

Homogeneity and Monotonicity of Distance-Rationalizable Voting Rules

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Abstract

Distance rationalizability is a framework for classifying voting rules by interpreting them in terms of distances and consensus classes. It also allows to design new voting rules with desired properties. A particularly natural and versatile class of distances that can be used for this purpose is that of *votewise* distances [12], which “lift” distances over individual votes to distances over entire elections using a suitable norm. In this paper, we continue the investigation of the properties of votewise distance-rationalizable rules initiated in [12]. We describe a number of general conditions on distances and consensus classes that ensure that the resulting voting rule is homogeneous or monotone. This complements the results of [12], where the authors focus on anonymity, neutrality and consistency. We also introduce a new class of voting rules, that can be viewed as “majority variants” of classic scoring rules, and have a natural interpretation in the context of distance rationalizability.

1 Introduction

In collaborative environments, agents often need to make joint decisions based on their preferences over possible outcomes. Thus, social choice theory emerges as an important tool in the design and analysis of multiagent systems [13]. However, voting procedures that have been developed for human societies are not necessarily optimal for artificial agents and vice versa. For instance, there are voting rules that allow for polynomial-time winner determination (and thus are suitable for autonomous agents), yet have been deemed too complicated to be comprehended by an average voter in many countries; an example is provided by Single Transferable Vote. Further, unlike an electoral committee in a human

society, the designer of a multi-agent voting system is usually unencumbered by legacy issues or the need to appeal to the general public, and can choose a voting rule that is most suitable for the application at hand, or, indeed, design a brand-new voting rule that satisfies the axioms that he deems important.

A recently proposed *distance rationalizability* framework [17, 10, 12, 11] is ideally suited for such settings. Under this framework, one can define a voting rule by a class of consensus elections and a distance over elections; the winners of an election are defined as the winners in the nearest consensus. In other words, for any election this rule seeks the most similar election with an obvious winner (where the similarity is measured by the given distance), and outputs its winner. Examples of natural consensus classes include *strong unanimity consensus*, where all voters agree on the ranking of all candidates, and *Condorcet consensus*, where there is a candidate that is preferred by a majority of voters to every other candidate. Combined with the *swap distance* (defined as the number of swaps of adjacent candidates that transforms one election into the other), these consensus classes produce, respectively, the Kemeny rule and the Dodgson rule.

The examples above illustrate that the distance rationalizability framework can be used to interpret (rationalize) existing voting rules in terms of a search for consensus (see [17] for a comprehensive list of results in this vein). It can also be applied to design new voting rules: for instance, in [10] the authors investigate the rule obtained by combining the Condorcet consensus with the Hamming distance. Further, by decomposing a voting rule into a consensus class and a distance we can hope to gain further insights into the structure of the rule. This decomposition is especially useful when the distance reflects changes in voters' opinions in a simple and transparent way like the so-called *votewise* distances introduced in [12]. These are distances over elections that are obtained by aggregating distances between individual votes using a suitable norm, such as ℓ_1 or ℓ_∞ . Indeed, paper [12] shows that one can derive conclusions about anonymity, neutrality and consistency of votewise rules (i.e., rules rationalized via votewise distances) from the basic properties of the underlying distances on votes, norms, and consensus classes.

In this paper we pick up this thread of research and study two important properties of voting rules not considered in [12], namely, monotonicity and homogeneity. Briefly put, monotonicity ensures that providing more support to a winning candidate cannot turn him into a loser, and homogeneity ensures that the result of an election depends on the proportions of particular votes and not on their absolute counts. Both properties are considered highly desirable for reasonable voting rules. (although, for example, single transferable vote and plurality run-off used in political elections in, respectively, Australia and France, are known not to be monotone). We focus on the four standard consensus classes considered in the previous work (strong unanimity \mathcal{S} , unanimity \mathcal{U} , majority \mathcal{M} and Condorcet \mathcal{C}) and ℓ_1 - and ℓ_∞ -norms, Our aim is to identify distances on votes that, combined with these norms and consensus classes, produce homogeneous and/or monotone rules.

Of the four consensus classes considered in this paper, the majority consensus \mathcal{M} received relatively little attention in the existing literature. Thus, in order to study the homogeneity and monotonicity of the rules that are distance-rationalizable with respect to

\mathcal{M} , we need to develop a better understanding of such rules. Our main result here is a characterization of all voting rules that are rationalizable with respect to \mathcal{M} via a neutral distance on votes and the ℓ_1 -norm. It turns out that such rules have a very natural interpretation: they are “majority variants” of classic scoring rules. This characterization enables us to analyze the homogeneity of the rules in this class, leading to a dichotomy result.

As argued above, a votewise distance-rationalizable rule can be characterized by three parameters: a distance on votes, a norm, and a consensus class. From this perspective, it is interesting to ask how much the voting rule changes if we vary one or two of these parameters. We provide two results that contribute to this agenda. First, we show that essentially any rule that is votewise-rationalizable with respect to \mathcal{M} can also be rationalized with respect to \mathcal{U} , by modifying the norm accordingly. This enables us to answer a question left open in [11]. Second, we show that, for any consensus class and any distance on votes, replacing the ℓ_1 -norm with the ℓ_∞ -norm produces a voting rule that is an n -approximation of the original rule, where n is the number of voters. For the Dodgson rule, this transformation produces a rule that is polynomial-time computable and homogeneous. This line of work also emphasizes the constructive aspect of the distance rationalizability framework: we are able to derive new voting rules with attractive properties by combining a known consensus class with a known distance measure in a novel way.

Related work. The formal theory of distance rationalizability was initiated by Meskanen and Nurmi [17], though the idea, in one shape or another, appeared in earlier papers as well (see, e.g., [18, 2, 16, 15]). The goal of Meskanen and Nurmi was to seek best possible distance-rationalizations of classical voting rules. This research program was advanced by Elkind, Faliszewski, and Slinko [10, 12, 11], who, in addition to further classification work, also suggested studying general properties of distance-rationalizable voting rules. In particular, in [11] they identified an interesting and versatile class of distances—which they called votewise distances—that lead to rules whose properties can be meaningfully studied.

