Exercise 0.1. Equation of state for a fluid composed of particles with non-zero volume (so not ideal gas) is called van der Waals equation, and has the following form:

$$p = \frac{RT}{V-b} - \frac{a}{V^2}$$

Where:

a, b – constants characteristic for given gas

T – temperature

V – gas volume

p - pressure

Find how the pressure of the gas changes if:

a) Temperature changes and volume stays constant
b) Volume changes and temperature stays constant

Solution

a)

$$\frac{\partial p}{\partial T} = \frac{T}{V-b}$$

b)

$$\frac{\partial p}{\partial V} = \frac{-RT}{(V-b)^2} + \frac{a}{V^3}$$

Exercise 0.2. As a result of an experiment, following relationships were found:

$$\frac{\partial p}{\partial V} = -n \cdot R \cdot T \cdot f(V)$$

$$\frac{\partial p}{\partial T} = \frac{n \cdot R}{V} - 2 \cdot n \cdot R \cdot T \cdot a$$

Where:

n – number of moles

a – known constant

f(V) – unknown function of volume

Find an equation of state for given gas (equation for pressure). **Hint**: Use Schwarz theorem to obtain f(V), then integrate one of the given equations and use the second one to calculate integration constant.

Solution

$$\frac{\partial^2 p}{\partial V \partial T} = -n \cdot R \cdot f(V)$$

$$\frac{\partial^2 p}{\partial T \partial V} = -\frac{n \cdot R}{V^2}$$

$$\frac{\partial^2 p}{\partial T \partial V} = \frac{\partial^2 p}{\partial V \partial T} \quad \Rightarrow \quad -n \cdot R \cdot f(V) = -\frac{n \cdot R}{V^2} \quad \Rightarrow \quad f(V) = \frac{1}{V^2}$$

$$\frac{\partial p}{\partial V} = \frac{-n \cdot R \cdot T}{V^2} \quad \Rightarrow \quad p = \int \frac{-n \cdot R \cdot T}{V^2} dV = \frac{n \cdot R \cdot T}{V} + f(T)$$

$$\frac{\partial p}{\partial T} = \frac{n \cdot R}{V} + \frac{d}{dT} f(T) = \frac{n \cdot R}{V} - 2 \cdot n \cdot R \cdot T \cdot a \quad \Rightarrow \quad \frac{d}{dT} f(T) = -2 \cdot n \cdot R \cdot T \cdot a$$

$$f(T) = \int -2 \cdot n \cdot R \cdot T \cdot a \, dT = -n \cdot R \cdot T^2 + Const$$

Finally:

$$p = \frac{n \cdot R \cdot T}{V} - n \cdot R \cdot T^2 + Const$$

Exercise 0.3. The concentration at any point in space is given by:

$$c(x, y, z) = A(xy + yz + zx)$$

Where A=constant

a) Find the direction in which concentration changes most rapidly with distance from the point (1,1,1). Determine the maximum rate of change at that point.

b) Calculate cosines of direction

Solution

a)

$$\nabla c = A\left[(y + z)\vec{\imath} + (x + z)\vec{\jmath} + (y + x)\vec{k}\right]$$

$$\nabla c(1,1,1) = 2A(\vec{\imath} + \vec{\jmath} + \vec{k})$$

Rate:

$$rate = |\nabla c(1,1,1)| = 2A\sqrt{1^2 + 1^2 + 1^2} = 2A\sqrt{3}$$

b)

Direction cosines:

$$\cos\alpha_i = \frac{\vec{v}\cdot\overrightarrow{e_x}}{|\vec{v}|}$$

$$cos\alpha = cos\beta = cos\gamma = \frac{2A}{2A\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}}$$

Exercise 0.4. Calculate gradient of the function:

$$f(x,y,x) = x^2 - yz + xz^2$$

In the points (1,1,1) and (3,2,1).

Solution

$$\nabla f = (2x + z^2, -z, 2z - y)$$

$$\nabla f(1,1,1) = (3,-1,1)$$

$$\nabla f(3,2,1) = (7,-2,0)$$

Exercise 0.5. Electrical potential is given by the function:

$$\phi(x,y,z) = \frac{\mu x}{4\pi\varepsilon_0(x^2+y^2+z^2)}$$

Where:

ε_0 – electric constant

μ – electric dipole moment

Calculate electric field.

