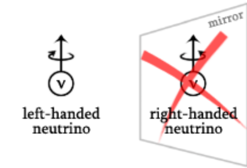

Symmetries (a bit) revisited and current current interaction picture

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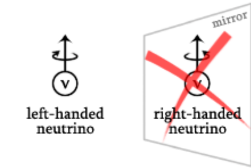
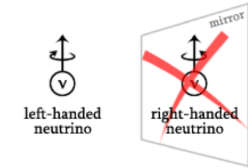
Symmetries



- They started to flourish thanks to Greek philosophers, and after a while they were completely lost for science (but were doing fine in arts and engineering though…)
- **Re-discovered** again at the end of 19th century
- So, why they are so cool and fundamental now? Why they occupy a central place in modern science? Well, it has to do with the **nice abstraction** they bring along – using symmetry **we are insensitive** to any **specific details** regarding given natural phenomenon
- Why theorist love symmetries so much? Well, if they search for new physics not knowing any details about it – symmetry can still guide them and impose important restrictions on reasonable models
- Important vocabulary:
  - **Symmetries** can be **exhibited** by physical **systems**
  - They are **associated with transformations** of such systems
  - If we **apply a transformation** and we find the system to be **indistinguishable from the original one** – we say we found a **symmetry** for this system – or just that the system has symmetry



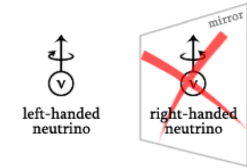
Transformations

- Now, what would happen if the symmetries were all perfect? **Nothing!** There would be no way to detect them – so, in other words the most interesting symmetries are the one that are broken…

- Since physical systems (in our line of duty) are described using Hilbert space objects (states, state space), we need to learn how to map one state into another
- This is done via a **unitary operator** $\mathcal{U}$:

$$|\psi\rangle \to |\psi'\rangle = \mathcal{U}|\psi\rangle, \mathcal{U}^\dagger\mathcal{U} = \mathcal{U}\mathcal{U}^\dagger = 1$$

- **The unitarity** condition here is **essential**, since in this way we can maintain the transformed state normalisation (conservation of probability)
- There is even a fundamental **theorem by Wigner**, stating that for each transformation as the above one, where: $|\langle\psi'|\phi'\rangle|^2 = |\langle\psi|\phi\rangle|^2$ the operators providing mapping **can always be chosen to be unitary** (or anti-unitary), or in other words, symmetry transformation is represented by an unitary operator



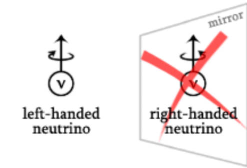
Transformations

- It is easy to check that indeed we can achieve the conservation of probab. Using unitary (anti-unitary) operators:

$$\langle\psi_1|\mathcal{U}^\dagger\mathcal{U}|\psi_2\rangle = \boldsymbol{\langle\psi_1|\psi_2\rangle}, \qquad \langle\psi_1|\mathcal{A}^\dagger\mathcal{A}|\psi_2\rangle = \boldsymbol{\langle\psi_1|\psi_2\rangle}^*$$

- … and remembering that only squared modulus quantities have meaning (they are measurable)
- One more interesting consequence: if we apply **different** unitary operators to the same state vector, the transformations differ from each other at most **by phase factors**!

- Two pictures: **active** and **passive** transformations
- Active: we consider two systems $|\psi\rangle$ and $|\psi'\rangle$ described by the same observer O (i.e., the same reference frame)
- Passive: we consider one system $|\psi\rangle$ that is described by two observers O and O' (in a transformed reference frame)



Transformations

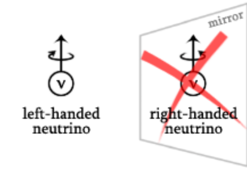
- Remember: in quantum physics we have two actors – **states** and **operators** (observables) – their relations is what matters (we can attach to these relations some physical meaning)
- Using active transformations we look for symmetries by **comparing properties** of the transformed system $|\psi'\rangle$ w.r.t. the original one $|\psi\rangle$
- Using passive way the symmetries can be found by checking if both observers using **the same equations** to describe the same system