The study of distance rationalizability is naturally related to the study of another—much older—framework, which is based on interpreting voting rules as maximum likelihood estimators (the MLE framework). This framework, which could be dated back to Condorcet and has been pursued by Young [21], and, more recently, in [8], [7], and—in the context of combinatorial domains—in [19]. To date, most of the research on the MLE framework was concerned with determining which of the existing voting rules can be interpreted as maximum likelihood estimators; however, paper [19] also shows that the MLE approach can be used to deduce new useful voting rules.

This paper is loosely related to the work of Caragiannis et al. [6], where the authors give a monotone, homogeneous voting rule that calculates scores which approximate candidates’ Dodgson scores up to an $O(m \log m)$ multiplicative factor, where m is the number of candidates. The relation to our work is twofold. First, we also focus on monotonicity and homogeneity, although our goal is to come up with a general method of constructing monotone and homogeneous rules and not to approximate particular rules. Second, in the course of our study we discover a homogeneous and polynomial-time computable voting rule that approximates the scores of candidates in Dodgson elections up to a multiplicative

factor of n , where n is the number of voters. While the number of voters is usually much bigger than the number of candidates, and thus our algorithm is usually inferior to that of [6], it illustrates the power of the distance rationalizability framework.

Organization of the paper. The paper is organized as follows. Section 2 contains preliminary definitions regarding voting rules in general and the distance-rationalizability framework specifically. In Section 3 we provide a detailed study of rules that are vote-wise rationalizable with respect to the majority consensus. Section 4 presents our results on homogeneity of votewise rules, showing that very often votewise rules indeed satisfy homogeneity, but that their subclasses (in particular, those rationalized via the majority consensus and the Condorcet consensus) may also fail it under certain conditions. In Section 4.1 we briefly depart from our path of studying homogeneity and monotonicity and show that ℓ_∞ -votewise rules form weak approximations of ℓ_1 -votewise rules. Finally, in Section 5 we present our results on monotonicity of votewise rules. We conclude in Section 6, giving a broader view of our results and mentioning several open problems.

2 Preliminaries

Basic notation. An *election* is a pair $E = (C, V)$, where $C = \{c_1, \dots, c_m\}$ is the set of *candidates* and $V = (v_1, \dots, v_n)$ is the set of *voters*. The *size* of an election is the number of voters in it, i.e., we write $|E| = |V|$. Voter v_i is identified with a total order \succ_i over C , which we will refer to as v_i 's *preference order*, or *ranking*. We write $c_j \succ_i c_\ell$ to denote that voter v_i prefers c_j to c_ℓ . We denote by $\mathcal{P}(C)$ the set of all preference orders over C . For a voter v , we denote by $\text{top}(v)$ the candidate ranked first by v , and set $\mathcal{P}(C, c) = \{v \in \mathcal{P}(C) \mid \text{top}(v) = c\}$. For any voter $v_i \in V$ and a candidate $c \in C$, we denote by $\text{rank}(v_i, c)$ the position of c in v_i 's ranking. For example, if $\text{top}(v_i) = c$ then $\text{rank}(v_i, c) = 1$. A *voting rule* is a mapping \mathcal{R} that for any election (C, V) outputs a non-empty subset of candidates $W \subseteq C$ called the *election winners*. Given an election $E = (C, V)$ and $s \in \mathbb{N}$, we denote by sE the election (C, sV) , where sV is obtained by concatenating s copies of V .

Two important properties of voting rules that will be studied in this paper are homogeneity and monotonicity.

Homogeneity. A voting rule \mathcal{R} is *homogeneous* if for each election $E = (C, V)$ and each positive $s \in \mathbb{N}$ we have $\mathcal{R}(E) = \mathcal{R}(sE)$.

Monotonicity. A voting rule \mathcal{R} is *monotone* if for every election $E = (C, V)$, every $c \in \mathcal{R}(E)$ and every $E' = (C, V')$ obtained from E by moving c up in some voters' rankings (but not changing their rankings in any other way) we have $c \in \mathcal{R}(E')$.

Voting rules. We will now define the classic voting rules discussed in this paper, namely, scoring rules, (Simplified) Bucklin, and Dodgson.

Scoring rules In this paper, we will use a somewhat nonstandard definition of a scoring rule. Any vector $\alpha = (\alpha_1, \dots, \alpha_m) \in (\mathbb{R}_+ \cup \{0\})^m$ defines a partial voting rule \mathcal{R}_α for elections with a fixed number m of candidates. Under this rule, for each preference order $u \in \mathcal{P}(C)$, $|C| = m$, a candidate $c \in C$ gets $\alpha_{\text{rank}(u,c)}$ points (as is standard) and these values are summed up together to obtain the score of c . However, we define the winners to be the candidates with the lowest score (rather than with the highest; as is typical when discussing scoring rules). A sequence of scoring vectors $(\alpha^{(m)})_{m \in \mathbb{N}}$, where $\alpha^{(m)} \in (\mathbb{R}_+ \cup \{0\})^m$, defines a voting rule $\mathcal{R}_{(\alpha^{(m)})}$ which is applicable for any number of alternatives.

For example, in this notation the Borda rule is defined by a family of scoring vectors $\alpha^{(m)} = (0, 1, \dots, m-1)$ and the k -approval is the family of scoring vectors given by $\alpha_i^{(m)} = 0$ for $i \leq k$, $\alpha_i^{(m)} = 1$ for $i > k$. The 1-approval rule is also known as Plurality. The traditional model, where the winners are the candidates with the highest score, can be converted to our notation by setting $\alpha'_i = \alpha_{\max} - \alpha_i$, where $\alpha_{\max} = \max_{i=1}^m \alpha_i$. The reason for this deviation is that in the context of this paper it will be much more convenient to speak of minimizing one's score. Note that, in general, we do not require $\alpha_1 \leq \dots \leq \alpha_m$, although this assumption is obviously required for monotonicity.

