isoGeometric Residual Minimization Method (iGRM)

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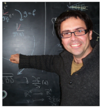

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Outline

- Unstable problems
- Factorization of Kronecker product matrices
- Residual minimization method
- Factorization of residual minimization problem matrix
- Iterative solver
- isoGeometric Residual Minimization method (iGRM)
- Numerical results: Manufactured solution problem, Eriksson-Johnson model problem
- Conclusions

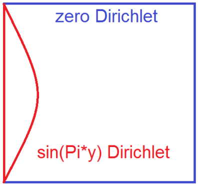
Eriksson-Johnson problem

Given $\Omega = (0,1)^2$, $\beta = (1,0)^T$, we seek the solution of the advection-diffusion problem

$$\frac{\partial u}{\partial x} - \epsilon \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0 \qquad (1)$$

with Dirichlet boundary conditions

$$u = 0 \text{ for } x \in (0,1), y \in \{0,1\} \qquad u = sin(\Pi y) \text{ for } x = 0$$



The problem is driven by the inflow Dirichlet boundary condition. It develops a boundary layer of width ϵ at the outflow $x = 1$.

Eriksson-Johnson problem

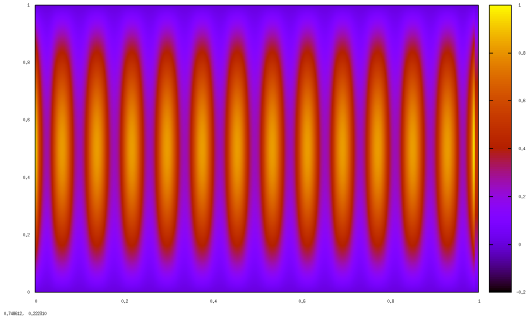


Figure: Galerkin FEM solution to Erikkson-Johnson problem, $Pe = 10^6$

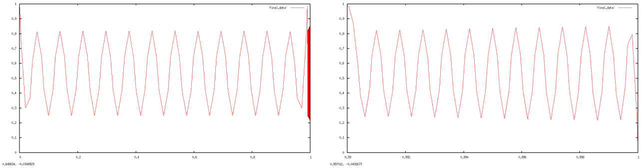
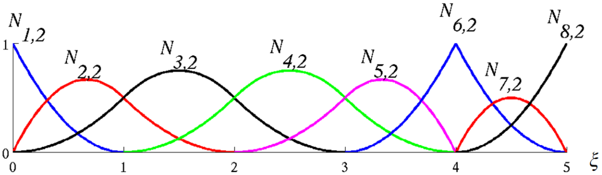


Figure: Cross-section and zoom

Higher continuity of isogeometric analysis

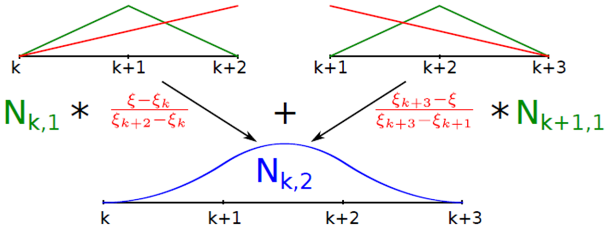


Quadratic basis functions for an open, non-uniform knot vector:

$$\Xi = \{0,0,0,1,2,3,4,4,5,5,5\}$$

Figure: Isogeometric analysis provided smooth approximations with lower number of degrees of freedom

Higher continuity of isogeometric analysis



$$N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \leq \xi < \xi_{i+1}, \\ 0 & \text{otherwise} \end{cases}$$

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi)$$

Figure: Recursive definition of B-splines

Higher continuity of isogeometric analysis

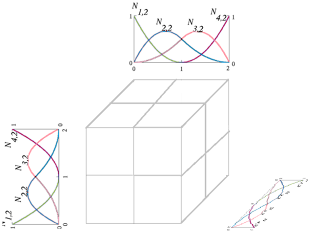


Figure: Tensor product structure of the 3D mesh

Isogeometric basis functions:

- 1D B-splines basis along x axis $B^x_{1,p}(x), \ldots, B^x_{N_x,p}(x)$
- 1D B-splines basis along y axis $B^y_{1,p}(y), \ldots, B^y_{N_y,p}(y)$
- 1D B-splines basis along z axis $B^z_{1,p}(z), \ldots, B^z_{N_z,p}(z)$
- In 2D we take tensor product basis $\{B^x_{i,p}(x)B^y_{j,p}(y)\}_{i=1,\ldots,N_x;j=1,\ldots,N_y}$
- In 3D we take tensor product basis $\{B^x_{i,p}(x)B^y_{j,p}(y)B^z_{k,p}(z)\}_{i=1,\ldots,N_x;j=1,\ldots,N_y;1,\ldots,N_z}$

Mass and stiffness matrices over 2D domain $\Omega = \Omega_x \times \Omega_y$

$$\mathcal{M} = (B_{ij}, B_{kl})_{L^2} = \int_\Omega B_{ij} B_{kl} \,\mathrm{d}\Omega =$$

$$\int_\Omega B_i^x(x) B_j^y(y) B_k^x(x) B_l^y(y) \,\mathrm{d}\Omega = \int_\Omega (B_i B_k)(x)\,(B_j B_l)(y) \,\mathrm{d}\Omega$$

$$= \left(\int_{\Omega_x} B_i B_k \,\mathrm{d}x \right) \left(\int_{\Omega_y} B_j B_l \,\mathrm{d}y \right) = \mathcal{M}^x \otimes \mathcal{M}^y$$

$$\mathcal{S} = (\nabla B_{ij}, \nabla B_{kl})_{L^2} = \int_\Omega \nabla B_{ij} \cdot \nabla B_{kl} \,\mathrm{d}\Omega =$$

$$\int_\Omega \frac{\partial (B_i^x(x) B_j^y(y))}{\partial x} \frac{\partial (B_k^x(x) B_l^y(y))}{\partial x} + \frac{\partial (B_i^x(x) B_j^y(y))}{\partial y} \frac{\partial (B_k^x(x) B_l^y(y))}{\partial y} \,\mathrm{d}\Omega$$