- „Active math": we act with an **operator $\mathcal{U}$ on the state vectors**, all Hermitian operators $\mathcal{O}$ corresponding to observed quantities are not changed (we also have a special name for that: **Schrödinger picture:** SP)

$$\boxed{|\psi\rangle \to |\psi'\rangle = \mathcal{U}|\psi\rangle, \mathcal{O} \to \mathcal{O}}$$

- „Passive math": two observers are looking at the same system represented by two different state vectors: $|\psi\rangle$ and $\mathcal{U}|\psi\rangle$

Transformations



- An equivalent approach to SP one is offered by, so called, Heisenberg picture – we allow the **state vectors** (SV) to **remain unchanged varying operators** instead in the opposite way

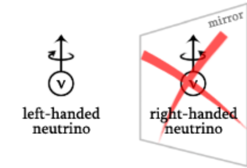
$$\boxed{|\psi\rangle \to |\psi\rangle, \mathcal{O} \to \mathcal{O}' = \mathcal{U}^{-1}\mathcal{O}\mathcal{U} = \mathcal{U}^{\dagger}\mathcal{O}\mathcal{U}}$$

- By the very definitione both pictures result in the same matrix elements:

$$\boxed{\langle \boldsymbol{\psi_1} | \boldsymbol{\mathcal{O}} | \boldsymbol{\psi_2} \rangle \to \langle \psi_1' | \mathcal{O} | \psi_2' \rangle_S = \langle \psi_1 | \mathcal{O}' | \psi_2 \rangle_H = \langle \boldsymbol{\psi_1} | \boldsymbol{\mathcal{U}^{\dagger}\mathcal{O}\mathcal{U}} | \boldsymbol{\psi_2} \rangle}$$

- And of course: if the unitary transformation is a symmetry of the system the new matrix element is equal to the original one!

- So, to get the same result we either transform the system in one way or the reference system oppositely. The SP is related to the state vectors transformations and the HP to the operators transformations
- In more formal way we could say, that SV transformation $|\psi\rangle \to \mathcal{U}|\psi\rangle$ could be considered to be equivalent to: $\mathcal{O} \to \mathcal{U}\mathcal{O}\mathcal{U}^{\dagger}$ (in the sense that if both transformations SP and HP are applied they cancel each other!)



Symmetries and conservation laws

- Building on all of that we can search for symmetries by examining **observables after transformation**: if a Hermitian operator commutes with the one describing the transformation, $[\boldsymbol{\mathcal{O}}, \boldsymbol{\mathcal{U}}] = \mathbf{0}$, the measured values do not change

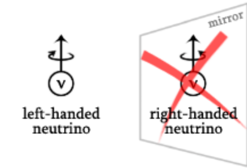
$$\langle \mathcal{O} \rangle = \langle \psi | \mathcal{O} | \psi \rangle \rightarrow \langle \psi | \mathcal{U}^{\dagger} \mathcal{O} \mathcal{U} | \psi \rangle = \langle \psi | \mathcal{U}^{\dagger} \mathcal{U} \mathcal{O} | \psi \rangle = \langle \mathcal{O} \rangle$$

- Can add Hamiltonian to the picture now: it drives the time evolution of a state, if $\mathcal{O}$ commutes with $\mathcal{H}$ the related observable is **constant in time**

$$i\hbar \frac{d}{dt} \langle \mathcal{O} \rangle = \langle [\mathcal{O}, \mathcal{H}] \rangle$$

- This allows both operators to be reduced to the diagonal form at the same time – they have a complete set of stationary states (energy e-states), which are also e-states of $\mathcal{O}$
- Note! The Hamiltonian must be invariant w.r.t. each (unitary) transformation $\mathcal{U}$ that represent a symmetry of a system $[\mathcal{U}, \mathcal{H}] = 0$
- In other words: **if a unitary operator $\boldsymbol{\mathcal{U}}$ commutes** with the Hamiltonian, the **energy and time evolution** are not affected at all by such transformation.

