$$\mathbf{u}_A(x,t) = \gamma u^1_{1,1}(x,t) + 2\gamma u^2_{1,1}(x,t) + 3\gamma u^3_{1,1}(x,t)$$

Numerical results: manufactured solution

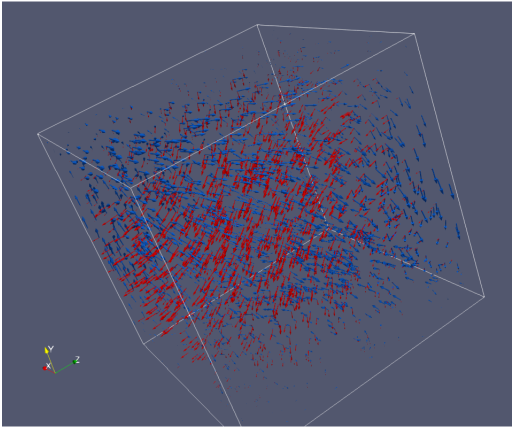


Figure: Electric (red) and magnetic (blue) vector fields, resulting from the problem with manufactured solution.

Numerical results: manufactured solution

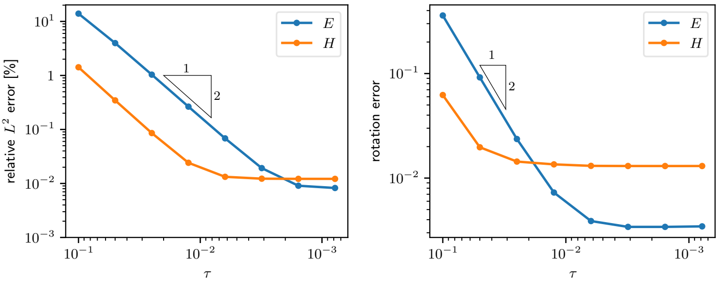


Figure: Order of the time integration scheme as measured in L2 (left) and H-curl (right) norms for electric (blue) and magnetic (orange) vector fields resulting from the solution of the problem with manufactured solution over the computational mesh with 16x16x16 elements.

Factorization of Kronecker product matrices

The direction splitting algorithm for the Kronecker product matrices implements three steps, which result is equivalent to the Gaussian elimination algorithm

$$(\mathcal{M})^{-1} = (\mathcal{A}^x \otimes \mathcal{B}^y \otimes \mathcal{C}^z)^{-1} = (\mathcal{A}^x)^{-1} \otimes (\mathcal{B}^y)^{-1} \otimes (\mathcal{C}^z)^{-1}$$

First, we solve along x direction

$$\begin{bmatrix} A^x_{11} & A^x_{12} & \cdots & 0 \\ A^x_{21} & A^x_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A^x_{kk} \end{bmatrix} \begin{bmatrix} z_{111} & z_{121} & \cdots & z_{1lm} \\ z_{211} & z_{221} & \cdots & z_{2lm} \\ \vdots & \vdots & \ddots & \vdots \\ z_{k11} & z_{k21} & \cdots & z_{klm} \end{bmatrix} = \begin{bmatrix} y_{111} & y_{121} & \cdots & y_{1lm} \\ y_{211} & y_{221} & \cdots & y_{2lm} \\ \vdots & \vdots & \ddots & \vdots \\ y_{k11} & y_{k21} & \cdots & y_{klm} \end{bmatrix}$$

Second, we solve along y direction

$$\begin{bmatrix} B^y_{11} & B^y_{12} & \cdots & 0 \\ B^y_{21} & B^y_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B^y_{ll} \end{bmatrix} \begin{bmatrix} y_{111} & y_{211} & \cdots & y_{k1m} \\ y_{121} & y_{221} & \cdots & y_{k2m} \\ \vdots & \vdots & \ddots & \vdots \\ y_{1l1} & y_{1l1} & \cdots & y_{klm} \end{bmatrix} = \begin{bmatrix} z_{111} & z_{211} & \cdots & z_{k1m} \\ z_{121} & z_{221} & \cdots & z_{k2m} \\ \vdots & \vdots & \ddots & \vdots \\ z_{1l1} & z_{2l1} & \cdots & z_{klm} \end{bmatrix}$$