Solution

$$\vec{E} = -\nabla\phi$$

$$\frac{\partial\phi}{\partial x} = \frac{\mu}{4\pi\varepsilon_0}\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2+z^2}\right) = \frac{\mu}{4\pi\varepsilon_0}\frac{x^2+y^2+z^2-2x^2}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial\phi}{\partial y} = \frac{\mu}{4\pi\varepsilon_0}\frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2+z^2}\right) = \frac{\mu}{4\pi\varepsilon_0}\frac{-2yx}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial\phi}{\partial z} = \frac{\mu}{4\pi\varepsilon_0}\frac{\partial}{\partial z}\left(\frac{x}{x^2+y^2+z^2}\right) = \frac{\mu}{4\pi\varepsilon_0}\frac{-2zx}{(x^2+y^2+z^2)^2}$$

$$\vec{E} = \frac{\mu}{4\pi\varepsilon_0}\left(\frac{-y^2 - z^2 + x^2}{(x^2 + y^2 + z^2)^2}, \frac{2yx}{(x^2 + y^2 + z^2)^2}, \frac{2zx}{(x^2 + y^2 + z^2)^2}\right)$$

Exercise 0.6. Calculate divergence of the electric field from the previous exercise

Solution

$$\frac{\partial E_x}{\partial x} = \frac{\mu}{4\pi\varepsilon_0}\frac{\partial}{\partial x}\left(\frac{-y^2 - z^2 + x^2}{(x^2 + y^2 + z^2)^2}\right) = \frac{\mu}{4\pi\varepsilon_0}\frac{2x(x^2 + y^2 + z^2) - (-y^2 - z^2 + x^2)4x}{(x^2 + y^2 + z^2)^3}$$

$$= \frac{\mu}{4\pi\varepsilon_0}\frac{-2x^3 + 6xy^2 + 6xz^2}{(x^2 + y^2 + z^2)^3}$$

$$\frac{\partial E_y}{\partial y} = \frac{\mu}{4\pi\varepsilon_0}\frac{\partial}{\partial y}\left(\frac{2yx}{(x^2 + y^2 + z^2)^2}\right) = \frac{\mu}{4\pi\varepsilon_0}\frac{2x(x^2 + y^2 + z^2) - 8y^2x}{(x^2 + y^2 + z^2)^3} = \frac{\mu}{4\pi\varepsilon_0}\frac{2x^3 + 2xz^2 - 6y^2x}{(x^2 + y^2 + z^2)^3}$$

$$\frac{\partial E_z}{\partial z} = \frac{\mu}{4\pi\varepsilon_0}\frac{\partial}{\partial z}\left(\frac{2zx}{(x^2 + y^2 + z^2)^2}\right) = \frac{\mu}{4\pi\varepsilon_0}\frac{2x(x^2 + y^2 + z^2) - 8z^2x}{(x^2 + y^2 + z^2)^3} = \frac{\mu}{4\pi\varepsilon_0}\frac{2x^3 + 2xy^2 - 6z^2x}{(x^2 + y^2 + z^2)^3}$$

$$div\vec{E} = \frac{\mu}{4\pi\varepsilon_0}\frac{2x^3 + 2xz^2 - 6y^2x + 2x^3 + 2xy^2 - 6z^2x - 2x^3 + 6xy^2 + 6xz^2}{(x^2 + y^2 + z^2)^3}$$

$$= \frac{\mu}{4\pi\varepsilon_0}\frac{2xz^2 + 2x^3 + 2xy^2}{(x^2 + y^2 + z^2)^3} = \frac{2x}{(x^2 + y^2 + z^2)^2}$$

Exercise 0.7. Calculate curl of a following vector field:

$$\vec{F}(x, y, z) = x^2yz\vec{\imath} + xz^2\vec{\jmath} + x^2y^3z\vec{k}$$

Solution

$$rot\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & xyz & x^2y^3z \end{vmatrix}$$

$$= \left(\frac{\partial x^2y^3z}{\partial y} - \frac{\partial xyz}{\partial z}\right)\vec{\imath} + \left(\frac{\partial x^2yz}{\partial z} - \frac{\partial x^2y^3z}{\partial x}\right)\vec{\jmath} + \left(\frac{\partial xyz}{\partial x} - \frac{\partial x^2yz}{\partial y}\right)\vec{k}$$

$$= (3y^2x^2z - 2xy)\vec{\imath} + (x^2y - 2xy^3z)\vec{\jmath} + (yz - x^2z)\vec{k}$$