Let’s do it step by step then…



- Dynamic Schrödinger equations (natural units):

$$i\frac{d|\psi\rangle}{dt} = \mathcal{H}|\psi\rangle, i\frac{d\mathcal{O}}{dt} = \mathcal{H}\mathcal{O}$$

- We mentioned two pictures of quantum state evolution, the relation between them can be summarised as follow:

$$\mathcal{O}^H(t) = \mathcal{U}^\dagger(t)\mathcal{O}^S\mathcal{U}(t), |\psi\rangle \to |\psi'\rangle = |\psi(t)\rangle = \mathcal{U}(t,t_0)|\psi(t_0)\rangle$$

$$\langle\mathcal{O}\rangle = \langle\psi'|\mathcal{O}|\psi'\rangle = \langle\psi(t)|\mathcal{O}|\psi(t)\rangle = \langle(\mathcal{U}\psi(t_0))|\mathcal{O}|\mathcal{U}\psi(t_0)\rangle =$$

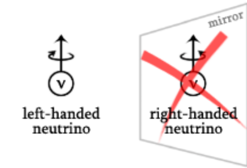
$$\langle\psi(t_0)|\mathcal{U}^\dagger\mathcal{O}\mathcal{U}|\psi(t_0)\rangle = \langle\psi|\mathcal{U}^\dagger\mathcal{O}\mathcal{U}|\psi\rangle$$

- If we **know our $\mathcal{H}$ e-basis**, we can make the notation even more explicit:

$$|\psi\rangle = \sum_k c_k|\epsilon_k\rangle \to |\psi'\rangle = \mathcal{U}|\psi\rangle = e^{-i\mathcal{H}t}|\psi\rangle = \sum_k c_k e^{-i\epsilon_k t}|\epsilon_k\rangle$$

- So, the evolved observable:

$$\langle\mathcal{O}\rangle = \sum_{k,l} c_k^* c_l\langle\epsilon_k|\mathcal{O}|\epsilon_l\rangle \to \sum_{k,l} c_k^* c_l\langle\epsilon_k|\mathcal{O}|\epsilon_l\rangle e^{-i(\epsilon_l-\epsilon_k)t}$$



Let's do it step by step then…

- Now, it is fairly easy to write down the equation using Heisenberg picture, which is useful for exposing an interesting fact:

$$\frac{d\mathcal{O}}{dt} = -i\left(\frac{\partial\mathcal{O}}{\partial t} + [\mathcal{O},\mathcal{H}]\right)$$

- If $\mathcal{O}$ does not depend explicitly on time, we have:

$$\frac{d\mathcal{O}}{dt} = -i[\mathcal{O},\mathcal{H}]$$

- So, if $\mathcal{O}$ and $\mathcal{H}$ commute, then $\mathcal{O}$ is constant. The quantity (observable) corresponding to the Hermitian operator $\mathcal{O}$ is **conserved**.

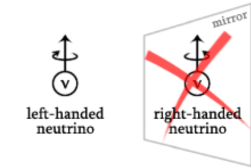
- Nice! Now, let's have a look at something more familiar – **translations**! We can start with infinitesimal one along x axis:

$$x \to x' = x + dx, |\psi\rangle \to |\psi(x')\rangle = |\psi(x+dx)\rangle$$

$$|\psi(x+dx)\rangle = |\psi(x)\rangle + dx\frac{\partial|\psi(x)\rangle}{\partial x} = \left(1 + dx\frac{\partial}{\partial x}\right)|\psi(x)\rangle = \delta\mathcal{D}_x|\psi(x)\rangle$$

- We say, that $\delta\mathcal{D}_x$ generates infinitesimal translations, using linear momentum representation we can write:

$$\delta D_x = 1 + \frac{i}{\hbar}dxp_x$$



Continuous transformations

- A finite translation Δx can be made as a series of small ones:

$$\mathcal{D}_x = \lim_{n\to\infty}\left(1+\frac{i}{\hbar}dxp_x\right)^n = exp\left(\frac{i}{\hbar}\Delta x p_x\right)$$

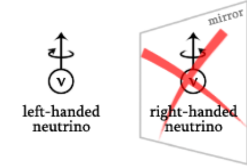
- Now, looking at this our previous discussion seems to be well justified…
  - We obtained a symmetry transformation that is represented by an unitary operator!