Note that vectors $(\alpha_1, \dots, \alpha_m)$ and $(\beta\alpha_1, \dots, \beta\alpha_m)$ define the same voting rule for any $\beta > 0$; the same is true for $(\alpha_1, \dots, \alpha_m)$ and $(\alpha_1 + \gamma, \dots, \alpha_m + \gamma)$ for any $\gamma \geq 0$. Thus, in what follows, we normalize the scoring vectors by requiring their smallest coordinate to be 0, and the smallest non-zero coordinate to be 1.

Bucklin The *Bucklin rule*¹ \mathcal{R}_B can be thought of as an adaptive version of k -approval. Under the *Bucklin rule*, we first determine the smallest value of k such that some candidate is ranked in top k positions by more than half of the voters. The winner(s) are the candidates that are ranked in the top k positions the maximum number of times. Under the *Simplified Bucklin rule* \mathcal{R}_{sB} , the winners are all candidates ranked in top k positions by a majority of voters. For any election E we have $\mathcal{R}_B(E) \subseteq \mathcal{R}_{sB}(E)$.

Dodgson To define the Dodgson rule, we need to introduce the concept of a *Condorcet winner*. A Condorcet winner is a candidate that is preferred to any other candidate by a majority of voters. The *Dodgson score* of a candidate c is the smallest number of swaps of adjacent candidates that have to be performed on the votes to make c the Condorcet winner. The winner(s) under the Dodgson rule are the candidates with the lowest Dodgson score.

Norms and Metrics. A *norm* on \mathbb{R}^n is a mapping $N : \mathbb{R}^n \rightarrow \mathbb{R}$ that has the following properties for all $x, y \in \mathbb{R}^n$:

- (1) $N(\alpha x) = |\alpha|N(x)$ for all $\alpha \in \mathbb{R}$;
- (2) $N(x) \geq 0$ and $N(x) = 0$ if and only if $x = (0, \dots, 0)$;

¹Also known as the majoritarian compromise.

$$(3) \quad N(x + y) \leq N(x) + N(y).$$

Two important properties of norms that will be of interest to us are symmetry and monotonicity. We say that a norm N is *symmetric* if for each permutation $\sigma : [1, n] \rightarrow [1, n]$ it holds that $N(x_1, \dots, x_n) = N(x_{\sigma(1)}, \dots, x_{\sigma(n)})$. For monotonicity, we make use of the definition proposed in [3]. Specifically, we say that a norm N is *monotone in the positive orthant*, or \mathbb{R}_+^n -*monotone*, if for any two vectors $(x_1, \dots, x_n), (y_1, \dots, y_n) \in \mathbb{R}_+^n$ such that $x_i \leq y_i$ for all $i \leq n$ we have $N(x_1, \dots, x_n) \leq N(y_1, \dots, y_n)$.

A well-studied class of norms are the ℓ_p -norms given by

$$\ell_p(x_1, \dots, x_n) = (|x_1|^p + \dots + |x_n|^p)^{\frac{1}{p}}$$

for $p \in \mathbb{N}$. This definition can be extended to $p = +\infty$ by setting $\ell_\infty(x_1, \dots, x_n) = \max\{x_1, \dots, x_n\}$. Observe that for any $p \in \mathbb{N} \cup \{+\infty\}$ the ℓ_p norm is, in fact, a family of norms, i.e., it is well-defined on \mathbb{R}^i for any $i \in \mathbb{N}$. Also, any such norm is clearly symmetric and monotone in the positive orthant.

A *metric*, or *distance*, on a set X is a mapping $d : X^2 \rightarrow \mathbb{R}$ that satisfies the following conditions for all $x, y, z \in X$:

- (1) $d(x, y) \geq 0$;
- (2) $d(x, y) = 0$ if and only if $x = y$;
- (3) $d(x, y) = d(y, x)$;
- (4) $d(x, z) \leq d(x, y) + d(y, z)$.

A function that satisfies conditions (1), (3) and (4), but not (2), is called a *pseudodistance*.

Given a distance d on X and a norm N on \mathbb{R}^n , we can define a distance $N \circ d$ on X^n by setting

$$(N \circ d)(\mathbf{x}, \mathbf{y}) = N(d(x_1, y_1), \dots, d(x_n, y_n))$$

for all vectors $\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n) \in X^n$. A distance defined in this manner is called a *product metric*.

In this paper, we will study distances over votes and their extensions to distances over elections via product metrics. Some examples of distances over votes are given by the *discrete distance* d_{discr} , the *swap distance* d_{swap} , and the *Sertel distance* d_{ser} , defined as follows. For any set of candidates C and any $u, v \in \mathcal{P}(C)$, we set $d_{\text{discr}}(u, v) = 0$ if $u = v$ and $d_{\text{discr}}(u, v) = 1$ otherwise. The swap distance d_{swap} is given by $d_{\text{swap}}(u, v) = \frac{1}{2} |\{(c, c') \in C^2 \mid c \succ_u c', c' \succ_v c\}|$, where \succ_u and \succ_v are the preference orders associated with u and v , respectively. The Sertel distance between u and v is defined as the smallest value of i such that for all $j > i$ voters u and v rank the same candidate in position j .

A distance d on $\mathcal{P}(C)$ is called *neutral* if for any $u, v \in \mathcal{P}(C)$ and any permutation $\pi : C \rightarrow C$ we have $d(u, v) = d(\pi(u), \pi(v))$, where $\pi(x)$ denotes the vote obtained from x by moving candidate c_i into position $\text{rank}(x, \pi(c_i))$, for $i = 1, \dots, |C|$. Clearly, all distances listed above are neutral.

Distance Rationalizability. Intuitively, a consensus class is a collection of elections with an obvious winner. Formally, a *consensus class* is a pair $(\mathcal{E}, \mathcal{W})$ where \mathcal{E} is a set of elections and $\mathcal{W}: \mathcal{E} \rightarrow C$ is a function that for each election $E \in \mathcal{E}$ outputs the alternative called the *consensus winner*. The following four consensus classes have been considered in the previous work on distance rationalizability:

Strong unanimity. Denoted \mathcal{S} , contains elections $E = (C, V)$ where all voters report the same preference order. The consensus winner is the candidate ranked first by all voters.

