

## 1.1. Fields

A field  $f(\vec{r})$  associates a physical quantity with a position  $\vec{r} = (x, y, z)$ . A field can be also time dependent, for example  $f(\vec{r}, t)$ . The simplest case is a scalar field, where given physical quantity can be described by one value at each point. Good example of such field is temperature  $T(\vec{r}, t)$ . A different story is a vector field, which requires specification of a magnitude and direction of a physical quantity in each point. Example of such field is a velocity field of the wind  $\vec{v}(\vec{r}, t)$  - in each point of the atmosphere, wind can have different direction and speed. Every sufficiently smooth scalar field has an associated natural vector field – gradient field, about which we will talk later.

## 1.2. Partial derivatives

Partial derivatives are very much alike the ordinary ones. Let's consider a partial derivative of function  $f=f(a_1, a_2, \dots, a_n)$  with respect to the  $i$ -th variable  $a_i$ , it can be defined as follows:

$$f_{a_i} = \frac{\partial f}{\partial a_i} = \lim_{\Delta a_i \rightarrow 0} \frac{f(a_1, a_2, \dots, a_i + \Delta a_i, \dots, a_n) - f(a_1, a_2, \dots, a_i, \dots, a_n)}{\Delta a_i} \quad (1.1)$$

### Example 1.1.

Calculate partial derivatives with respect to the variables  $x$  and  $y$  of a function :

$$f(x, y) = 2x^3 + 6xy + y^2$$

using definition (0.1).

### Solution

$$\begin{aligned} f_x &= \frac{\partial f}{\partial x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{2(x + \Delta x)^3 + 6(x + \Delta x)y + y^2 - (2x^3 + 6xy + y^2)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{2x^3 + 6x^2\Delta x + 6x\Delta x^2 + 2\Delta x^3 + 6xy + 6y\Delta x + y^2 - 2x^3 - 6xy - y^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{6x^2\Delta x + 6x\Delta x^2 + 2\Delta x^3 + 6y\Delta x}{\Delta x} = \lim_{\Delta x \rightarrow 0} 6x^2 + 6x\Delta x + 2\Delta x^2 + 6y = \mathbf{6x^2 + 6y} \\ f_y &= \frac{\partial f}{\partial y} = \lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{2x^3 + 6x(y + \Delta y) + (y + \Delta y)^2 - (2x^3 + 6xy + y^2)}{\Delta y} \\ &= \lim_{\Delta y \rightarrow 0} \frac{2x^3 + 6xy + 6x\Delta y + y^2 + 2y\Delta y + \Delta y^2 - 2x^3 - 6xy - y^2}{\Delta y} = \lim_{\Delta y \rightarrow 0} \frac{6x\Delta y + 2y\Delta y + \Delta y^2}{\Delta y} \\ &= \lim_{\Delta y \rightarrow 0} 6x + 2y + \Delta y = \mathbf{6x + 2y} \end{aligned}$$

Although the results are correct, using every time definition (1.1) do not seem very practical. This is why for practical calculations we will rather use a conclusion which can be made out of (1.1):

**Partial derivative can be calculated the same way as ordinary one, with assumption that all the other variables except the one we are calculating with respect to, are treated as constants.**

This rule is also true, for derivatives of higher order.

**Example 1.2.**

Calculate first and second partial derivative with respect to the variables  $x$  and  $y$  of a function :

$$f = x^2y^3 + e^{-x} \sin y$$

**Solution**

$$\frac{\partial f}{\partial x} = 2xy^3 - e^{-x} \sin y$$

$$\frac{\partial^2 f}{\partial x^2} = 2y^3 + e^{-x} \sin y$$

$$\frac{\partial f}{\partial y} = 3x^2y^2 + e^{-x} \cos y$$

$$\frac{\partial^2 f}{\partial y^2} = 6x^2y - e^{-x} \sin y$$

There are numerous theorems connected to partial derivatives. One of the most useful and important is **Schwarz' theorem**:

If  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  has continuous second partial derivatives at any given point in  $\mathbb{R}^n$ , say  $(a_1, \dots, a_n)$  then  $\wedge i, j \in \mathbb{N} \setminus \{0\}: i, j \leq n$ :

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} \quad (1.2)$$

It means, that partial derivatives of this function are commutative at that point.

**1.3. Gradient**

Gradient is an analog of ordinary derivative for functions of several variables. It represents the slope (=tangent) of the graph of the function. If  $f(x_1, x_2, \dots, x_n)$  is a differentiable function of several variables (so called **scalar field**), its **gradient is the vector** of the  $n$  partial derivatives of  $f$ , which points in the direction of the greatest rate of increase of the function. Length of this vector tell us about the rate of increase. Definition of a gradient in the 3-D Cartesian coordinate system is given by:

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k} \quad (1.3)$$

Where  $\vec{i}, \vec{j}, \vec{k}$  are the standard unit vectors

Symbol  $\nabla$  is a symbol of operator nabla, which defines an operation (in 3-D Cartesian coordinate system):

$$\nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \quad (1.4)$$

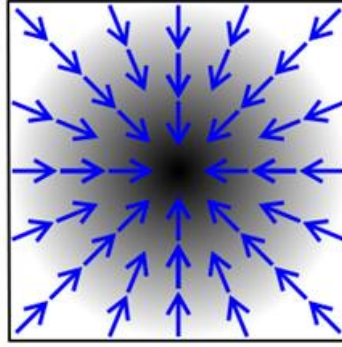


Fig.1.1. Values of the function are here represented by black and white colors, the darker the color, the higher function value. Blue arrows show direction of the function's gradient.