Third, we solve along z direction,

$$\begin{bmatrix} C^z_{1,1} & C^z_{1,2} & \cdots & 0 \\ C^z_{2,1} & C^z_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C^z_{m,m} \end{bmatrix} \begin{bmatrix} x_{111} & x_{211} & \cdots & x_{kl1} \\ x_{112} & x_{212} & \cdots & x_{kl2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{11m} & x_{21m} & \cdots & x_{klm} \end{bmatrix} = \begin{bmatrix} b_{111} & b_{211} & \cdots & b_{kl1} \\ b_{112} & b_{212} & \cdots & b_{kl2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{11m} & b_{21m} & \cdots & b_{klm} \end{bmatrix}$$

Kronecker product structure in time-dependent Maxwell equations

For example for the first sub-step, the update of electric fields

$$\begin{bmatrix} M_x \otimes \left(M_y + \frac{\tau^2}{4\epsilon\mu} S_y\right) \otimes M_z E_1^{n+\frac{1}{2}} \\ M_x \otimes M_y \otimes \left(M_z + \frac{\tau^2}{4\epsilon\mu} S_z\right) E_2^{n+\frac{1}{2}} \\ \left(M_x + \frac{\tau^2}{4\epsilon\mu} S_x\right) \otimes M_y \otimes M_z E_3^{n+\frac{1}{2}} \end{bmatrix}$$

$$= \begin{bmatrix} M_x \otimes M_y \otimes M_z E_1^n \\ M_x \otimes M_y \otimes M_z E_2^n \\ M_x \otimes M_y \otimes M_z E_3^n \end{bmatrix} + \begin{bmatrix} -\frac{\tau}{2\epsilon} M_x \otimes M_y \otimes A_z H_2^n \\ \frac{\tau}{2\epsilon} M_x \otimes M_y \otimes A_z H_1^n \\ -\frac{\tau}{2\epsilon} M_x \otimes A_y \otimes M_z H_1^n \end{bmatrix}$$

$$+ \begin{bmatrix} \frac{\tau}{2\epsilon} M_x \otimes A_y \otimes M_z H_3^n \\ -\frac{\tau}{2\epsilon} A_x \otimes M_y \otimes M_z H_3^n \\ \frac{\tau}{2\epsilon} A_x \otimes M_y \otimes M_z H_2^n \end{bmatrix} + \begin{bmatrix} \frac{\tau^2}{4\epsilon\mu} A_x \otimes B_y \otimes M_z E_2^n \\ \frac{\tau^2}{4\epsilon\mu} M_x \otimes A_y \otimes B_z E_3^n \\ \frac{\tau^2}{4\epsilon\mu} B_x \otimes M_y \otimes A_z E_1^n \end{bmatrix}$$

Kronecker product structure in time-dependent Maxwell

$$\begin{bmatrix} \mathcal{M}_1^1 E_1^{n+\frac{1}{2}} \\ \mathcal{M}_2^1 E_2^{n+\frac{1}{2}} \\ \mathcal{M}_3^1 E_3^{n+\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} \mathcal{M} E_1^n \\ \mathcal{M} E_2^n \\ \mathcal{M} E_3^n \end{bmatrix} + \begin{bmatrix} \mathcal{F}_1^1 H_2^n \\ \mathcal{F}_2^1 H_1^n \\ \mathcal{F}_3^1 H_1^n \end{bmatrix} + \begin{bmatrix} \mathcal{F}_1^2 H_3^n \\ \mathcal{F}_2^2 H_3^n \\ \mathcal{F}_3^2 H_2^n \end{bmatrix} + \begin{bmatrix} \mathcal{F}_1^3 E_2^n \\ \mathcal{F}_2^3 E_3^n \\ \mathcal{F}_3^3 E_1^n \end{bmatrix} = \begin{bmatrix} \mathcal{RHS}_1 \\ \mathcal{RHS}_2 \\ \mathcal{RHS}_3 \end{bmatrix}$$

where the entries of each matrix are

$$\mathcal{M}_{ijk,lmo} = \int_{\Omega_x} B_{i,p}(x) B_{l,p}(x) dx \int_{\Omega_y} B_{j,p}(y) B_{m,p}(y) dy \int_{\Omega_z} B_{k,p}(z) B_{o,p}(z) dz$$

where
$i = 1, ..., N_x$, $j = 1, ..., N_y$, $k = 1, ..., N_z$ span over trial space dimensions,
$l = 1, ..., \tilde{N}_x$, $m = 1, ..., \tilde{N}_y$, $n = 1, ..., \tilde{N}_z$ span over test space dimensions.
The matrices on the RHS are multiplied by the solution vectors from previous time step, so as the result on the RHS we have a vectors $\mathcal{RHS}_{1lmo}$, $\mathcal{RHS}_{2lmo}$, and $\mathcal{RHS}_{3lmo}$,
where again $l = 1, ..., \tilde{N}_x$, $m = 1, ..., \tilde{N}_y$, $o = 1, ...\tilde{N}_z$.