Unanimity. Denoted \mathcal{U} , contains all elections $E = (C, V)$ where all voters rank the same candidate first. The consensus winner is the candidate ranked first by all voters.

Majority. Denoted \mathcal{M} , contains all elections $E = (C, V)$ where more than half of the voters rank the same candidate first. The consensus winner is the candidate ranked first by the majority of voters.

Condorcet. Denoted \mathcal{C} , contains all elections $E = (C, V)$ with a Condorcet winner. The consensus winner is the Condorcet winner.

We say that a voting rule \mathcal{R} is *compatible* with a consensus class \mathcal{K} if for any consensus election $E \in \mathcal{K}$ it holds that $\mathcal{W}(E) = \mathcal{R}(E)$. Similarly, \mathcal{R} is said to be *weakly compatible* with \mathcal{K} if for any $E \in \mathcal{K}$ we have $\mathcal{W}(E) \in \mathcal{R}(E)$. Essentially all well-known voting rules are weakly compatible with \mathcal{S} , \mathcal{U} and \mathcal{M} , but there are rules that are not compatible with any of these consensus classes (e.g., k -approval for $k > 1$). The rules that are compatible with \mathcal{C} are also known as *Condorcet-consistent* rules; we use the term “compatibility” rather than “consistency” to avoid confusion with the consistency property of voting rules.

We are now ready to define the concept of distance rationalizability. Our definition below is taken from [12], which itself was inspired by [17, 10].

Definition 2.1. *Let d be a distance over elections and let $\mathcal{K} = (\mathcal{E}, \mathcal{W})$ be a consensus class. The (\mathcal{K}, d) -score of a candidate c in an election E is the distance (according to d) between E and a closest election $E' \in \mathcal{E}$ such that $c \in \mathcal{W}(E')$. A voting rule \mathcal{R} is distance-rationalizable via a consensus class \mathcal{K} and a distance d over elections (is (\mathcal{K}, d) -rationalizable) if for each election E the set $\mathcal{R}(E)$ consists of all candidates with the smallest (\mathcal{K}, d) -score.*

A particularly useful class of distances to be used in distance rationalizability constructions is that of *vote-wise* distances, which are obtained by combining a distance over votes with a suitable norm. Formally, given a set of candidates C , consider a distance d over $\mathcal{P}(C)$ and a family of norms $\mathcal{N} = (N_i)_{i=1}^{\infty}$, where N_i is a norm over \mathbb{R}^i . We define a distance $\widehat{d}^{\mathcal{N}}$ over elections with the set of candidates C as follows: for any $E = (C, V)$, $E' = (C, V')$, we set $\widehat{d}^{\mathcal{N}}(E, E') = (N_i \circ d)(V, V')$ if $|V| = |V'| = i$, and $\widehat{d}^{\mathcal{N}}(E, E') = +\infty$ if $|V| \neq |V'|$. A voting rule \mathcal{R} is said to be \mathcal{N} -*vote-wise distance-rationalizable* (or simply \mathcal{N} -*vote-wise*) with respect to a consensus class \mathcal{K} if there exists a distance d over votes such that \mathcal{R} is $(\mathcal{K}, \widehat{d}^{\mathcal{N}})$ -rationalizable. When \mathcal{N} is the ℓ_p -norm for some $p \in \mathbb{N} \cup \{+\infty\}$, we write \widehat{d}^p instead of \widehat{d}^{ℓ_p} , and when $\mathcal{N} = \ell_1$, we omit the index altogether and write \widehat{d} . It is known that any

voting rule is distance-rationalizable with respect to any consensus class that it is compatible with [12]. However, some voting rules are not \mathcal{N} -votewise distance-rationalizable with respect to standard consensus classes for any reasonable norm \mathcal{N} [11].

Let us now consider some examples of distance-rationalizations of voting rules. Nitzan [18] was the first to show that Plurality is $(\mathcal{U}, \widehat{d}_{\text{discr}})$ -rationalizable and Borda is $(\mathcal{U}, \widehat{d}_{\text{swap}})$ -rationalizable. It is easy to see that Dodgson is $(\mathcal{C}, \widehat{d}_{\text{swap}})$ -rationalizable and Kemeny is $(\mathcal{S}, \widehat{d}_{\text{swap}})$ -rationalizable. The distance $\widehat{d}_{\text{ser}}^\infty$, combined with the majority consensus, yields the Simplified Bucklin rule [12].

For any set of candidates C with $|C| = m$ and a scoring vector $\alpha = (\alpha_1, \dots, \alpha_m)$, paper [12] defines a (pseudo)distance $d_\alpha(u, v)$ on $\mathcal{P}(C)$ as $d_\alpha(u, v) = \sum_{j=1}^m |\alpha_{\text{rank}(u, c_j)} - \alpha_{\text{rank}(v, c_j)}|$, and shows that if $\alpha_1 = 0$ then \mathcal{R}_α is $(\mathcal{U}, \widehat{d}_\alpha)$ -(pseudo)distance-rationalizable. The following lemma, proved as part of Theorem 3 of [10],² will be useful for us later on.

Lemma 2.2 ([10]). *Let $C = \{c_1, \dots, c_m\}$ be a set of candidates, $\alpha = (\alpha_1, \dots, \alpha_m)$ be a normalized scoring vector, and c be a candidate. For each vote v over C it holds that $\min\{d_\alpha(v, u) \mid u \in \mathcal{P}(C, c)\} = 2|\alpha_{\text{rank}(v, c)} - \alpha_1|$.*

3 \mathcal{M} -Scoring Rules

The majority consensus is a very natural notion of agreement in the society. However, it has received little attention in the literature so far. Here we will show that it leads to a series of interesting rules with nice properties.