### Example 1.3.

Calculate gradient of a function:

$$r = \sqrt{x^2 + 2y^2 + 3z^2}$$

**Solution**

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2 + 2y^2 + 3z^2}}$$

$$\frac{\partial r}{\partial y} = \frac{4y}{2\sqrt{x^2 + 2y^2 + 3z^2}}$$

$$\frac{\partial r}{\partial z} = \frac{6z}{2\sqrt{x^2 + 2y^2 + 3z^2}}$$

$$\nabla r = \left( \frac{2x}{2\sqrt{x^2 + 2y^2 + 3z^2}}, \frac{4y}{2\sqrt{x^2 + 2y^2 + 3z^2}}, \frac{6z}{2\sqrt{x^2 + 2y^2 + 3z^2}} \right)$$

### 1.4. Divergence

Divergence is a vector operator that **measures the magnitude of a vector field's** source or sink at a given point **in terms of a signed scalar**. We can say that divergence represents the volume density of the outward flux of a vector field from an infinitesimal volume around a given point. If the divergence sign is positive at a given point, it means that vectors of the field are going out of this point. On the other hand, if the sign is negative it means that vectors of the field are coming into this point.

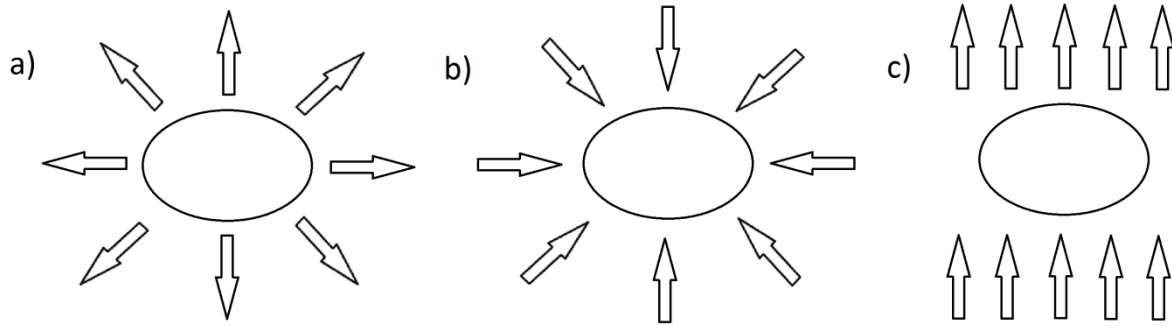


Fig.1.2. a) positive value of divergence, b) negative value of divergence, c) divergence equal to zero.

In a 3-D Cartesian coordinates the definition of divergence is as follows:

For a continuously differentiable vector field  $\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$  divergence is equal to the scalar-valued function:

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \quad (1.5)$$

#### Example 1.4.

Calculate divergence of a field:

$$\vec{F}(x, y, z) = xy\vec{i} + yz^2\vec{j} + x^2yz\vec{k}$$

**Solution**

$$\nabla \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz^2) + \frac{\partial}{\partial z}(x^2yz) = y + z^2 + x^2y$$

### 1.5. Curl

**Curl is a vector operator** that describes the infinitesimal **rotation of a 3-D vector field**. At every point in the field the **curl field is represented by a vector**. Length and direction of this vector characterize the rotation at that point. In general, we can say that curl operator acting at vector field creates another vector field, which describes density of circulation of the original vector field. In 3-D Cartesian coordinates, curl has a character of following determinant:

For a continuously differentiable vector field  $\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$

$$\text{rot } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \left( \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \vec{i} + \left( \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \vec{j} + \left( \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \vec{k} \quad (1.6)$$

**Example 1.5.**

Calculate curl for a vector field:

$$\vec{F}(x, y, z) = xyz\vec{i} + yz^2\vec{j} + x^2y^2z\vec{k}$$

**Solution**

$$\begin{aligned} \text{rot}\vec{F} = \nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & yz^2 & x^2y^2z \end{vmatrix} \\ &= \left( \frac{\partial x^2y^2z}{\partial y} - \frac{\partial yz^2}{\partial z} \right) \vec{i} + \left( \frac{\partial xyz}{\partial z} - \frac{\partial x^2y^2z}{\partial x} \right) \vec{j} + \left( \frac{\partial yz^2}{\partial x} - \frac{\partial xyz}{\partial y} \right) \vec{k} \\ &= (2yx^2z - 2yz)\vec{i} + (xy - 2xy^2z)\vec{j} + (0 - xz)\vec{k} \end{aligned}$$

**1.6. Laplacian**

Laplace operator is a differential operator given by the divergence of the gradient of a function. The Laplacian  $\Delta f(p)$  of a function  $f$  at a point  $p(x_1, \dots, x_n)$  can be interpreted as a rate at which the average value of a function  $f$  over spheres centered at  $p$ , deviates from  $f(p)$  as the radius of a sphere grows. We can define Laplacian as:

$$\Delta f = \nabla^2 f = \nabla \cdot \nabla f = \text{div grad } f \quad (1.7)$$

In Cartesian coordinates:

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \quad (1.8)$$

**Example 1.6.**

Calculate Laplacian of a function:

$$f = x^2yz + xy^3z^2$$

**Solution**

$$\begin{aligned} \Delta f &= \frac{\partial^2}{\partial x^2}(x^2yz + xy^3z^2) + \frac{\partial^2}{\partial y^2}(x^2yz + xy^3z^2) + \frac{\partial^2}{\partial z^2}(x^2yz + xy^3z^2) = \\ &= 2yz + 6xyz^3 + 2xy^3 \end{aligned}$$