Kronecker product structure in time-dependent Maxwell

$$\begin{bmatrix}\mathcal{M}_1^1 E_1^{n+\frac{1}{2}}\\ \mathcal{M}_2^1 E_2^{n+\frac{1}{2}}\\ \mathcal{M}_3^1 E_3^{n+\frac{1}{2}}\end{bmatrix} = \begin{bmatrix}\mathcal{M} E_1^n\\ \mathcal{M} E_2^n\\ \mathcal{M} E_3^n\end{bmatrix} + \begin{bmatrix}\mathcal{F}_1^1 H_2^n\\ \mathcal{F}_2^1 H_1^n\\ \mathcal{F}_3^1 H_1^n\end{bmatrix} + \begin{bmatrix}\mathcal{F}_1^2 H_3^n\\ \mathcal{F}_2^2 H_3^n\\ \mathcal{F}_3^2 H_2^n\end{bmatrix} + \begin{bmatrix}\mathcal{F}_1^3 E_2^n\\ \mathcal{F}_2^3 E_3^n\\ \mathcal{F}_3^3 E_1^n\end{bmatrix} = \begin{bmatrix}\mathcal{RHS}_1\\ \mathcal{RHS}_2\\ \mathcal{RHS}_3\end{bmatrix}$$

where the entries of each matrix are

$$\mathcal{M}^1_{1\,ijk,lmo} = \int_{\Omega_x} B_{i,p}(x) B_{l,p}(x) dx$$

$$\int_{\Omega_y} \left(B_{j,p}(y) B_{m,p}(y) + \frac{\tau^2}{4\epsilon\mu} \frac{\partial B_{j,p}(y)}{\partial y} \frac{\partial B_{m,p}(y)}{\partial y} \right) dy \int_{\Omega_z} B_{k,p}(z) B_{o,p}(z)$$

$$\mathcal{M}^1_{2\,ijk,lmo} = \int_{\Omega_x} B_{i,p}(x) B_{l,p}(x) dx \int_{\Omega_y} B_{j,p}(y) B_{m,p}(y) dy$$

$$\int_{\Omega_z} \left(B_{k,p}(z) B_{n,p}(z) + \frac{\tau^2}{4\epsilon\mu} \frac{\partial B_{k,p}(z)}{\partial z} \frac{\partial B_{o,p}(z)}{\partial z} \right) dz$$

$$\mathcal{M}^1_{3\,ijk,lmo} = \int_{\Omega_x} \left(B_{i,p}(x) B_{l,p}(x) + \frac{\tau^2}{4\epsilon\mu} \frac{\partial B_{i,p}(x)}{\partial x} \frac{\partial B_{l,p}(x)}{\partial x} \right) dx$$

Kronecker product structure in time-dependent Maxwell

$$\begin{bmatrix} \mathcal{M}_1^1 E_1^{n+\frac{1}{2}} \\ \mathcal{M}_2^1 E_2^{n+\frac{1}{2}} \\ \mathcal{M}_3^1 E_3^{n+\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} \mathcal{M} E_1^n \\ \mathcal{M} E_2^n \\ \mathcal{M} E_3^n \end{bmatrix} + \begin{bmatrix} \mathcal{F}_1^1 H_2^n \\ \mathcal{F}_2^1 H_1^n \\ \mathcal{F}_3^1 H_1^n \end{bmatrix} + \begin{bmatrix} \mathcal{F}_1^2 H_3^n \\ \mathcal{F}_2^2 H_3^n \\ \mathcal{F}_3^2 H_2^n \end{bmatrix} + \begin{bmatrix} \mathcal{F}_1^3 E_2^n \\ \mathcal{F}_2^3 E_3^n \\ \mathcal{F}_3^3 E_1^n \end{bmatrix} = \begin{bmatrix} \mathcal{RHS}_1 \\ \mathcal{RHS}_2 \\ \mathcal{RHS}_3 \end{bmatrix}$$