Definition 3.1. *For any scoring vector $\alpha = (\alpha_1, \dots, \alpha_m)$, let $\mathcal{M}\text{-}\mathcal{R}_\alpha$ be a partial voting rule defined on the profiles with m alternatives as follows. Given an election $E = (C, V)$ with $|C| = m$ and $V = (v_1, \dots, v_n)$, for each candidate $c \in C$, we define the \mathcal{M} -score of c as the sum of $\lfloor \frac{n}{2} \rfloor + 1$ lowest values among $\alpha_{\text{rank}(v_1, c)}, \dots, \alpha_{\text{rank}(v_n, c)}$. The winners are the candidates with the lowest $\mathcal{M}\text{-}\mathcal{R}_\alpha$ scores. As in the classic case, a family of scoring vectors $(\alpha^{(i)})_{i \in \mathbb{N}}$ defines an \mathcal{M} -scoring rule $\mathcal{M}\text{-}\mathcal{R}_{(\alpha^{(i)})}$.*

We will refer to voting rules from Definition 3.1 as \mathcal{M} -scoring rules. Such rules (or their slight modifications) are often used for score aggregation in real-life settings; for example, it is not unusual for a professor to grade the students on the basis of their five best assignments out of six or in some sport competitions to award winners on the basis of their several best attempts.

It is not hard to see that \mathcal{M} -Plurality is equivalent to Plurality: under both rules, the winners are the candidates with the maximum number of first-place votes. However, essentially all other scoring rules differ from their \mathcal{M} -counterparts.

Proposition 3.2. *Consider a normalized scoring vector $\alpha = (\alpha_1, \dots, \alpha_m)$. The rule $\mathcal{M}\text{-}\mathcal{R}_\alpha$ coincides with \mathcal{R}_α if and only if (i) $\alpha_1 = \dots = \alpha_m$ or (ii) $\alpha_i = 0$ for some $i \in \{1, \dots, m\}$ and $\alpha_j = 1$ for all $j \neq i$.*

²The proof in [10] assumes that—in our notation— $\alpha_1 \leq \dots \leq \alpha_m$, but it is not hard to see that it goes through as long as we require $\alpha_1 = 0 \leq \alpha_k$ for all $k > 1$.

Proof. Clearly, if all coordinates of the scoring vector are equal, both $\mathcal{M}\text{-}\mathcal{R}_\alpha$ and \mathcal{R}_α output the set of all candidates on any preference profile. Further, we have already argued that if $\alpha_1 = 0$, $\alpha_j = 1$ for each $j > 1$, then $\mathcal{M}\text{-}\mathcal{R}_\alpha$ is the Plurality rule. Clearly, this argument also applies if the only 0 appears in a different position of the scoring vector.

We will now show that the converse direction is also true. Note that we can assume without loss of generality that $\alpha_1 \leq \dots \leq \alpha_m$: it is not hard to see that for any permutation $\sigma : [1, m] \rightarrow [1, m]$ it holds that $\mathcal{M}\text{-}\mathcal{R}_\alpha$ is equivalent to \mathcal{R}_α if and only if $\mathcal{M}\text{-}\mathcal{R}_{\sigma(\alpha)}$ is equivalent to $\mathcal{R}_{\sigma(\alpha)}$, where $\sigma(\alpha)$ is the scoring vector given by $(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(m)})$. Thus for any scoring rule that satisfies neither (i) nor (ii) we can assume that either (a) $\alpha_1 = \alpha_2 = 0$, $\alpha_m > 0$ or (b) $\alpha_1 = 0$, $\alpha_2 = 1$, and $\alpha_m > 1$. We will argue that in both of these cases $\mathcal{M}\text{-}\mathcal{R}_\alpha$ is not equivalent to \mathcal{R}_α .

Indeed, consider two candidates c and w and an election E with n voters where $\lfloor \frac{n}{2} \rfloor + 1$ voters rank c first and w second, and the remaining voters rank w first and c last.

In case (a), the $\mathcal{M}\text{-}\mathcal{R}_\alpha$ -score of both c and w is 0, so both of them are among the winners under $\mathcal{M}\text{-}\mathcal{R}_\alpha$. On the other hand, c 's \mathcal{R}_α -score is at least $\lceil \frac{n}{2} \rceil - 1$, while w 's \mathcal{R}_α -score is zero, so w is among the winners under \mathcal{R}_α and c is not. Thus, we have $\mathcal{M}\text{-}\mathcal{R}_\alpha(E) \neq \mathcal{R}_\alpha(E)$.

In case (b), c is the unique winner under $\mathcal{M}\text{-}\mathcal{R}_\alpha$. On the other hand, under \mathcal{R}_α candidate c gets $\alpha_m(\lceil \frac{n}{2} \rceil - 1)$ points, and candidate w gets $\lfloor \frac{n}{2} \rfloor + 1$ points. Since $\alpha_m > 1$, for large enough values of n (it suffices to pick $n > \frac{\alpha_m + 1}{\alpha_m - 1}$) candidate w has a lower score under \mathcal{R}_α , i.e., c cannot be the winner of E . \square

The \mathcal{M} -scoring rules tend to ignore extremely negative opinions. Therefore, intuitively, they are less susceptible to manipulation: if a voter v ranks a candidate c lower than the majority of other voters, v cannot manipulate against c by moving her to the bottom of their ranking. In this section we will show that these rules are also very interesting from the distance rationalizability point of view: it turns out that they essentially coincide with the class of rules that are ℓ_1 -votewise rationalizable with respect to \mathcal{M} .

We will first need to generalize a result from [12]. to pseudodistances and weak compatibility.

Proposition 3.3. *Any voting rule that is pseudodistance-rationalizable with respect to a consensus class \mathcal{K} is weakly compatible with \mathcal{K} .*

Proof. Consider a \mathcal{K} -consensus $E = (C, V)$ with winner c and a (\mathcal{K}, d) -rationalizable voting rule \mathcal{R} , where d is a pseudodistance. We have $d(E, E) = 0$, so $d(E, E) \leq d(E, E')$ for any election E' . Therefore, $c \in \mathcal{R}(E)$. \square

Now, we can characterize \mathcal{M} -scoring rules that are (pseudo)distance-rationalizable with respect to \mathcal{M} .

