Draft of the lecture

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1 Topology Design, Dimensioning with Modular Capacity, Linearization of Non-linear Goal Functions

1.1 Network design problems with non-continuous variables

1. Topological Design with a Fixed Charge Model (TDFCM), L-P formulation:

   - Indices:
     - ⋆ $e = 1, 2, \ldots, E$ candidate links;
     - ⋆ $d = 1, 2, \ldots, D$ demands;
     - ⋆ $p = 1, 2, \ldots, P_d$ candidate paths of flows realizing demand d.
   - Variables:
     - ⋆ x_{dp} flow realizing demand d on path p,
     - ⋆ y_e capacity to be installed on link e,
     - ⋆ u_e $= 1$ if link e is to be installed;
       $= 0$, otherwise (**binary** decision variable).
   - Constants:
     - ⋆ δ_{edp} $= 1$ if link e belongs to path p realizing demand d;
       $= 0$ otherwise;
     - ⋆ h_d volume of demand d;
     - ⋆ ξ_e fixed unit cost of capacity usage on link e;
     - ⋆ κ_e cost of 'opening' (i.e., installation) of link e;
     - ⋆ W 'big W' (sufficiently large value).
   - Goal function: min $\mathbf{F}(\mathbf{y}, \mathbf{u}) = \sum_e (\xi_e y_e + \kappa_e u_e)$.
   - Constraints:
     - ⋆ $\sum_p x_{dp} = h_d$ $\quad d = 1, 2, \ldots, D$;
     - ⋆ $\sum_d \sum_p \delta_{edp} x_{dp} = y_e$ $\quad e = 1, 2, \ldots, E$;
     - ⋆ $y_e \leq W u_e$ $\quad e = 1, 2, \ldots, E$;
     - ⋆ $\mathbf{y}$ and $\mathbf{x}$ — non-negative continuous and $\mathbf{u}$ — binary.

2. Topology design with candidate locations for nodes:

- add binary decision variable u_v indicating installation of node v;
- add u_v to the goal function (with the respective installation cost);
- add costants describing incidence of links and nodes (the same as a_{ev} and b_{ev} in the N-L formulation);
- add constraints: $\sum_e (a_{ev} + b_{ev}) u_e \leq W u_v \quad v = 1, 2, \ldots, V$.

3. Network design with modular capacity (M: module size of the link capacity, e.g., $M = 10\,\text{Gb/s}$):

   - y_e — integer variable, MI(L)P problem (x_{dp} are continuous):
     - ⋆ LINK constraints: $\sum_d \sum_p \delta_{edp} x_{dp} \leq M y_e \; e = 1, 2, \ldots, E$;
     - ⋆ PATH constraints: no changes.
   - u_{dp} — binary variables (flows are non-bifurcated, single path routing for a demand), y_e — integer variable, I(L)P problem:
     - ⋆ LINK constraints: $\sum_d \sum_p \delta_{edp} h_d u_{dp} \leq M y_e \; e = 1, 2, \ldots, E$;
     - ⋆ PATH constraints: $\sum_p u_{dp} = 1 \; d = 1, 2, \ldots, D$.

4. Modular dimensioning:

   - Multiple modules: M_k, $k = 1, 2, \ldots, K$:
     - ⋆ finding of link capacity the same as summarizing boxes;
     - ⋆ ξ_{ek} cost of capacity module k on link e (constant);
     - ⋆ M_k size of module k (constant);
     - ⋆ y_{ek} number of modules k to be installed on link e (variable);
     - ⋆ goal function: min $\mathbf{F}(\mathbf{y}) = \sum_e \sum_k \xi_{ek} y_{ek}$;
     - ⋆ $\sum_d \sum_p \delta_{edp} x_{dp} \leq \sum_k M_k y_{ek} \quad e = 1, 2, \ldots, E$;
     - ⋆ y_{ek} — integer variable.
   - Incremental modular function: m_k, $k = 1, 2, \ldots, K$:
     - ⋆ finding of link capacity due to the well defined modularity order;
     - ⋆ u_{ek} determines if capacity module k is used on link e (variable);
     - ⋆ goal function: min $\mathbf{F}(\mathbf{u}) = \sum_e \sum_k \xi_{ek} u_{ek}$,
     - ⋆ $\sum_d \sum_p \delta_{edp} x_{dp} \leq \sum_k m_k u_{ek} \quad e = 1, 2, \ldots, E$,
     - ⋆ $u_{e1} \geq u_{e2} \geq \cdots \geq u_{eK} \quad e = 1, 2, \ldots, E$,
     - ⋆ u_{ek} — binary variable.
   - The values of ξ_{ek} or M_k/m_k may be different in both approaches.