$$\mathcal{D}_x^\dagger \mathcal{D}_x = 1$$

  - We also say, that the linear momentum is the generator of the translation (operator). If the $\mathcal{H}$ is invariant w.r.t. space translations (along x axis), we have

$$[\mathcal{D}_x, \mathcal{H}] = 0 \rightarrow [p_x, \mathcal{H}] = 0$$

  - And that is a pretty heavy stuff – since p_x is a Hermitian operator, and $\frac{dp_x}{dt} = 0$ we see that the momentum is **conserved**!
- The following statements are true (and can always be extended to other quantities):
  - The Hamiltonian is **invariant** w.r.t. space translations
  - The linear momentum operator **commutes** with the Hamiltonian
  - The linear momentum is **conserved**



Gear up… Groups

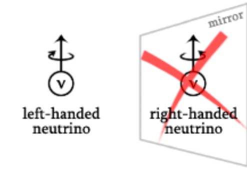
- That was nice! But we could do even better… Let's go more abstract

- A set of elements $\{g_1, g_2, \dots, g_n\}$ is called a group $\mathbb{G}$, if the following is true
  - **Closure** property $\mathbb{G} \ni (g_j, g_l) \to g_j \odot g_l = g_k \in \mathbb{G}$
  - A **unit element** exists in $\mathbb{G}$: $\mathbb{G} \ni e \to \forall\ g_j \in \mathbb{G}: e \odot g_j = g_j \odot e = g_j$
  - The **associative low** holds:$(g_j \odot g_l) \odot g_k = g_j \odot (g_l \odot g_k)$
  - **Inverse element**: $\forall\ g_j \in \mathbb{G} \wedge g_j^{-1} \in \mathbb{G}: g_j \odot g_j^{-1} = g_j^{-1} \odot g_j = e$

- **Symmetry transformations** in physics satisfy those axioms, for instance space translations $x \to x' = x + dx$, phase (or gauge) transformations $|\psi\rangle \to e^{i\alpha}|\psi\rangle$
- A special type of groups are called **Lie groups** – they are **continuous ones** and depend analytically on a **finite number of real parameters** (both above examples are Lie groups)

- Imagine we have a group of continuous symmetry transformations and we are able to construct its unitary representation. Now, in the neighbourhood of the identity:

$$\mathcal{U}(\alpha_1, \alpha_2, \dots, \alpha_n) = exp\left\{\sum_l i\alpha_l \mathcal{G}_l\right\}$$



Gear up… Groups

- Here (last slide actually…): $\alpha_l, l = 1, \dots, n$ are the group **real parameters**, $\mathcal{G}_l$ are operators, that have a special name: the **group generators**
- We know, that the operators representing symmetry transformations are **unitary**, which implies that **generators must be Hermitian**!! (remember $\delta\mathcal{D}_x$…?)

$$\mathcal{U} = 1 + i\varepsilon\mathcal{G}$$

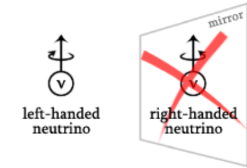
$$1 = \mathcal{U}\mathcal{U}^\dagger = (1 + i\varepsilon\mathcal{G}\ )\left(1 - i\varepsilon\mathcal{G}^\dagger\right) = 1 + i\varepsilon\left(\mathcal{G} - \mathcal{G}^\dagger\right) + O(\varepsilon^2)$$

- The generator is Hermitian and corresponds to an observable
- The symmetry implies the existence of an unitary operator that commutes with the Hamiltonian
- This leads to (we know that already for the momentum!) $[\mathcal{G}, \mathcal{H}] = 0$
- Thus, the **expectation value of the generator is constant**, or a **symmetry transformation** leads to a **conservation law** for the **corresponding generator**

$$\mathcal{H} \to \mathcal{H}' = \mathcal{U}^\dagger(\alpha)\mathcal{H}\mathcal{U}(\alpha) = \mathcal{H}$$

$$(1 - i\alpha\mathcal{G})\mathcal{H}(1 + i\alpha\mathcal{G}) \approx \mathcal{H} - i\alpha[\mathcal{G}, \mathcal{H}] = \mathcal{H}$$

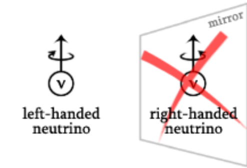
- So, again – the generator is the **constant of motion**