Proposition 3.4. *Let $\alpha = (\alpha_1, \dots, \alpha_m)$ be a normalized scoring vector. The rule $\mathcal{M}\text{-}\mathcal{R}_\alpha$ is ℓ_1 -votewise distance-rationalizable with respect to \mathcal{M} if and only if $\alpha_1 = 0$, $\alpha_j > 0$ for all $j \neq 1$. Further, $\mathcal{M}\text{-}\mathcal{R}_\alpha$ is ℓ_1 -votewise pseudodistance-rationalizable with respect to \mathcal{M} if and only if $\alpha_1 = 0$.*

Proof. Suppose first that $\alpha_1 \neq 0$. Since α is normalized, there exists a $j \neq 1$ such that $\alpha_j = 0$. Consider a preference profile in which some candidate c is ranked first by everyone, and some other candidate w is ranked in the j -th position by everyone. Clearly, c is the majority winner, but under $\mathcal{M}\text{-}\mathcal{R}_\alpha$ w is a winner, and c is not. Thus, by Proposition 3.3 no such rule can be pseudodistance-rationalizable with respect to \mathcal{M} .

Now, suppose that $\alpha_1 = 0$. Consider the pseudodistance d_α , an election $E = (C, V)$, a candidate $c \in C$, and a voter $v \in V$ that ranks c in the j -th position. By Lemma 2.2, $\min\{d_\alpha(v, u) \mid u \in \mathcal{P}(C, c)\} = 2\alpha_j$. This implies that in E for any candidate $c \in C$ his $\mathcal{M}\text{-}\mathcal{R}_\alpha$ -score is twice the distance to the nearest \mathcal{M} -consensus with winner c . Hence, the rule $\mathcal{M}\text{-}\mathcal{R}_\alpha$ is $(\mathcal{M}, \widehat{d}_\alpha)$ -rationalizable.

Clearly, d_α is not necessarily a distance. Indeed, if we have $\alpha_j = 0 = \alpha_1$ for some $j \neq 1$, the distance between a vote v and the vote obtained from v by swapping the candidates in the first and the j -th position is 0. This argument also shows that in this case $\mathcal{M}\text{-}\mathcal{R}_\alpha$ is not distance-rationalizable. Indeed, if all voters rank c first and rank w in the j -th position, then both c and w are winners under $\mathcal{M}\text{-}\mathcal{R}_\alpha$, even though c is the unique majority winner. Now, suppose that $\alpha_j \neq 0$ for all $j \neq 1$. It may still happen that $\alpha_j = \alpha_k$ for some $j, k \in \{2, \dots, m\}$, in which case d_α is still a pseudodistance. However, in this case we can set $\varepsilon = \min\{|\alpha_j - \alpha_k| \mid \alpha_j \neq \alpha_k\}$ and let $d'_\alpha(u, v) = 0$ if $u = v$ and $d'_\alpha(u, v) = \min\{d_\alpha(u, v), \varepsilon\}$ otherwise. It is not hard to see that d'_α is a distance; in particular, we have $d'_\alpha(u, v) \neq 0$ for $u \neq v$ by construction, and the triangle inequality is satisfied by our choice of ε . Further, consider a vote v that ranks c in the j -th position, $j > 1$, and the nearest (with respect to d_α) vote u that ranks c first. We have $d_\alpha(v, u) = 2\alpha_j > 0$, so $d'_\alpha(u, v) = d_\alpha(u, v)$. Therefore, the argument showing that $\mathcal{M}\text{-}\mathcal{R}_\alpha$ is $(\mathcal{M}, \widehat{d}_\alpha)$ -rationalizable applies to \widehat{d}'_α as well, and hence $\mathcal{M}\text{-}\mathcal{R}_\alpha$ is ℓ_1 -votewise distance-rationalizable. \square

We remark that our proof generalizes to scoring rules and \mathcal{U} , thus answering a question left open in [10], where the authors ask whether scoring rules with $\alpha_i = \alpha_j$ for $i, j > 1$ can be distance-rationalized (rather than pseudodistance-rationalized). Further, in [10] the authors consider only monotone scoring rules, i.e., rules that satisfy—in our notation— $\alpha_1 \leq \dots \leq \alpha_m$, while our result holds for all scoring vectors.

The following lemma explains how to find an \mathcal{M} -consensus that is nearest to a given election with respect to a given ℓ_1 -votewise distance.

Lemma 3.5. *Let \mathcal{R} be a voting rule that is $(\mathcal{M}, \widehat{d})$ -rationalized. Let $E = (C, V)$ be an arbitrary election where $V = (v_1, \dots, v_n)$ and let $E' = (C, U)$ be an \mathcal{M} -consensus such that $\widehat{d}(E, E')$ is minimal among all n -voter \mathcal{M} -consensuses over C . Let $c \in C$ be the consensus winner of (C, U) . Then, for each $i = 1, \dots, n$, either $u_i \in \arg \min_{x \in \mathcal{P}(C, c)} d(x, v_i)$ or $u_i = v_i$.*

Combining Lemma 3.5 with the argument in the proof of Theorem 4.9 in [12], we can show that the converse of Proposition 3.4 is also true: any voting rule that can be pseudodistance-rationalized via \mathcal{M} and a neutral ℓ_1 -votewise pseudodistance is, in fact, an \mathcal{M} -scoring rule. Also, any \mathcal{M} -scoring rule is obviously neutral. We can summarize these observations in the following theorem.

Theorem 3.6. *Let \mathcal{R} be a voting rule. There exists a neutral ℓ_1 -votewise pseudodistance \widehat{d} such that \mathcal{R} is $(\mathcal{M}, \widehat{d})$ -rationalizable if and only if \mathcal{R} can be defined as an \mathcal{M} -scoring rule $\mathcal{M}\text{-}\mathcal{R}_{(\alpha^{(i)})}$ such that $\alpha_1^{(i)} \leq \alpha_j^{(i)}$ for all $j > 1$ and all $i \in \mathbb{N}$.*

The discussion above suggests that using the majority consensus to rationalize a voting rule is similar to using the unanimity consensus, except that we only take into account the best “half-plus-one” votes. In fact, it turns out that under very weak assumptions we can translate a votewise rationalization of a rule with respect to \mathcal{M} to a votewise rationalization of that rule with respect to \mathcal{U} .