$$= \int_\Omega \frac{\partial B_i^x(x)}{\partial x} B_j^y(y) \frac{\partial B_k^x(x)}{\partial x} B_l^y(y) + B_i^x(x) \frac{\partial B_j^y(y)}{\partial y} B_k^x(x) \frac{\partial B_l^y(y))}{\partial y} \,\mathrm{d}\Omega$$

$$= \int_{\Omega_x} \frac{\partial B_i}{\partial x} \frac{\partial B_k}{\partial x} \,\mathrm{d}x \int_{\Omega_y} B_j B_l \,\mathrm{d}y + \int_{\Omega_x} B_i B_k \,\mathrm{d}x \int_{\Omega_y} \frac{\partial B_j}{\partial y} \frac{\partial B_l}{\partial y} \,\mathrm{d}y$$

$$= \mathcal{S}^x \otimes \mathcal{M}^y + \mathcal{M}^x \otimes \mathcal{S}^y$$

Derivation of Spatial Direction Splitting

Idea exploit Kronecker product structure of

$$\mathbf{M}\mathbf{x} = \mathbf{b}$$

with $\mathbf{M} = \mathbf{A} \otimes \mathbf{B}$, where $\mathbf{A}$ is $n \times n$, $\mathbf{B}$ is $m \times m$

Definition of Kronecker (tensor) product:

$$\mathbf{M} = \mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} \mathbf{A}\,B_{11} & \mathbf{A}\,B_{12} & \cdots & \mathbf{A}\,B_{1m} \\ \mathbf{A}\,B_{21} & \mathbf{A}\,B_{22} & \cdots & \mathbf{A}\,B_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}\,B_{m1} & \mathbf{A}\,B_{m2} & \cdots & \mathbf{A}\,B_{mm} \end{bmatrix}$$

Derivation of Spatial Direction Splitting

RHS and solution are partitioned into m blocks of size n each

$$\mathbf{x}_i = (x_{i1}, \ldots, x_{in})^\mathsf{T}$$
$$\mathbf{b}_i = (b_{i1}, \ldots, b_{in})^\mathsf{T}$$

We can rewrite the system as a block matrix equation:

$$\begin{cases} \mathbf{A}B_{11}\mathbf{x}_1 + \mathbf{A}B_{12}\mathbf{x}_2 + \cdots + \mathbf{A}B_{1m}\mathbf{x}_m = \mathbf{b}_1 \\ \mathbf{A}B_{21}\mathbf{x}_1 + \mathbf{A}B_{22}\mathbf{x}_2 + \cdots + \mathbf{A}B_{2m}\mathbf{x}_m = \mathbf{b}_2 \\ \qquad \vdots \qquad\qquad \vdots \qquad \vdots \qquad\qquad \vdots \\ \mathbf{A}B_{m1}\mathbf{x}_1 + \mathbf{A}B_{m2}\mathbf{x}_2 + \cdots + \mathbf{A}B_{mm}\mathbf{x}_m = \mathbf{b}_m \end{cases}$$

Derivation of Spatial Direction Splitting

Factor out $\mathbf{A}$:

$$\begin{cases} \mathbf{A}\big(B_{11}\mathbf{x}_1 + B_{12}\mathbf{x}_2 \ + \cdots + B_{1m}\mathbf{x}_m\big) \ = \mathbf{b}_1 \\ \mathbf{A}\big(B_{21}\mathbf{x}_1 + B_{22}\mathbf{x}_2 \ + \cdots + B_{2m}\mathbf{x}_m\big) \ = \mathbf{b}_2 \\ \qquad\quad \vdots \qquad\qquad \vdots \qquad \vdots \qquad\qquad \vdots \\ \mathbf{A}\big(B_{m1}\mathbf{x}_1 + B_{m2}\mathbf{x}_2 + \cdots + B_{mm}\mathbf{x}_m\big) = \mathbf{b}_m \end{cases}$$

Wy multiply by $\mathbf{A}^{-1}$ and define $\mathbf{y}^i = \mathbf{A}^{-1}\mathbf{b}^i$
(we have one 1D problem here $\mathbf{A}\ \mathbf{y}^i = \mathbf{b}^i$ with multiple RHS)

$$\begin{cases} B_{11}\mathbf{x}_1 + B_{12}\mathbf{x}_2 \ + \cdots + B_{1m}\mathbf{x}_m \ = \mathbf{y}_1 \\ B_{21}\mathbf{x}_1 + B_{22}\mathbf{x}_2 \ + \cdots + B_{2m}\mathbf{x}_m \ = \mathbf{y}_2 \\ \qquad \vdots \qquad\qquad \vdots \qquad \vdots \qquad\qquad \vdots \\ B_{m1}\mathbf{x}_1 + B_{m2}\mathbf{x}_2 + \cdots + B_{mm}\mathbf{x}_m = \mathbf{y}_m \end{cases}$$

Derivation of Spatial Direction Splitting

Consider each component of $\mathbf{x}_i$ and $\mathbf{y}_i$ $\Rightarrow$ family of linear systems

$$\begin{cases} B_{11}x^{1i} + B_{12}x^{2i} + \cdots + B_{1m}x^{mi} = y_{1i} \\ B_{21}x^{1i} + B_{22}x^{2i} + \cdots + B_{2m}x^{mi} = y_{2i} \\ \qquad\vdots \qquad\qquad \vdots \qquad \vdots \qquad\qquad \vdots \\ B_{m1}x^{1i} + B_{m2}x^{2i} + \cdots + B_{mm}x^{mi} = y_{mi} \end{cases}$$

for each $i = 1, \ldots, n$

$\Rightarrow$ linear systems with matrix $\mathbf{B}$ (We have another 1D problem here with multiple RHS $\mathbf{B}\ \mathbf{x}^i = \mathbf{y}^i$)

Residual minimization method

$$b(u,v) = l(v) \quad \forall v \in V \tag{2}$$

For our weak problem (2) we define the operator $B : U \to V'$ such as $< Bu, v >_{V' \times V} = b\,(u,v)$.