And now, the cool stuff (résumé)

- Two essential aspects of our discussion so far:
  - **Invariance** (symmetry) of the equations used to describe a system under some transformation (e.g., translation, rotation, …)
  - **Conservation** of the related physical quantities (translation - linear momentum)
- Important to understand/remember: the invariance properties are **abstract features** of the math we use to describe physics.
- **Invariance means conservation**: homogeneity of space means that the linear momentum is conserved. Formally, this is summarised by **Noether's theorem** (each conserved quantity corresponds to an invariant)
- The way we use that is: each interaction must obey various invariance requirements, that means each interaction obeys corresponding conservation laws – this **pose strong limits** on its possible mathematical description!

- Transformations, that lead to symmetries, can be **continuous or discrete**. This is important distinction that leads to **additive** and **multiplicative** conservation laws respectively
- It is also pedagogical to look at the classical invariance principles, since they are expressed using Lagrange equations, that play vital role in quantum theories



Invariance in classical physics

- The state of a system with n degrees of freedom can be described by a **Lagrangian** that contains n generalised coordinates q_k for which n conjugated momenta p_k can be derived

$$L(q_1, \dots, q_n) = E_{kin} - E_{pot}, q_k\colon k = \{1, \dots, n\}, p_k = \frac{\partial L}{\partial \dot{q}_k}$$

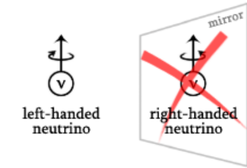
$$L(q_1, \dots, q_n) = \frac{dp_k}{dt} - \frac{\partial L}{\partial q_k} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k}$$

- Similarly to the Heisenberg equation, using the Lagrangian **some things are exposed** in a very natural way: if the Lagrangian does not depend (or is symmetric) on the q_j we see at once that $\frac{\partial L}{\partial q_k} = 0$, thus the conjugated momentum $\frac{dp_k}{dt} = 0$
- For free Lagrangian, we get for translational symmetry:

$$L = E_{kin} = \frac{1}{2} m\dot{x}^2, p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

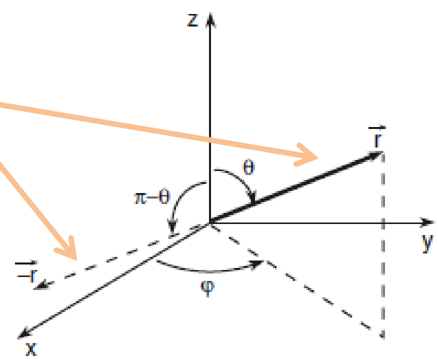
- **A conserved quantity is associated to a continuous symmetry** (the reverse is also true) or in other words: any **symmetry constraints the Lagrangian** (its form)
- Adding relativistic theory, we get also **Poincare invariance principle**: invariance under Lorentz transformations (boosts) and space-time translations requires that the Lagrangian function **transforms as a scalar** (we are going to revisit this many times)

Parity



- This is the first symmetry transformation that belongs to this strange gang of discrete operations – cannot be obtained as a sequence of infinitesimal transformations, as a consequence the discrete transformations **do not have generators**
- So, this one is special!
- We say, that **the parity** (space inversion) **converts a right handed** coordinate system into **left handed** one

Note! Here we actually have an active transformation, we changed the state!



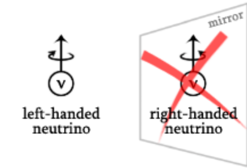
- Let see, what happens if we act with the parity operator on states in Hilbert space

$$|\psi\rangle \rightarrow |\psi'\rangle = \mathcal{P}|\psi\rangle, \mathcal{P}^{\dagger}\mathcal{P} = \mathbf{1}$$

- For position operator we require:

$$\langle\psi'|\mathcal{X}|\psi'\rangle = \langle(\mathcal{P}\psi)|\mathcal{X}\mathcal{P}|\psi\rangle = \langle\psi|\mathcal{P}^{\dagger}\mathcal{X}\mathcal{P}|\psi\rangle = -\langle\psi|\mathcal{X}|\psi\rangle$$