Definition 3.7. *Let $\mathcal{N} = (N_i)_{i=1}^\infty$ be a family of functions where for each i , $i \geq 1$, N_i is a mapping from \mathbb{R}^i to \mathbb{R} . We define a family $\mathcal{N}^{\mathcal{M}} = (N_i^{\mathcal{M}})_{i=1}^\infty$ as follows. For each $i \geq 1$, $N_i^{\mathcal{M}}$ is a mapping from \mathbb{R}^i to \mathbb{R} given by*

$$N_i^{\mathcal{M}}(x_1, \dots, x_i) = N_{\lfloor \frac{i}{2} \rfloor + 1}(|x_{\pi(1)}|, \dots, |x_{\pi(\lfloor \frac{i}{2} \rfloor + 1)}|),$$

where π is a permutation of $[1, i]$ such that $|x_{\pi(1)}| \geq |x_{\pi(2)}| \geq \dots \geq |x_{\pi(i)}|$.

For a family of symmetric norms $\mathcal{N} = (N_i)_{i=1}^\infty$ that are monotone in the positive orthant, the family $\mathcal{N}^{\mathcal{M}}$ is also a family of norms, which we will call the *majority variant* of \mathcal{N} .

Proposition 3.8. *Let $\mathcal{N} = (N_i)_{i=1}^\infty$ be a family of norms, where each N_i is a symmetric norm on \mathbb{R}^i that is monotone in the positive orthant. Then the family $\mathcal{N}^{\mathcal{M}} = (N_i^{\mathcal{M}})_{i=1}^\infty$ is also a family of symmetric norms that are monotone in the positive orthant.*

Proof. Let us fix a positive integer n . We will first show that $N_n^{\mathcal{M}}$ is a norm. It is easy to see that since $N_{\lfloor \frac{n}{2} \rfloor}$ is a norm, for every $(x_1, \dots, x_n) \in \mathbb{R}^n$ it holds that (a) $N_n^{\mathcal{M}}(x_1, \dots, x_n) \geq 0$, (b) $N_n^{\mathcal{M}}(x_1, \dots, x_n) = 0$ if and only if $x_1 = \dots = x_n = 0$, and (c) for each $\alpha \in \mathbb{R}$ it holds that $N_n^{\mathcal{M}}(\alpha x_1, \dots, \alpha x_n) = |\alpha| N_n^{\mathcal{M}}(x_1, \dots, x_n)$.

Let us now show that $N_n^{\mathcal{M}}$ satisfies the triangle inequality. Let (x_1, \dots, x_n) and (y_1, \dots, y_n) be two vectors in \mathbb{R}^n . We need to show that $N_n^{\mathcal{M}}(x_1 + y_1, \dots, x_n + y_n) \leq N_n^{\mathcal{M}}(x_1, \dots, x_n) + N_n^{\mathcal{M}}(y_1, \dots, y_n)$. Let π , σ_x and σ_y be permutations of $[1, n]$ such that $|x_{\pi(1)} + y_{\pi(1)}| \geq \dots \geq |x_{\pi(n)} + y_{\pi(n)}|$, $|x_{\sigma_x(1)}| \geq \dots \geq |x_{\sigma_x(n)}|$, $|y_{\sigma_y(1)}| \geq \dots \geq |y_{\sigma_y(n)}|$. Let $h = \lfloor \frac{n}{2} \rfloor + 1$. We have

$$\begin{aligned} N_n^{\mathcal{M}}(x_1 + y_1, \dots, x_n + y_n) &= N_h(|x_{\pi(1)} + y_{\pi(1)}|, \dots, |x_{\pi(h)} + y_{\pi(h)}|) \\ &\leq N_h(|x_{\pi(1)}| + |y_{\pi(1)}|, \dots, |x_{\pi(h)}| + |y_{\pi(h)}|) \\ &\leq N_h(|x_{\pi(1)}|, \dots, |x_{\pi(h)}|) + N_h(|y_{\pi(1)}|, \dots, |y_{\pi(h)}|) \\ &\leq N_h(|x_{\sigma_x(1)}|, \dots, |x_{\sigma_x(h)}|) + N_h(|y_{\sigma_y(1)}|, \dots, |y_{\sigma_y(h)}|) \\ &= N_n^{\mathcal{M}}(x_1, \dots, x_n) + N_n^{\mathcal{M}}(y_1, \dots, y_n), \end{aligned}$$

where the second inequality follows by triangle inequality for N_h , and the third one follows by N_h 's symmetry and monotonicity in the positive orthant. As a result, $N_n^{\mathcal{M}}$ is a norm.

By construction, $\mathcal{M}\text{-}N_n$ is both symmetric and monotone in the positive orthant. This completes the proof. \square

As an immediate corollary we get the following result.

Corollary 3.9. *Let \mathcal{N} be a family of symmetric norms that are monotone in the positive orthant and let d be a distance over votes. Let \mathcal{R} be a voting rule that is $(\mathcal{M}, \widehat{d^{\mathcal{N}}})$ -rationalized. Then \mathcal{R} is $(\mathcal{U}, \widehat{d^{\mathcal{N}, \mathcal{M}}})$ -rationalized.*

This discussion illustrates that when a rule can be rationalized in several different ways, the right choice of a consensus class plays an important role, as it may greatly simplify the underlying norm and hence the distance. This is why it pays to keep a variety of consensus classes available and search for best distance rationalizations possible. Corollary 3.9 also has a useful application: Paper [11] shows that STV³ cannot be rationalized with respect to \mathcal{S} , \mathcal{C} or \mathcal{U} by any neutral \mathcal{N} -votewise distance, where \mathcal{N} is a family of symmetric norms monotone in the positive orthant. Corollary 3.9 allows us to extend this result to \mathcal{M} , thus showing that STV cannot be rationalized by a “reasonable” votewise distance with respect to any of the standard consensus classes.