$$B : U \to V' \tag{3}$$

such that

$$\langle Bu, v \rangle_{V' \times V} = b(u,v) \tag{4}$$

so we can reformulate the problem as

$$Bu - l = 0 \tag{5}$$

We wish to minimize the residual

$$u_h = argmin_{w_h \in U_h} \frac{1}{2} \| Bw_h - l \|^2_{V'} \tag{6}$$

Residual minimization method

We introduce the Riesz operator being the isometric isomorphism

$$R_V : V \ni v \to (v, .) \in V' \tag{7}$$

We can project the problem back to V

$$u_h = argmin_{w_h \in U_h} \frac{1}{2} \| R_V^{-1}(Bw_h - l) \|_V^2 \tag{8}$$

The minimum is attained at u_h when the Gâteaux derivative is equal to 0 in all directions:

$$\langle R_V^{-1}(Bu_h - l), R_V^{-1}(B\, w_h) \rangle_V = 0 \quad \forall\, w_h \in U_h \tag{9}$$

We define the residual $r = R_V^{-1}(Bu_h - l)$ and we get

$$\langle r, R_V^{-1}(B\, w_h) \rangle = 0 \quad \forall\, w_h \in U_h \tag{10}$$

which is equivalent to

$$\langle Bw_h, r \rangle = 0 \qquad \forall w_h \in U_h. \tag{11}$$

From the definition of the residual we have

$$(r, v)_V = \langle Bu_h - l, v \rangle \quad \forall v \in V. \tag{12}$$

Find $(r, u_h)_{V\times U_h}$ such as

$$\begin{aligned} (r, v)_V - \langle Bu_h - l, v\rangle &= 0 \quad \forall v \in V \\ \langle Bw_h, r\rangle &= 0 \quad \forall w_h \in U_h \end{aligned} \tag{13}$$

We discretize the test space $V_m \in V$ to get the discrete problem:
Find $(r_m, u_h)_{V_m\times U_h}$ such as

$$\begin{aligned} (r_m, v_m)_{V_m} - \langle Bu_h - l, v_m\rangle &= 0 \quad \forall v_m \in V_m \\ \langle Bw_h, r_m\rangle &= 0 \quad \forall w_h \in U_h \end{aligned} \tag{14}$$

where $(*, *)_{V_m}$ is an inner product in V_m, $\langle Bu_h, v_m\rangle = b\,(u_h, v_m)$, $\langle Bw_h, r_m\rangle = b\,(w_h, r_m)$.

Remark

We define the discrete test space V_m in such a way that it is as close as possible to the abstract V space, to ensure stability, in a sense that the discrete inf-sup condition is satisfied. In our method it is possible to gain stability enriching the test space V_m without changing the trial space U_h.

Discretization of the residual minimization method

We approximate the solution with tensor product of one dimensional B-splines basis functions of order p

$$u_h = \sum_{i,j} u_{i,j} B^x_{i;p}(x) B^y_{j;p}(y). \tag{15}$$

We test with tensor product of one dimensional B-splines basis functions, where we enrich the order in the direction of the x axis from p to r ($r \geq p$, and we enrich the space only in the direction of the alternating splitting)

$$v_m \leftarrow B^x_{i;r}(x) B^y_{j;p}(y). \tag{16}$$

We approximate the residual with tensor product of one dimensional B-splines basis functions of order p

$$r_m = \sum_{s,t} r_{s,t} B^x_{s;r}(x) B^y_{t;p}(y), \tag{17}$$

and we test again with tensor product of 1D B-spline basis functions of order r and p, in the corresponding directions

$$w_h \leftarrow B^x_{k;p}(x) B^y_{l;p}(y). \tag{18}$$

Decomposition into Kronecker product structure

$$A = A_y \otimes A_x; B = B_x \otimes B_y; B^T = B_y^T \otimes B_x^T; A_y = B_y \quad (19)$$

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} = \begin{pmatrix} A_y & 0 \\ 0 & A_y^T \end{pmatrix} \begin{pmatrix} A_x & B_x \\ B_x^T & 0 \end{pmatrix} = \begin{pmatrix} A_x A_y & B_x A_y \\ B_x^T A_y^T & 0 \end{pmatrix}. \quad (20)$$

Both matrices $\begin{pmatrix} A_y & 0 \\ 0 & A_y^T \end{pmatrix}$ and $\begin{pmatrix} A_x & B_x \\ B_x^T & 0 \end{pmatrix}$ can be factorized in a linear $\mathcal{O}(N)$ computational cost.

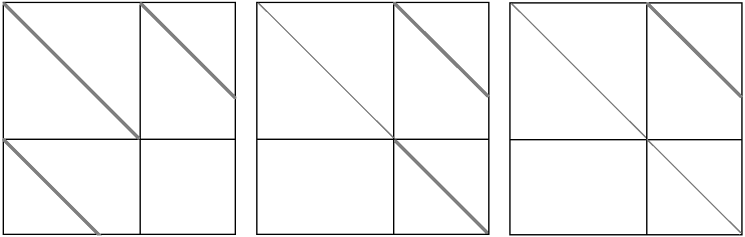


Figure: Factorization of second block.