Parity



- So, we must have:

$$\mathcal{P}^\dagger \mathcal{X} \mathcal{P} = -\mathcal{X} \rightarrow \mathcal{P}\mathcal{X} = -\mathcal{X}\mathcal{P}$$

- Parity and position operators anti-commute. Let $|x\rangle$ be position operator e-state:

$$\mathcal{X}|x\rangle = x|x\rangle \rightarrow \mathcal{X}\mathcal{P}|x\rangle = -\mathcal{P}\mathcal{X}|x\rangle = (-x)\mathcal{P}|x\rangle$$

$$\mathcal{P}|x\rangle = e^{i\varphi}|-x\rangle \rightarrow \mathcal{P}|x\rangle = |-x\rangle$$

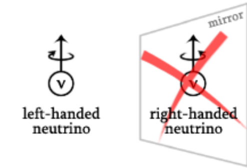
$$\mathcal{P}\mathcal{P}|x\rangle = \mathcal{P}^2|x\rangle = \mathcal{P}|-x\rangle = |x\rangle \rightarrow \mathcal{P}^2 = 1$$

- The parity operator has e-values: ± 1, and $\mathcal{P}^{-1} = \mathcal{P}^\dagger = \mathcal{P}$
- What about other quantities…, for instance momentum. In a second we understand that this one is tricky. We basically need to have a particle/state that has momentum in the first place…
- First, let's do a smart trick and add translation generator in the picture: the following operators should be equivalent:
  - Translation followed by space inversion – $\mathcal{P}\delta\mathcal{D}_x$
  - Space inversion followed by translation in the opposite direction – $\delta\mathcal{D}_{-x}\mathcal{P}$

$$\mathcal{P}\delta\mathcal{D}_x = \delta\mathcal{D}_{-x}\mathcal{P} \rightarrow \delta\mathcal{D}_x = 1 - \frac{i}{\hbar} dx \wp_x$$

$$\{\mathcal{P}, \wp_x\} = 0 \rightarrow \mathcal{P}^\dagger \wp \mathcal{P} = -\wp$$

Parity



- Let's have a look at the momentum for a bit longer. For the time being we assume that parity is an exact symmetry. Let's start with a single particle

$$\mathcal{P}|\psi(x,t)\rangle = p_\psi e^{i\varphi}|\psi(-x,t)\rangle$$

$$\mathcal{P}^2|\psi(x,t)\rangle = |\psi(x,t)\rangle \rightarrow P_\psi = \pm 1$$

- That we already knew, however what happens if we explicitly use momentum e-functions

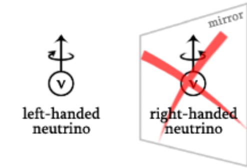
$$|\psi_p(x,t)\rangle = e^{i(px-Et)}$$

$$\mathcal{P}|\psi_p(x,t)\rangle = P_\psi|\psi_p(-x,t)\rangle = P_\psi|\psi_{-p}(x,t)\rangle$$

- We see, that a particle (state) can be an e-state of the parity operator with the e-value P_ψ only if the particle is at rest! For this reason we call P_ψ the intrinsic parity of a particle.
- If we deal with a particle systems we make the following generalisation:

$$\mathcal{P}|\psi(x_1,x_2,\dots,x_n,t)\rangle = P_1 P_2 \cdots P_n|\psi(-x_1,-x_2,\dots,-x_n,t)\rangle$$

- So, here we see that we have **multiplicative conservation** rule!



Parity and angular momentum

- Write it down first and use what we already know about the parity

$$\vec{L} = \vec{x} \times \vec{p}$$

$$\mathcal{P}^{\dagger}\vec{L}\mathcal{P} = \mathcal{P}^{\dagger}\vec{x} \times \vec{p}\mathcal{P} = \mathcal{P}^{\dagger}\vec{x}\mathcal{P} \times \mathcal{P}^{\dagger}\vec{p}\mathcal{P} = (-\vec{x})(-\vec{p}) = \vec{L}$$

- So, the parity and the angular momentum do commute!

$$[\mathcal{P}, \vec{L}] = 0$$

- In 3-dim space (isomorphic with $\mathbb{R}^3$) the parity operator can be represented as a matrix