where the entries of each matrix are

$$\mathcal{F}^1_{1\,ijk,lmo} = -\frac{\tau}{2\epsilon}\int_\Omega B_{i,p}(x)B_{j,p}(y)\frac{\partial B_{k,p}(z)}{\partial z}B_{l,p}(x)B_{m,p}(y)B_{o,p}(z)dxdydz$$

$$\mathcal{F}^1_{2\,ijk,lmo} = \frac{\tau}{2\epsilon}\int_\Omega B_{i,p}(x)B_{j,p}(y)\frac{\partial B_{k,p}(z)}{\partial z}B_{l,p}(x)B_{m,p}(y)B_{o,p}(z)dxdydz$$

$$\mathcal{F}^1_{3\,ijk,lmo} = -\frac{\tau}{2\epsilon}\int_\Omega B_{i,p}(x)\frac{\partial B_{j,p}(y)}{\partial y}B_{k,p}(x)B_{l,p}(x)B_{m,p}(y)B_{o,p}(z)dxdydz$$

$$\mathcal{F}^2_{1\,ijk,lmo} = \frac{\tau}{2\epsilon}\int_\Omega B_{i,p}(x)\frac{\partial B_{j,p}(y)}{\partial y}B_{k,p}(x)B_{l,p}(x)B_{m,p}(y)B_{o,p}(z)dxdydz$$

$$\mathcal{F}^2_{2\,ijk,lmo} = -\frac{\tau}{2\epsilon}\int_\Omega \frac{\partial B_{i,p}(x)}{\partial x}B_{j,p}(y)B_{k,p}(x)B_{l,p}(x)B_{m,p}(y)B_{o,p}(z)dxdydz$$

Kronecker product structure in time-dependent Maxwell

$$\begin{bmatrix} \mathcal{M}_1^1 E_1^{n+\frac{1}{2}} \\ \mathcal{M}_2^1 E_2^{n+\frac{1}{2}} \\ \mathcal{M}_3^1 E_3^{n+\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} \mathcal{M} E_1^n \\ \mathcal{M} E_2^n \\ \mathcal{M} E_3^n \end{bmatrix} + \begin{bmatrix} \mathcal{F}_1^1 H_2^n \\ \mathcal{F}_2^1 H_1^n \\ \mathcal{F}_3^1 H_1^n \end{bmatrix} + \begin{bmatrix} \mathcal{F}_1^2 H_3^n \\ \mathcal{F}_2^2 H_3^n \\ \mathcal{F}_3^2 H_2^n \end{bmatrix} + \begin{bmatrix} \mathcal{F}_1^3 E_2^n \\ \mathcal{F}_2^3 E_3^n \\ \mathcal{F}_3^3 E_1^n \end{bmatrix} = \begin{bmatrix} \mathcal{RHS}_1 \\ \mathcal{RHS}_2 \\ \mathcal{RHS}_3 \end{bmatrix}$$

where the entries of each matrix are

$$\mathcal{F}^2_{3\,ijk,lmo} = -\frac{\tau}{2\epsilon} \int_\Omega \frac{\partial B_{i,p}(x)}{\partial x} B_{j,p}(y) B_{k,p}(x) B_{l,p}(x) B_{m,p}(y) B_{o,p}(z) dxdydz$$

$$\mathcal{F}^3_{1\,ijk,lmo} = \frac{\tau^2}{4\epsilon\mu} \int_\Omega \frac{\partial B_{i,p}(x)}{\partial x} B_{k,p}(x) B_{j,p}(y) B_{l,p}(x) \frac{\partial B_{m,p}(y)}{\partial y} B_{o,p}(z) dxdydz$$

$$\mathcal{F}^3_{2\,ijk,lmo} = \frac{\tau^2}{4\epsilon\mu} \int_\Omega B_{i,p}(x) \frac{\partial B_{j,p}(y)}{\partial y} B_{k,p}(z) B_{l,p}(x) B_{m,p}(y) \frac{\partial B_{o,p}(z)}{\partial z} dxdydz$$

$$\mathcal{F}^3_{3\,ijk,lmo} = \frac{\tau^2}{4\epsilon\mu} \int_\Omega B_{i,p}(x) B_{j,p}(y) \frac{\partial B_{k,p}(z)}{\partial z} \frac{\partial B_{l,p}(x)}{\partial x} B_{m,p}(y) B_{o,p}(z) dxdydz$$