Towards iterative solver

$$\begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} r \\ u \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix} \tag{21}$$

$$A = M + \eta K$$

$$\begin{aligned} M &= M_x \otimes M_y, \\ K &= K_x \otimes M_y + M_x \otimes K_y. \end{aligned}$$

$$\begin{aligned} A &= M + \eta K \\ &= (M_x + \eta K_x) \otimes (M_y + \eta K_y) - \eta^2 K_x \otimes K_y = \tilde{A} - \tilde{K} \end{aligned}$$

$$\begin{bmatrix} \tilde{A} & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} r \\ u \end{bmatrix} = \begin{bmatrix} F + \tilde{K} r \\ 0 * u \end{bmatrix} \tag{22}$$

so we iterate

$$\begin{bmatrix} r^{k+1} \\ u^{k+1} \end{bmatrix} = \begin{bmatrix} \tilde{A} & B \\ B^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} F + \tilde{K} r^k \\ 0 * u^k \end{bmatrix} \tag{23}$$

Towards iterative solver

Firstly, consider the residual error

$$
\begin{aligned}
r^k &= F - Aw^k - Bu^k = F + \tilde{K}w^k - \tilde{A}w^k - Bu^k, \\
s^k &= -B^T w^k.
\end{aligned}
\tag{24}
$$

To minimize these residual-type error, we build a dual problem, which resembles the features of preconditioner.

$$
\begin{bmatrix} \tilde{A} & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} d^k \\ c^k \end{bmatrix} = \begin{bmatrix} r^k \\ s^k \end{bmatrix},
\tag{25}
$$

Using (24), solving these equations gives

$$
\begin{aligned}
d^k &= \tilde{A}^{-1}(F + \tilde{K}w^k - Bu^k) - w^k - \tilde{A}^{-1}Bc^k \\
&:= \tilde{d}^k - w^k - \tilde{A}^{-1}Bc^k, \\
B^T\tilde{A}^{-1}Bc^k &= -B^T\tilde{d}^k + B^T w^k + s^k = -B^T\tilde{d}^k,
\end{aligned}
\tag{26}
$$

where the first equation is a primal update and the second equation is to be solved by a Conjugate-Gradient (CG) type method.

Inner loop

Initialize $\{q^{(0)} = p^{(0)} = \tilde{c}^k; u^{(0)} = u^k\}$

for $j = 1$ until convergence

$$\theta^{(j)} = Bp^{(j)};$$

$$\delta^{(j)} = \tilde{A}^{-1}\theta^{(j)};$$

$$\alpha^{(j)} = \frac{(p^{(j)}, q^{(j)})}{(\theta^{(j)}, \delta^{(j)})};$$

$$u^{(j+1)} = u^{(j)} + \alpha^{(j)} p^{(j)};$$

$$q^{(j+1)} = q^{(j)} - \alpha^{(j)} B^T \delta^{(j)};$$

$$\beta^{(j+1)} = \frac{(q^{(j+1)}, q^{(j+1)})}{(q^{(j)}, q^{(j)})};$$

$$p^{(j+1)} = q^{(j+1)} + \beta^{(j+1)} p^{(j)};$$

Algorithm 1: Inner CG

Iterative solver

To determine the convergence, for the inner loop,
we iterate until $\alpha^{(j+1)} \leq tolerance$. We denote this j as j_c.
The outer iteration calculates

$$\begin{aligned} c^k &= u^{(j_c)} - u^{(0)}; \\ u^{k+1} &= u^{(j_c)}; \\ w^{k+1} &= \tilde{d}^k + \tilde{A}^{-1} B c^k. \end{aligned} \tag{27}$$

The outer iteration stops at $c^{k+1} \leq tolerance$.

Towards iterative solver

Initialize $\{u^{(0)} = 0; w^{(0)} = 0\}$

for $k = 1, ..., N$ until convergence

Inner loop

$$
\begin{aligned}
c^k &= u^{(N_k)} - u^{(0)}; \\
u^{k+1} &= u^{(N_k)}; \\
w^{k+1} &= \tilde{d}^k + \tilde{A}^{-1} B c^k; \\
k &= k + 1; \\
\tilde{d}^{k+1} &= \tilde{A}^{-1}(F + \tilde{K} w^{k+1} - B u^{k+1}); \\
\tilde{c}^{k+1} &= -B^T \tilde{d}^{k+1}.
\end{aligned}
$$

Algorithm 2: Inner-Outer CG

isoGeometric Residual Minimization Method (iGRM)

- Residual minimization
  (unconditional stability in space)
- Discretization with B-spline basis functions
  (higher continuity smooth solutions)
- Kronecker product structure of the inner product matrix
  (linear cost $\mathcal{O}(N)$ preconditioner for iterative solver)
- Symmetric positive definite system
  (convergence of the conjugated gradient method)

A manufactured solution problem: strong form

We focus on a model problem with a manufactured solution. For a unitary square domain $\Omega = (0,1)^2$, the advection vector $\beta = (1,1)^T$, and $Pe = 100, \epsilon = 1/Pe$ we seek the solution of the advection-diffusion equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} - \epsilon \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = f \tag{28}$$

with Dirichlet boundary conditions $u = g$ on the whole of $\Gamma = \partial\Omega$. We utilize a manufactured solution

$$u(x,y) = (x + \frac{e^{Pe*x} - 1}{1 - e^{Pe}})(y + \frac{e^{Pe*y} - 1}{1 - e^{Pe}})$$

enforced by the right-hand side, and we use homogeneous Dirichlet boundary conditions on $\partial\Omega$.

A manufactured solution problem: weak form

$$b(u,v) = l(v) \quad \forall v \in V \tag{29}$$

$$\begin{aligned} b(u,v) = {} & \left(\frac{\partial u}{\partial x}, v\right)_\Omega + \left(\frac{\partial u}{\partial y}, v\right)_\Omega + \epsilon\left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right)_\Omega + \epsilon\left(\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}\right)_\Omega \\ & - \left(\epsilon\frac{\partial u}{\partial x} n_x, v\right)_\Gamma - \left(\epsilon\frac{\partial u}{\partial y} n_y, v\right)_\Gamma \\ & - (u, \epsilon\nabla v \cdot n)_\Gamma - (u, \beta \cdot n v)_\Gamma - \left(u, 3p^2\epsilon/hv\right)_\Gamma \end{aligned}$$