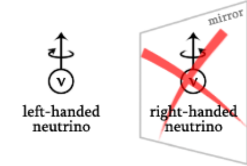
$$\mathcal{P}^{(\mathbb{R}^3)} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rightarrow \mathcal{P}^{(\mathbb{R}^3)}\mathcal{R} = \mathcal{R}\mathcal{P}^{(\mathbb{R}^3)}, \mathcal{R} \in \mathbb{O}(3)$$

- What if we operate in the Hilbert space? Does the relation above still hold?

$$\mathcal{P}\delta\mathcal{D}_{\varepsilon} = \delta\mathcal{D}_{\varepsilon}\mathcal{P}, \delta\mathcal{D}_{\varepsilon} = 1 - \frac{i}{\hbar}\varepsilon\varepsilon^{0} \cdot \vec{J}$$

$$[\mathcal{P}, \vec{J}] = 0 \rightarrow \mathcal{P}^{\dagger}\vec{J}\mathcal{P} = \vec{J}$$

- We say, under rotations x and $\vec{J}$ behave like vectors (tensors of rank 1), but with different e-values (**odd – vector** and **even – pseudo-vector**)



Parity and scalar products

- There is something very interesting, when we look at different scalar products, let's consider momentum-position and spin-position

$$\mathcal{P}^\dagger \vec{x} \cdot \vec{p} \mathcal{P} = \mathcal{P}^\dagger \vec{x} \mathcal{P} \cdot \mathcal{P}^\dagger \vec{p} \mathcal{P} = (-\vec{x}) \cdot (-\vec{p}) = \vec{x} \cdot \vec{p}$$

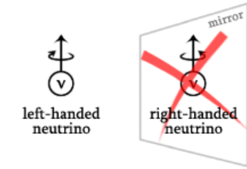
$$\mathcal{P}^\dagger \vec{x} \cdot \vec{S} \mathcal{P} = \mathcal{P}^\dagger \vec{x} \mathcal{P} \cdot \mathcal{P}^\dagger \vec{S} \mathcal{P} = (-\vec{x}) \cdot (\vec{S}) = -\vec{x} \cdot \vec{S}$$

- We say, that quantities behaving under rotations as scalars (tensors of rank 0) under space inversion operation can be even (scalars) or odd (pseudo-scalars)

- Parity turned out to be very important and is used as quantum number for particle classification (PDG – Particle Data Group notation J^P)

- Parity can always be assigned to bosons without ambiguity.
- In case of fermions, the angular momentum plays a role via their spin – they must always be produced in pairs. For that reason we assume a convention where the proton's parity is set to be $P_p = +1$ and the other fermions have the parity assigned relatively to the proton
- QFT requires that fermions and anti-fermions have the opposite parity, boson and anti-bosons have the same parity. We also assume that all quarks have positive parity

Parity 2-particle states



- Let's consider a system with 2 particles of known intrinsic parities P_1, P_2. The system can only be a parity e-state in the centre of mass system. We can describe it using two bases:

$$|p, \theta, \varphi\rangle = |\vec{p}, -\vec{p}\rangle$$

$$|p, l, m\rangle$$

$$|p, l, m\rangle = \sum_{\theta,\varphi} |p, \theta, \varphi\rangle\langle p, \theta, \varphi | p, l, m\rangle = \sum_{\theta,\varphi} Y_l^m(\theta, \varphi)|p, \theta, \varphi\rangle$$

- Ok, now inversion in polar coordinates: $(r, \theta, \varphi) \rightarrow (r, \pi - \theta, \pi + \varphi)$

$$Y_l^m(\theta, \varphi) \rightarrow Y_l^m(\pi - \theta, \pi + \varphi) = (-1)^l Y_l^m(\theta, \varphi)$$

$$\mathcal{P}|p, l, m\rangle = P_1 P_2 \sum_{\theta,\varphi} Y_l^m(\pi - \theta, \pi + \varphi)|p, \theta, \varphi\rangle =$$

$$= P_1 P_2 (-1)^l \sum_{\theta,\varphi} Y_l^m(\theta, \varphi)|p, \theta, \varphi\rangle = P_1 P_2 (-1)^l |p, l, m\rangle$$

$$\boldsymbol{P_{2-part} = P_1 P_2 (-1)^l}$$