1.2 Design with a non-linear goal function

1. For instance, function $\mathbf{F}(\mathbf{y}) = \sum_e \xi_e f(y_e)$:

   - $f(z)$: convex function (delay, penalty), convex set, epigraph, strictly convex function;
   - $f(z)$: concave function (link dimensioning with the 'economies of scales' due to diminishing marginal return, utility function), concave set.

2. Convex programming (CXP): global minimum identical with local minimum, no duality gap.

3. Concave programming (CVP): much more complex to obtain the global minimum than for CXP.

4. Network design with penalty-like goal function, bifurcated solution, linearization.

5. Network design with utility-like goal function, linearization with usage of binary variables.

6. Yaged's method: a heuristic for solving of a concave problem.

1.3 Exercises

1. A fragment of an optimization task is given below. In the given form it is not a proper instance of MILP. **(a)** Why? **(b)** Correct it, so that it is a proper MILP task. **(c)** Interpret the meaning of variables u_{dp} and the constraints.

   - Indices:
     - ⋆ $e = 1, 2, \ldots, E$ links;
     - ⋆ $d = 1, 2, \ldots, D$ demands;
     - ⋆ $p, q = 1, 2, \ldots, P_d$ candidate paths for flows that can realize demand d.
   - Constants:
     - ⋆ $\delta_{edp} = 1$ if link e belongs to path p that can realize demand d; otherwise 0
   - Variables:
     - ⋆ w_e weight of link e, non-negative continuous value;
     - ⋆ u_{dp} binary value;
   - Constraints:
     - ⋆ $\forall_{d=1,2,\ldots,D}$ $\sum_p u_{dp} = 1;$
     - ⋆ $\forall_{d=1,2,\ldots,D} \forall_{q=1,2,\ldots,P_d}$ $\sum_{p=1,2,\ldots,P_d} \sum_e \delta_{edp} w_e u_{dp} \leq \sum_e \delta_{edq} w_e.$

2. Formulate a set of equalities/inequalities (using some additional variables, e.g., integral, or some constants, if necessary) that can be used to describe the relationship between value of variable x (giving the link load, $0 \leq x \leq 40$) and value $y = f(x)$, i.e., a cost of the capacity usage on this link (having those equalities/inequalities, and using a selected value of x, we should be able to calculate the value of y). The constraints should be formulated according to the rules related to formulation of Mixed Integer Linear Programming (MILP) problems, in which a goal function (a cost of the link usage) is minimized. The relationship between y and x is given in Figure 1.

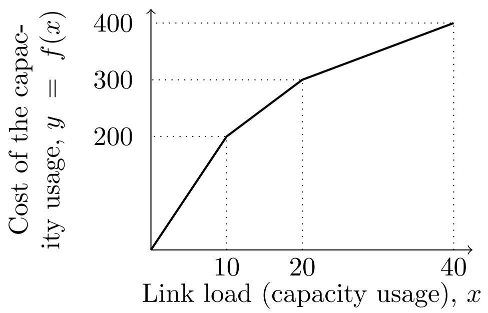

Figure 1: An example of a concave tariff, linear within segments.

3. An optimization problem has been defined below (with the A/N-L formulation). It can be used to find flows in a network and additionally to configure the network. **(a)** What kind of configuration is found by the optimal solution of this problem? **(b)** Provide the missing interpretations of constants and variables. **(c)** Explain the meaning of the goal function and all the constraints.