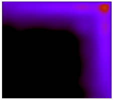
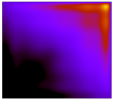
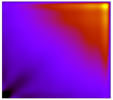
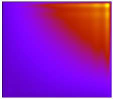
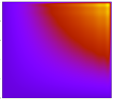
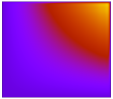
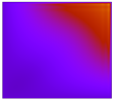
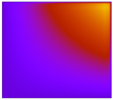
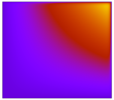
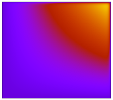
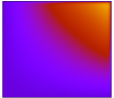
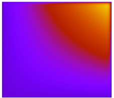
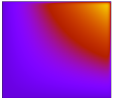
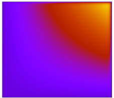
$n = (n_x, n_y)$ is versor normal to Γ, and h is element diameter,

$$l(v) = (f, v)_\Omega - (g, \epsilon\nabla v \cdot n)_\Gamma - (g, \beta \cdot n v)_\Gamma - \left(g, 3p^2\epsilon/hv\right)_\Gamma \tag{30}$$

red terms correspond to weak imposition of the Dirichlet b.c. on Γ with $g = 0$, f is the manufactured solution, blue terms are the integration by parts, gray terms the penalty terms. We seek the solution in space $U = V = H^1(\Omega)$. The inner product in V is

$$(u,v)_V = (u,v)_{L_2} + \left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right)_{L_2} + \left(\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}\right)_{L_2} \tag{31}$$

A manufactured solution results

n	trial(2,1) test(2,0)	trial(3,2) test(2,0)	trial(4,3) test(2,0)	trial(5,4) test(2,0)
#DOF	389	410	433	458
L2	192	151	78	28
H1	101	74	44	32
8×8				
#DOF	1413	1450	1489	1530
L2	80	16	3.29	1.48
H1	59	29	18	10
16×16				
#DOF	5381	5450	5521	5594
L2	32	1.33	0.27	0.056
H1	31	9.77	3.16	0.82
32×32				
#DOF	20997	21130	21265	21402
L2	7.66	0.07	0.01	0.003
H1	9.86	1.67	0.26	0.068
64×64				

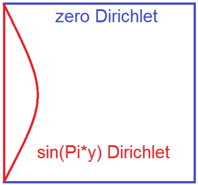
Eriksson-Johnson problem strong form

Given $\Omega = (0,1)^2$, $\beta = (1,0)^T$, we seek the solution of the advection-diffusion problem

$$\frac{\partial u}{\partial x} - \epsilon \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0 \qquad (32)$$

with Dirichlet boundary conditions

$$u = 0 \text{ for } x \in (0,1), y \in \{0,1\} \qquad u = sin(\Pi y) \text{ for } x = 0$$



The problem is driven by the inflow Dirichlet boundary condition. It develops a boundary layer of width ϵ at the outflow $x = 1$.

Eriksson-Johnson problem weak form

We introduce first the weak formulation for the Eriksson-Johnson problem

$$b(u,v) = l(v) \quad \forall v \in V \tag{33}$$

$$b(u,v) = \left(\frac{\partial u}{\partial x}, v\right) + \epsilon\left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right) + \epsilon\left(\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}\right)$$
$$- \left(\epsilon \frac{\partial u}{\partial x} n_x, v\right)_\Gamma - \left(\epsilon \frac{\partial u}{\partial y} n_y, v\right)_\Gamma$$
$$- (u, \epsilon \nabla v \cdot n)_\Gamma - (u, \beta \cdot n v)_\Gamma - \left(u, 3p^2\epsilon/hv\right)_\Gamma$$

where the blue, red, and gray terms correspond to the weak imposition of the Dirichlet b.c. on the boundary Γ, and $n = (n_x, n_y)$ is the versor normal to the boundary,

$$l(v) = -(g, \epsilon \nabla v \cdot n)_{\Gamma^-} - (g, \beta \cdot n v)_{\Gamma^-} - \left(g, 3p^2\epsilon/hv\right)_{\Gamma^-} \tag{34}$$

where the red and gray terms correspond to the weak introduction of the Dirichlet b.c. on the boundary Γ.
We plug the weak form and the inner product into the iGRM setup and we use the preconditioned CG solver.

Residual minimization method for Eriksson-Johnson problem

Find $(r_m, u_h)_{V_m \times U_h}$ such as

$$(r_m, v_m)_{V_m} - \left(\frac{\partial u_h}{\partial x}, v_m\right) - \epsilon\left(\frac{\partial u_h}{\partial x}, \frac{\partial v_m}{\partial x} + \frac{\partial u_h}{\partial y}, \frac{\partial v_m}{\partial y}\right) = (f, v_m) \quad \forall v_m \in V_m$$

$$\left(\frac{\partial w_h}{\partial x}, r_m\right) + \epsilon\left(\frac{\partial w_h}{\partial x}, \frac{\partial r_m}{\partial x} + \frac{\partial w_h}{\partial y}, \frac{\partial r_m}{\partial y}\right) = 0 \quad \forall w_h \in U_h$$

where $(r_m, v_m)_{V_m} = (r_m, v_m) + (\frac{\partial r_m}{\partial x}, \frac{\partial v_m}{\partial x}) + (\frac{\partial r_m}{\partial y}, \frac{\partial v_m}{\partial y})$ is the H^1 norm induced inner product.

Remark

We will use trial space U_h as quadratic B-splines with C^1 continuity and test space V_h as cubic B-splines with C^2 continuity

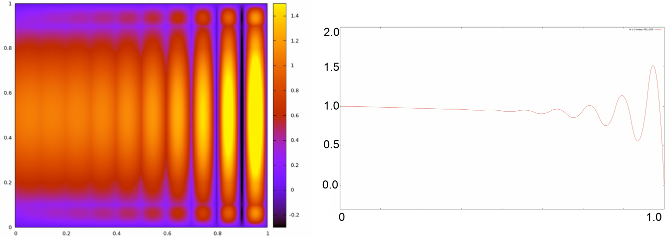


Figure: **Left panel:** Solution to the Erikkson-Johnsson problem with Galerkin method with $\epsilon = 10^{-2}$, with trial = test = quadratic B-splines on a uniform mesh of 20×20 elements. **Right panel:** Cross-section at $y = 0.5$.