The alternating directions solver decomposes this system into three one-dimensional systems with multiple right-hand-sides

$$\begin{bmatrix} \mathcal{A}_1 F_1^{n+\frac{1}{2}} \\ \mathcal{A}_2 F_2^{n+\frac{1}{2}} \\ \mathcal{A}_3 F_3^{n+\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} \mathcal{RHS}_1 \\ \mathcal{RHS}_2 \\ \mathcal{RHS}_3 \end{bmatrix}$$

where

$$\mathcal{A}_{1\,i,l} = \int_{\Omega_x} B_{i,p}(x) B_{l,p}(x) dx$$

$$\mathcal{A}_{2\,i,l} = \int_{\Omega_x} B_{i,p}(x) B_{l,p}(x) dx$$

$$\mathcal{A}_{3\,i,l} = \int_{\Omega_x} \left(B_{i,p}(x) B_{l,p}(x) + \frac{\tau^2}{4\epsilon\mu} \frac{\partial B_{i,p}(x)}{\partial x} \frac{\partial B_{l,p}(x)}{\partial x} \right) dx$$

and the RHS vectors $\mathcal{RHS}_{1\,i,jk}$, $\mathcal{RHS}_{2\,i,jk}$, $\mathcal{RHS}_{3\,i,jk}$ have been reordered into matrices with N_x rows and $N_y N_z$ columns, by ordering blocks of N_x consecutive rows, one after another.

After solving the first one-dimensional system with multiple RHSs we solve the second system

$$\begin{bmatrix} \mathcal{B}_1 G_1^{n+\frac{1}{2}} \\ \mathcal{B}_2 G_2^{n+\frac{1}{2}} \\ \mathcal{B}_3 G_3^{n+\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} F_1^{n+\frac{1}{2}} \\ F_2^{n+\frac{1}{2}} \\ F_3^{n+\frac{1}{2}} \end{bmatrix}$$

where

$$\mathcal{B}_{1\,j,m} = \int_{\Omega_y} \left(B_{j,p}(y) B_{m,p}(y) + \frac{\tau^2}{4\epsilon\mu} \frac{\partial B_{j,p}(y)}{\partial y} \frac{\partial B_{m,p}(y)}{\partial y} \right) dy$$

$$\mathcal{B}_{2\,j,m} = \int_{\Omega_y} B_{j,p}(y) B_{m,p}(y) dy$$

$$\mathcal{B}_{3\,j,m} = \int_{\Omega_y} B_{j,p}(y) B_{m,p}(y) dy$$

Finally, we solve the third system with multiple RHSs

$$\begin{bmatrix} \mathcal{C}_1 E_1^{n+\frac{1}{2}} \\ \mathcal{C}_2 E_2^{n+\frac{1}{2}} \\ \mathcal{C}_3 E_3^{n+\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} G_1^{n+\frac{1}{2}} \\ G_2^{n+\frac{1}{2}} \\ G_3^{n+\frac{1}{2}} \end{bmatrix}$$

where

$$\mathcal{C}_{1\,k,o} = \int_{\Omega_z} B_{k,p}(z) B_{o,p}(z) dz$$

$$\mathcal{C}_{2\,k,o} = \int_{\Omega_z} \left(B_{k,p}(z) B_{o,p}(z) + \frac{\tau^2}{4\epsilon\mu} \frac{\partial B_{k,p}(z)}{\partial z} \frac{\partial B_{o,p}(z)}{\partial z} \right) dz$$

$$\mathcal{C}_{3\,k,o} = \int_{\Omega_z} B_{k,p}(z) B_{o,p}(z) dz$$

Similar considerations apply for other sub-steps

Incorporating non-regular material data

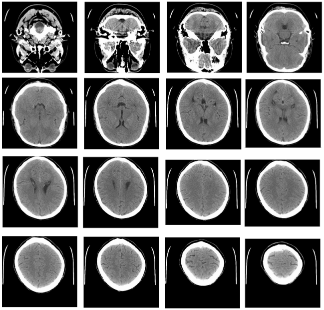


Figure: Exemplary MRI scans of the head of Maciej Paszyński.

