1 Basic definitions from probability theory

Let Ω be a given set.

Definition 1 (σ-field)**.** Let $\mathcal{F} \subset 2^{\Omega}$. We call $\mathcal{F}$ a σ-field if it satisfies the following conditions

1. $\emptyset \in \mathcal{F}$,
2. $A \in \mathcal{F} \Rightarrow A' \in \mathcal{F}$,
3. $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{+\infty} \in \mathcal{F}$.

Definition 2 (σ-field generated by a family of subsets)**.** Let $\mathcal{A} \subset 2^{\Omega}$. Then the smallest (with respect to inclusion) σ-field containing $\mathcal{A}$ is called a σ-field generated by $\mathcal{A}$ and is denoted by $\boxed{\sigma(\mathcal{A})}$.

Definition 3 (σ-field of borel sets)**.** Let $\Omega = \mathbb{R}$. By $\boxed{\mathcal{B}(\mathbb{R})}$ we denote the σ-field generated by $\mathcal{A} = \{(a, b) : a < b, a, b \in \mathbb{R}\}$.

Definition 4 (measurable space)**.** Let $\mathcal{F} \subset 2^{\Omega}$ be a σ-field. Then a pair $(\Omega, \mathcal{F})$ will be called a measurable space.

Definition 5 (measurable set)**.** Let $(\Omega, \mathcal{F})$ be a measurable space. Any $A \in \mathcal{F}$ will be called a measurable set.

Definition 6 (partition)**.** Let $(\Omega, \mathcal{F})$ be a measurable space and let $\mathcal{P} = \{A_1, A_2, \ldots, A_n\}$ be a finite family of measurable sets. Then $\mathcal{P}$ will be called a partition of Ω if

1. $A_i \cap A_j = \emptyset$ for any $i \neq j$,
2. $\Omega = \bigcup_{i=1}^{n} A_i$.

Definition 7 (finer/coarses partitions)**.** Let $(\Omega, \mathcal{F})$ be a measurable space and let $\mathcal{P}_1, \mathcal{P}_2$ be partitions of Ω. We say that $\mathcal{P}_2$ is finer than $\mathcal{P}_1$ (or equivalently that $\mathcal{P}_1$ is coarser than $\mathcal{P}_2$) if

$$\forall A \in \mathcal{P}_1 \exists B_1, B_2, \ldots, B_n \in \mathcal{P}_2 : A = \bigcup_{i=1}^{n} B_i.$$

Definition 8 (filtration)**.** Let $(\Omega, \mathcal{F})$ be a measurable space and let $\mathbb{T} = \{0, 1, \ldots, N\}$. Let $(\mathcal{F}_t)_{t \in \mathbb{T}}$ be a family of σ-fields, such that $\mathcal{F}_t \subset \mathcal{F}$ for each t. Then $(\mathcal{F}_t)_{t \in \mathbb{T}}$ is called a filtration if

$$\forall s, t \in \mathbb{T}, s \leq t : \mathcal{F}_s \subset \mathcal{F}_t.$$

Lemma 9. *Let $(\Omega, \mathcal{F})$ be a measurable space and let $\mathbb{T} = \{0, 1, \ldots, N\}$. Assume $\{\mathcal{P}_t\}_{t \in \mathbb{T}}$ is a sequence of partitions, satisfying*

$$\forall t \in \mathbb{T}, t < N : \mathcal{P}_{t+1} \text{ is finer then } \mathcal{P}_t.$$

Define a sequence of σ-fields $\{\mathcal{F}_t\}_{t\in\mathbb{T}}$ *by*

$$\mathcal{F}_t = \sigma(\mathcal{P}_t).$$

Then $\{\mathcal{F}_t\}_{t\in\mathbb{T}}$ *is a filtration.*

Definition 10 (measurable function)**.** Let $(\Omega, \mathcal{F})$, $(\Omega', \mathcal{F}')$ be measurable spaces. We call $X : \Omega \to \Omega'$ a measurable function if

$$\forall A \in \mathcal{F}' : X^{-1}(A) \in \mathcal{F}.$$

Definition 11 (borel function)**.** Let $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a measurable function. Then X will be called a borel function.

Remark 12. The property of being a borel function depends on the σ-field $\mathcal{F}$ from the domain of X. Therefore to stress this dependence the function X satisfying the previous definition is often said to be $\mathcal{F}$-measurable.

Theorem 13. *Let* $\mathcal{P} = \{A_1, \dots, A_n\}$ *be a partition of* Ω *and let* $\mathcal{F} = \sigma(\mathcal{P})$. *Then* $X : \Omega \to \mathbb{R}$ *is* $\mathcal{F}$*-measurable if and only if* X *is constant on each* A_i.