Numerical results for the Eriksson-Johnson problem

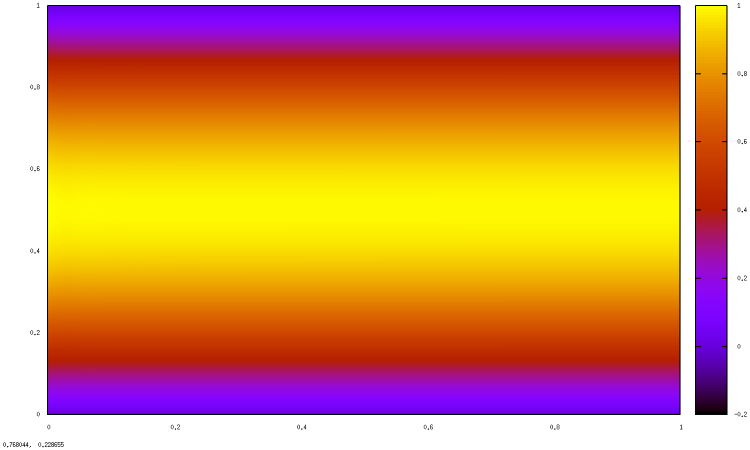


Figure: Solution to the Erikkson-Johnsson problem with residual minimization method with $\epsilon = 10^{-2}$, with trial = quadratic B-splines, test = cubic B-splines on a uniform mesh of 20×20 elements.

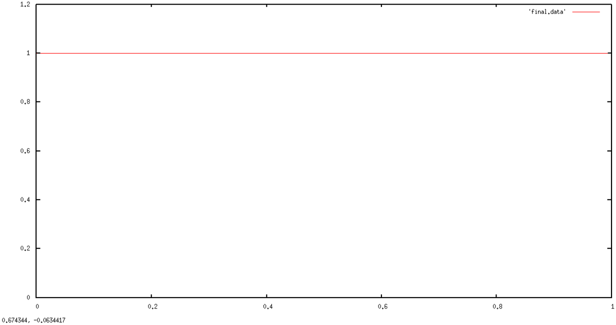


Figure: Cross-section at $y = 0.5$ of the solution to the Erikkson-Johnsson problem with residual minimization method with $\epsilon = 10^{-2}$, with trial = quadratic B-splines, test = cubic B-splines
on a uniform mesh of 20×20 elements.

Numerical results for the Eriksson-Johnson problem

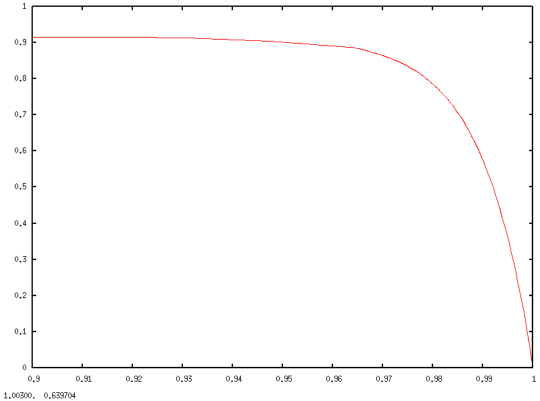


Figure: Zoom of the cross-section at $y = 0.5$ of the solution at (0.99,1.0) of the Erikkson-Johnsson problem with residual minimization method with $\epsilon = 10^{-2}$, with trial = quadratic B-splines, test = cubic B-splines on a uniform mesh of 20×20 elements.

Eriksson-Johnson problem weak form, SUPG method

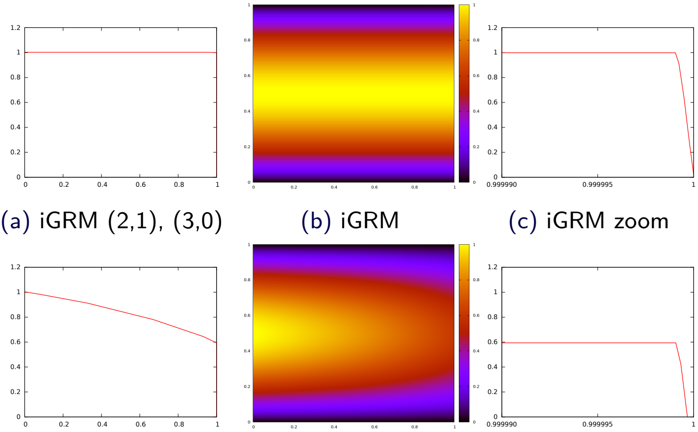
$$b(u,v) + (R(u), \tau\beta \cdot \nabla v) = l(v) \quad \forall v \in V \tag{35}$$

where $R(u) = \frac{\partial u}{\partial x} + \epsilon \Delta u$, and $\left(\frac{\beta_x}{h_x} + \frac{\beta_y}{h_y}\right) + 3\epsilon \frac{1}{h_x^2 + h_y^2}$,
and in our case diffusion term $\epsilon = 10^{-6}$, and convection term $\beta = (1,0)$, and h_x and h_y are dimensions of an element.

$$b_{SUPG}(u,v) = l(v) \quad \forall v \in V \tag{36}$$

$$\begin{aligned} b_{SUPG}(u,v) = & \left(\frac{\partial u}{\partial x}, v\right) + \epsilon \left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right) + \epsilon \left(\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}\right) \\ & - \left(\epsilon \frac{\partial u}{\partial x} n_x, v\right)_\Gamma - \left(\epsilon \frac{\partial u}{\partial y} n_y, v\right)_\Gamma \\ & - (u, \epsilon \nabla v \cdot n)_\Gamma - (u, \beta \cdot n v)_\Gamma - \left(u, 3p^2 \epsilon / hv\right)_\Gamma \\ & + \left(\frac{\partial u}{\partial x} + \epsilon \Delta u, \left(\frac{1}{h_x} + 3\epsilon \frac{1}{h_x^2 + h_y^2}\right)^2 \frac{\partial v}{\partial x}\right) \end{aligned}$$