Maciej Paszyński, Marcin Łoś, Judit Muñoz-Matute, Alternating directions implicit higher-order finite element method for simulations of time-dependent electromagnetic wave propagation in non-regular biological tissues, **https://arxiv.org/abs/2103.06998**

Incorporating non-regular material data

For example, if we want to modify material data $\tau = \hat{\tau}$, $\epsilon = \hat{\epsilon}$, $\mu = \hat{\mu}$ for test B-spline "*rst*", namely $B_{r,p}(x)B_{s,p}(y)B_{t,p}(z)$
In the first system, we extract the three equations (three rows) for the three components of the electric field for row $i = r$,
and the suitable columns from the RHS $l = r, m = s, o = t$, where we modify the material data

$$\sum_{l=1,\ldots,N_x} \int_{\Omega_x} B_{r,p}(x)B_{l,p}(x)dx \, F_1^{n+\frac{1}{2}}{}_{lst} = \mathcal{R}\hat{\mathcal{H}}\mathcal{S}_{1rst}$$

$$\sum_{l=1,\ldots,N_x} \int_{\Omega_x} B_{r,p}(x)B_{l,p}(x)dx \, F_2^{n+\frac{1}{2}}{}_{lst} = \mathcal{R}\hat{\mathcal{H}}\mathcal{S}_{2rst}$$

$$\sum_{l=1,\ldots,N_x} \int_{\Omega_x} \left(B_{r,p}(x)B_{l,p}(x) + \frac{\hat{\tau}^2}{4\hat{\epsilon}\hat{\mu}} \frac{\partial B_{r,p}(x)}{\partial x} \frac{\partial B_{l,p}(x)}{\partial x} \right) dx \, F_3^{n+\frac{1}{2}}{}_{lst} = \mathcal{R}\hat{\mathcal{H}}\mathcal{S}_{3rst}$$

The $\mathcal{R}\hat{\mathcal{H}}\mathcal{S}_{1rst}$, $\mathcal{R}\hat{\mathcal{H}}\mathcal{S}_{2rst}$, $\mathcal{R}\hat{\mathcal{H}}\mathcal{S}_{3rst}$ represent the right-hand sides with material data parameters $\tau = \hat{\tau}$, $\epsilon = \hat{\epsilon}$, $\mu = \hat{\mu}$. The other rows and columns in the first system remain unchanged.

Incorporating non-regular material data

Similarly, in the second system, we extract the equation for row $j = s$ and columns $l = r, m = s, n = t$

$$\sum_{m=1,\dots,N_y} \int_{\Omega_y} \left(B_{s,p}(y) B_{m,p}(y) + \frac{\hat{\tau}^2}{4\hat{\epsilon}\hat{\mu}} \frac{\partial B_{s,p}(y)}{\partial y} \frac{\partial B_{m,p}(y)}{\partial y} \right) dy \, {G_1^{n+\frac{1}{2}}}_{rmt} = {F_1^{n+\frac{1}{2}}}_{rst}$$

$$\sum_{m=1,\dots,N_y} \int_{\Omega_y} B_{s,p}(y) B_{m,p}(y) dy \, {G_2^{n+\frac{1}{2}}}_{rmt} = {F_2^{n+\frac{1}{2}}}_{rst}$$

$$\sum_{m=1,\dots,N_y} \int_{\Omega_y} B_{s,p}(y) B_{m,p}(y) dy \, {G_3^{n+\frac{1}{2}}}_{rmt} = {F_3^{n+\frac{1}{2}}}_{rst}$$

and we modify the material data. The other rows and columns in the second system remain unchanged.

Finally, in the third system, we extract the equation for row $k = t$ and columns $l = r, m = s, n = t$

$$\sum_{o=1,\dots,N_z} \int_{\Omega_z} B_{t,p}(z) B_{o,p}(z) dz {E_1^{n+\frac{1}{2}}}_{rso} = {G_1^{n+\frac{1}{2}}}_{rst}$$

$$\sum_{o=1,\dots,N_z} \int_{\Omega_z} \left(B_{t,p}(z) B_{o,p}(z) + \frac{\hat{\tau}^2}{4\hat{\epsilon}\hat{mu}} \frac{\partial B_{t,p}(z)}{\partial z} \frac{\partial B_{o,p}(z)}{\partial z} \right) dz {E_3^{n+\frac{1}{2}}}_{rso} = {G_2^{n+\frac{1}{2}}}_{rst}$$

$$\sum_{o=1,\dots,N_z} \int_{\Omega_z} B_{o,p}(z) B_{o,p}(z) dz {E_3^{n+\frac{1}{2}}}_{rso} = {G_3^{n+\frac{1}{2}}}_{rst}$$

and we modify the material data.