   - Indices:
     - ⋆ $e = 1, 2, \ldots, E$ arcs in the network;
     - ⋆ $s, t, v = 1, 2, \ldots, V$ nodes in the network.
   - Constants:
     - ⋆ h_{vt} (*explain the meaning of these constants*);
     - ⋆ $i(e)$ starting node of arc e;
     - ⋆ $j(e)$ teminating node of arc e;
     - ⋆ c_e capacity installed on link e;
     - ⋆ M a very large number.
   - Non-negative continuous variables:
     - ⋆ Z (*explain the meaning of this variable*);
     - ⋆ w_e weight (metric) of arc e;
     - ⋆ r_{vt} length (weight) of the shortest path from node v to node t ($r_{vv} \equiv 0$);
     - ⋆ x_{et} flow to node t on link e.
     - ⋆ y_{vt} (*explain the meaning of these variables*).
   - Binary variables:
     - ⋆ u_{et} variable equal to 1 if and only if arc e belongs to the shortest path leading to node t.
   - Goal function (*explain its meaning*): min Z.
   - Constraints (*explain the meaning of each constraint*):
     - ⋆ $\sum_{\{e:j(e)=t\}} x_{et} = \sum_{s \neq t} h_{st}$ $\quad t = 1, 2, \ldots, V$;

- ⋆ $\sum_{\{e:i(e)=v\}} x_{et} - \sum_{\{e:j(e)=v\}} x_{et} = h_{vt}$ $t, v = 1, 2, \ldots, V \quad t \neq v$;
- ⋆ $\sum_t x_{et} \leq Z c_e$ $e = 1, 2, \ldots, E$;
- ⋆ $0 \leq r_{j(e)t} + w_e - r_{i(e)t} \leq (1 - u_{et})\, M$ $e = 1, 2, \ldots, E \quad t = 1, 2, \ldots, V$;
- ⋆ $1 - u_{et} \leq r_{j(e)t} + w_e - r_{i(e)t}$ $e = 1, 2, \ldots, E \quad t = 1, 2, \ldots, V$;
- ⋆ $x_{et} \leq u_{et} \sum_{v \neq t} h_{vt}$ $e = 1, 2, \ldots, E \quad t = 1, 2, \ldots, V$;
- ⋆ $0 \leq y_{i(e)t} - x_{et} \leq (1 - u_{et}) \sum_{v \neq t} h_{vt}$ $e = 1, 2, \ldots, E, t = 1, 2, \ldots, V$;
- ⋆ $w_e \geq 1$ $e = 1, 2, \ldots, E$;
- ⋆ $0 \leq Z \leq 1$.

4. Formulate as an (M)ILP problem the task described as follows: an operator has to configure traffic flows in its network with respect to the summarized flow capacity maximization. Network is given as a graph $G(V, E)$, where V represents the nodes, and E represents the arcs (or links). The volume of the demand between any pair of nodes (n, m) (where $n, m \in V$, $n \neq m$) is given as $|n - m| \times 10$ Gb/s (for instance, traffic between nodes 2 and 7 would be $|7 - 2| \times 10 = 50$ Gb/s). Each link contains one optical fiber, where each fiber carries up to 40 wavelengths (optical channels). Each optical channel has capacity equal to 10 Gb/s. There is no wavelength conversion in the network, therefore traffic between a given pair of nodes uses the same wavelengths (optical frequencies), e.g., if we decide that the traffic from node a to node b is transmitted by node a on wavelengths λ_7 and λ_{15}, then this traffic uses wavelengths λ_7 and λ_{15} along the entire path to node b. The operator has a given budget for the data transport. The budget is equal to 100 000 USD. Distance between any pair of nodes (n, m) (where $n, m \in V$, $n \neq m$, and $(n, m) \in E$) is equal to $|n - m| \times 100$ km. The cost of transmitting 10 Gb/s of traffic over the distance of 100 km is equal to 1000 USD.

1.4 Reading

1.4.1 Contents of the lecture

Problems described in this lecture are generally dealt with in the following position:

- Michał Pióro and Deepankar Medhi. *Routing, Flow and Capacity Design in Communication and Computer Networks.* Morgan Kaufmann Publishers—Elsevier, San Francisco, CA, 2004: chapter 4.3, 5.5-5.6, 6.1-6.3.

1.4.2 Auxiliary references

- Michał Pióro and Deepankar Medhi. *Routing, Flow and Capacity Design in Communication and Computer Networks.* Morgan Kaufmann Publishers—Elsevier, San Francisco, CA, 2004: basic network design problems applying non-continuous variables.
- Poompat Saengudomlert. *Optimization for Communications and Networks.* CRC Press/Science Publishers, Boca Raton, FL, 2012: methods for solving various types of network design problems.

- Laurence A. Wolsey. *Integer Programming.* John Wiley & Sons, Inc., New York, NY, 1998: textbook on discrete (integer) programming.