Numerical results for the Eriksson-Johnson problem

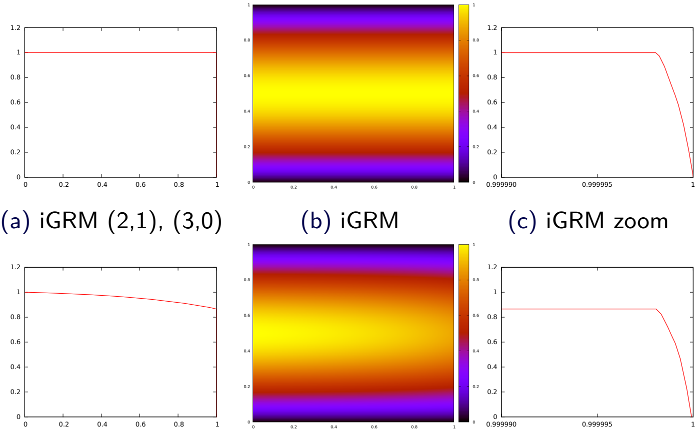


(a) iGRM (2,1), (3,0) (b) iGRM (c) iGRM zoom

(d) SUPG trial (3,2) (e) SUPG (f) SUPG zoom

Figure: Comparison of solutions of the Erikkson-Johnsson by using iGRM and SUPG methods for Pe=1,000,000 on 2x2 mesh, using (2,1) for trial and (3,0) for testing.

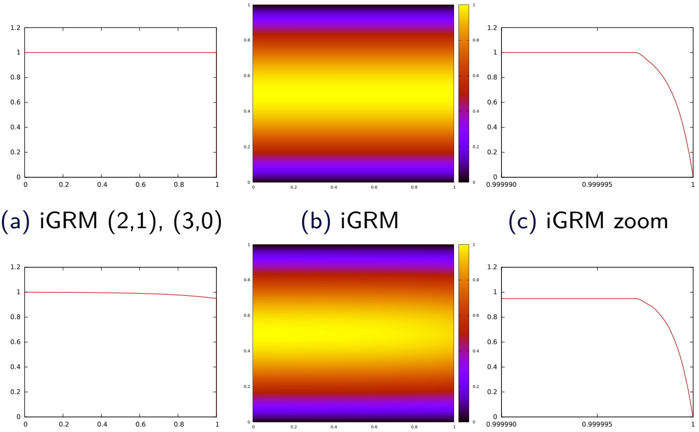
Numerical results for the Eriksson-Johnson problem



(a) iGRM (2,1), (3,0) (b) iGRM (c) iGRM zoom

(d) SUPG trial (3,2) (e) SUPG (f) SUPG zoom

Figure: Comparison of solutions of the Erikkson-Johnsson by using iGRM and SUPG methods for Pe=10000 on 4x4 mesh,
using (2,1) for trial and (3,0) for testing.

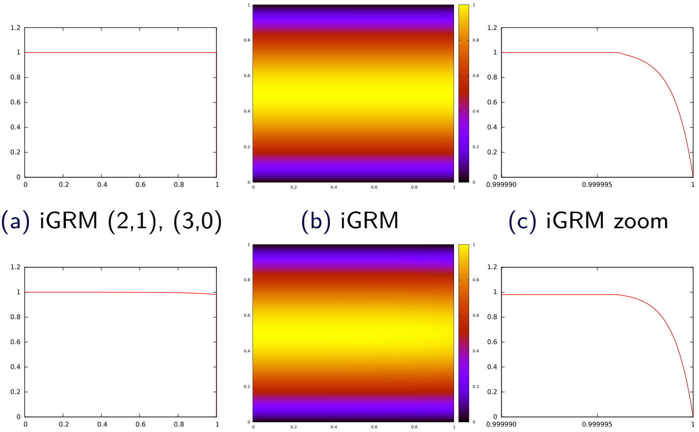


(a) iGRM (2,1), (3,0) (b) iGRM (c) iGRM zoom

(d) SUPG trial (3,2) (e) SUPG (f) SUPG zoom

Figure: Comparison of solutions of the Erikkson-Johnsson by using iGRM and SUPG methods for Pe=1,000,000 on 8x8 mesh, using (2,1) for trial and (3,0) for testing.

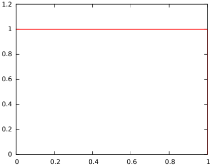
Numerical results for the Eriksson-Johnson problem



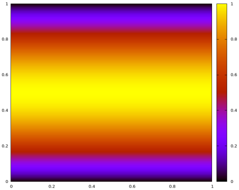
(a) iGRM (2,1), (3,0) (b) iGRM (c) iGRM zoom

(d) SUPG trial (3,2) (e) SUPG (f) SUPG zoom

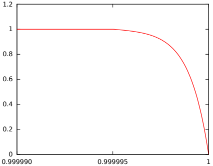
Figure: Comparison of solutions of the Erikkson-Johnsson by using iGRM and SUPG methods for Pe=1,000,000 on 16x16 mesh, using (2,1) for trial and (3,0) for testing.



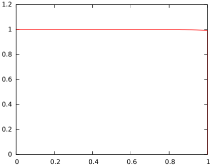
(a) iGRM (2,1), (3,0)



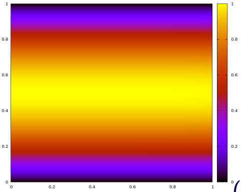
(b) iGRM



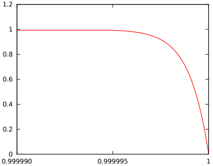
(c) iGRM zoom



(d) SUPG trial (3,2)



(e) SUPG crossection



(f) SUPG crossection zoom

Figure: Comparison of solutions of the Erikkson-Johnsson by using iGRM and SUPG methods for Pe=1,000,000 on 32x32 mesh, using (2,1) for trial and (3,0) for testing.

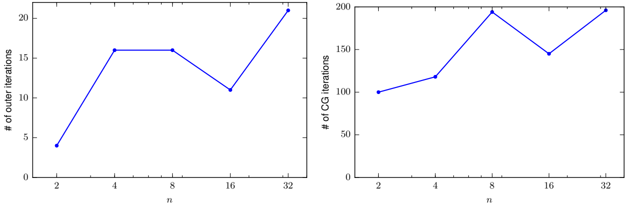


Figure: **Left panel:** Number of iterations of outer loop for. **Right panel:** Number of iterations of inner loop (CG).