The other rows and columns in the third system remain unchanged.

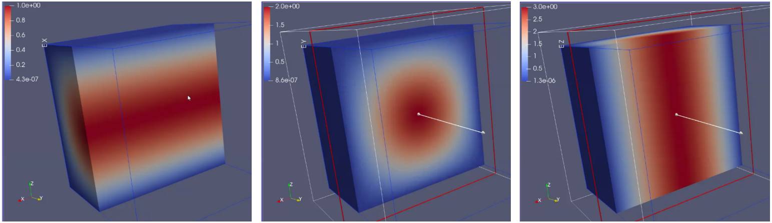


Figure: Electromagnetic waves propagation on the human head. Electric wave vector field. Cross sections along OYZ, OXZ, OXY. Time moment 0.0.

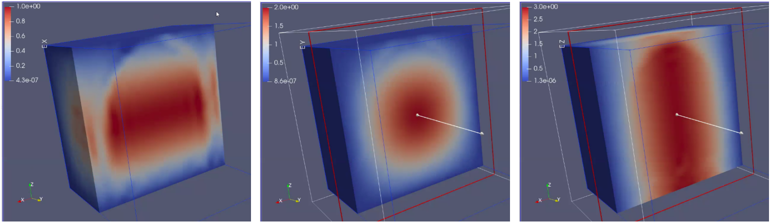


Figure: Electromagnetic waves propagation on the human head. Electric wave vector field. Cross sections along OYZ, OXZ, OXY. Time moment 0.25.

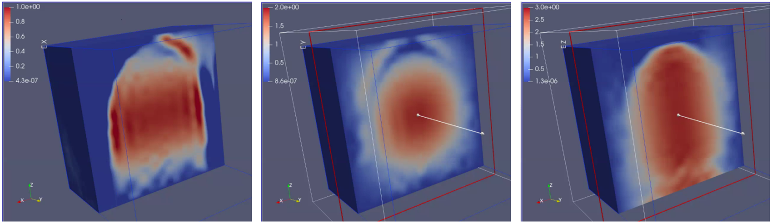


Figure: Electromagnetic waves propagation on the human head. Electric wave vector field. Cross sections along OYZ, OXZ, OXY. Time moment 0.5.

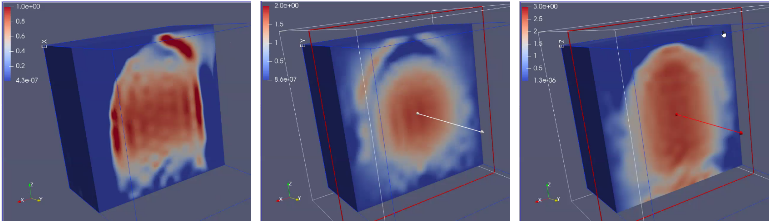


Figure: Electromagnetic waves propagation on the human head. Electric wave vector field. Cross sections along OYZ, OXZ, OXY. Time moment 0.75.

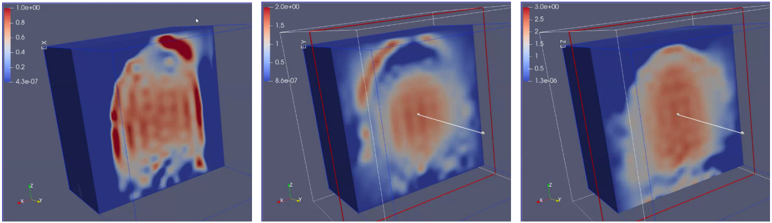


Figure: Electromagnetic waves propagation on the human head. Electric wave vector field. Cross sections along OYZ, OXZ, OXY. Time moment 1.0.

Conclusions

isoGeometric Residual Minimization Method (iGRM)
for time-dependent problems

- 2^{nd} order time schemes (unconditional stability in time)
- Residual minimization for each time step (stability in space)
- Discretization with B-spline basis functions
  (higher continuity smooth solutions)
- Kronecker product structure of the matrix
  (linear cost $\mathcal{O}(N)$ of direct solver)
- Linear computational cost solver for time-dependent Maxwell problems
- Parallelization in GALOIS environment
  (all simulations on a regular laptop)
- Future work: application as preconditioner for other engineering problems

National Science Centre, Poland grant no. 2017/26/M/ST1/00281
(HARMONIA)