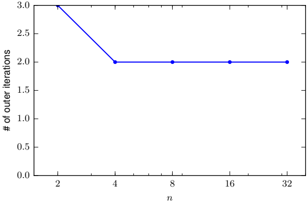


Figure: Number of iterations of outer loop, with inner loop treated by MUMPS solver.

$$\beta_x \frac{\partial u}{\partial x} + \beta_y \frac{\partial u}{\partial y} - \epsilon \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0$$

over the rectangular domain $\Omega = (0,1) \times (-1,1)$, with zero right-hand side $f = 0$, $Pe = 1,000,000$, the advection vector $\beta(x,y) = (\beta_x(x,y), \beta_y(x,y)) = \psi(\frac{-y}{(x^2+y^2)^{\frac{1}{2}}}, \frac{x}{(x^2+y^2)^{\frac{1}{2}}})$ modeling the circular wind, where ψ is the wind force coefficient.

$\Gamma_1 = \{(x,y) : x = 0, 0.5 \leq y \leq 1.0\}$,
$\Gamma_2 = \{(x,y) : x = 0, 0.0 \leq y \leq 0.5\}$,
$\Gamma_3 = \{(x,y) : x = 0, -0.5 \leq y \leq 0.0\}$,
$\Gamma_4 = \{(x,y) : x = 0, -1.0 \leq y \leq -0.5\}$,
We utilize the Dirichlet boundary conditions $u = g$ on $\Gamma = \partial\Omega$ where

$$g = \frac{1}{2} \left(\tanh \left((|y| - 0.35) \frac{b}{\epsilon} \right) + 1 \right), \text{ for } x \in \Gamma_2 \cup \Gamma_3$$

$$g = \frac{1}{2} \left(0.65 - \tanh \left((|y|) \frac{b}{\epsilon} \right) + 1 \right), \text{ for } x \in \Gamma_1 \cup \Gamma_4$$

$$g = 0, \text{ for } x \in \Gamma \setminus \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$$

Circular wind problem

The weak formulation

$$b(u,v) = l(v) \quad \forall v \in V$$

$$\begin{aligned} b(u,v) = \\ \left(\beta_x \frac{\partial u}{\partial x}, v\right)_\Omega + \left(\beta_y \frac{\partial u}{\partial y}, v\right)_\Omega + \epsilon \left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right)_\Omega + \epsilon \left(\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}\right)_\Omega \\ - \left(\epsilon \frac{\partial u}{\partial x} n_x, v\right)_\Gamma - \left(\epsilon \frac{\partial u}{\partial y} n_y, v\right)_\Gamma \\ - (u, \epsilon \nabla v \cdot n)_\Gamma - (u, \beta \cdot n v)_\Gamma - \left(u, 3p^2\epsilon/h v\right)_\Gamma \end{aligned}$$

where $n = (n_x, n_y)$ is the versor normal to Γ,

$$l(v) = -(g, \epsilon \nabla v \cdot n)_\Gamma - (g, \beta \cdot n v)_\Gamma - \left(g, 3p^2\epsilon/h v\right)_\Gamma$$

$n = (n_x, n_y)$ is the versor normal to the boundary, and the right-hand side forcing is equal to 0.

Circular wind problem

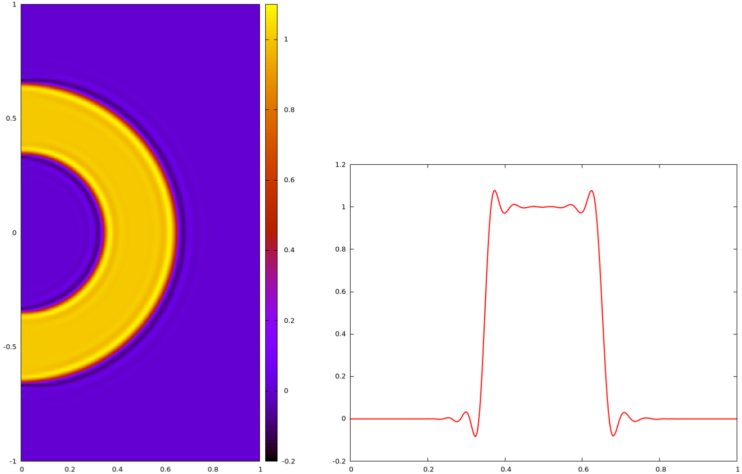


Figure: Solution to the circular wind problem on the mesh of 128×128 elements with trial(2,1),test(2,0), for Pecklet number $Pe = 1,000,000$, wind force $b = 1$. Horizontal cross-section at $x = 0$.

Conclusions and open questions

Conclusions

- We use B-spline basis functions on tensor product patches of elements and we do not break the test spaces
- We employ the residual minimization method with fixed trial space, and we enrich the test space to improve the stability
- To preserve the Kronecker product structure of the residual minimization system, we enrich the test space in the alternating direction manner (e.g. (3,2) for x direction, and $(2,3)$ for y direction) to preserve the Kronecker product structure of the matrix, and linear cost $\mathcal{O}(N)$ factorization
- For a linear cost $\mathcal{O}(N)$ Kronecker product precondtioner we can enrich the test space in arbitrary way, since we only need a Kronecker product structure of the Gramm matrix

Future work

- Stokes, Oseen, Navier-Stokes and Maxwell equations

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Papers

M. Los, Q.Deng, I. Muga, V.M.Calo, M. Paszynski, Isogeometric Residual Minimization Method (iGRM) with Direction Splitting Preconditoner for Stationary Advection-Diffusion Problems, submitted to *Computer Methods in Applied Mechanics and Engineering* (2019) **IF: 4.441**

Code based on:
M. Los, M. Wozniak, M. Paszynski, A. Lenharth, K. Pingali, IGA-ADS : Isogeometric Analysis FEM using ADS solver, *Computer & Physics Communications*, 217 (2017), pp. 99-116 **IF: 3.748**