Theorem 3.10. *For three candidates, STV (together with any intermediate tie-breaking rule) is not distance-rationalizable with respect to the majority consensus and any anonymous neutral \mathcal{N} -votewise distance, where \mathcal{N} is monotone in the positive orthant.*

4 Homogeneity

Homogeneity is a very natural property of voting rules. It can be interpreted as a weaker form of another appealing property, namely, consistency. Recall that a voting rule \mathcal{R} is said to be *consistent* if for any two elections $E_1 = (C, V_1)$ and $E_2 = (C, V_2)$ with $\mathcal{R}(E_1) \cap \mathcal{R}(E_2) \neq \emptyset$ it holds that $\mathcal{R}(C, V_1 + V_2) = \mathcal{R}(E_1) \cap \mathcal{R}(E_2)$, where $V_1 + V_2$ denotes the concatenation of V_1 and V_2 . Thus, loosely speaking, homogeneity imposes the same requirement as consistency, but only for the restricted case $V_1 = V_2$. Now, consistency is known to be hard to achieve: by Young’s theorem [20], the only voting rules that are simultaneously anonymous, neutral and consistent are the scoring rules (or their compositions). In contrast, we will now argue that for many consensus classes and many values of $p \in \mathbb{N} \cup \{+\infty\}$, the rules that are ℓ_p -votewise rationalizable with respect to these classes are homogeneous. We start by showing that this is the case for ℓ_p , $p \in \mathbb{N}$, and consensus classes \mathcal{S} and \mathcal{U} . We then provide a complete characterization of all homogeneous rules that are ℓ_1 -votewise distance rationalizable with respect to \mathcal{M} , assuming that the underlying distance on votes is neutral. Next, we show that combining ℓ_∞ with \mathcal{S} , \mathcal{U} or \mathcal{M} results in homogeneous rules, too. However, for \mathcal{C} this is not the case, and we conclude the section by discussing the homogeneity (or lack thereof) of the rules that are votewise rationalizable with respect to \mathcal{C} .

Theorem 4.1. *For any distance d on votes, the voting rule \mathcal{R} that is $(\mathcal{K}, \widehat{d^p})$ -rationalizable for $\mathcal{K} \in \{\mathcal{S}, \mathcal{U}\}$ and $p \in \mathbb{N}$ is homogeneous.*

³We skip the description of STV due to space constraints, but we mention that STV is one of the very few nontrivial election systems that are in practical use in real-life political systems.

Proof. Let us consider the case of \mathcal{U} and some ℓ_p -votewise distance \widehat{d}^p first. Let \mathcal{R} be $(\mathcal{U}, \widehat{d}^p)$ -rationalizable and let $E = (C, V)$ be an election with $C = \{c_1, \dots, c_m\}$ and $V = (v_1, \dots, v_n)$. Let s be an arbitrary positive integer. We will show that $\mathcal{R}(E) = \mathcal{R}(sE)$.

Let c be a candidate in $\mathcal{R}(E)$ and let (C, U) , where $U = (u_1, \dots, u_n)$, be a \mathcal{U} -consensus witnessing this fact. For the sake of contradiction assume that $c \notin \mathcal{R}(sE)$. Let d be some \mathcal{R} -winner of sE and let (C, W) be a \mathcal{U} -consensus witnessing this fact. It is easy to see that we can pick W so that it is of the form sW' , where $W' = (w_1, \dots, w_n)$. Since c is not a winner of sE , it holds that

$$\left(\sum_{i=1}^n s(d(v_i, u_i))^p \right)^{\frac{1}{p}} > \left(\sum_{i=1}^n s(d(v_i, w_i))^p \right)^{\frac{1}{p}}.$$

Since c is a winner of E , we also have

$$\left(\sum_{i=1}^n (d(v_i, u_i))^p \right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^n (d(v_i, w_i))^p \right)^{\frac{1}{p}}.$$

However, it is easy to see that these two inequalities are contradictory, and hence $c \in \mathcal{R}(sE)$. Using the same reasoning we can show that any winner of sE must be a winner of E .

For the consensus class \mathcal{S} we can use essentially the same argument as for \mathcal{U} . Indeed, in the case of \mathcal{S} we simply have $u_1 = u_2 = \dots = u_n$ and $w_1 = w_2 = \dots = w_n$, and the rest of the argument goes through without change. \square

In contrast, \mathcal{M} -Borda, i.e., the rule obtained by combining \mathcal{M} with $\widehat{d}_{\text{swap}}$, is not homogeneous.

Example 4.2. Let \mathcal{R} be \mathcal{M} -Borda rule which is rationalized by \mathcal{M} and $\widehat{d}_{\text{swap}}$. Consider the following election.

v_1	v_2	v_3	v_4	v_5	v_6
b	a	c	c	c	d
a	b	b	b	d	a
d	d	a	a	a	b
e	e	e	e	e	e
c	c	d	d	b	c

It can be verified that in this election b is an \mathcal{M} -Borda winner, but if we replace each voter by two identical ones, the winner is c . A simple calculation shows that to become a majority winner a needs four swaps, b needs three swaps, c needs four swaps, and d needs five swaps. Thus, b is a winner according to \mathcal{M} -Borda. However, if we replace each voter by two identical ones, it turns out that b needs five swaps to become a majority winner, but c requires only four (and, in fact, is the \mathcal{M} -Borda winner of the election).

For \mathcal{M} , the conclusion of Theorem 4.1 is no longer true. However, we can fully characterize homogeneous rules that can be rationalized via \mathcal{M} and a neutral ℓ_1 -votewise pseudodistance (recall that by Theorem 3.6 all such rules are necessarily \mathcal{M} -scoring rules). For convenience, we state the following theorem for scoring vectors that satisfy $\alpha_1 \leq \dots \leq \alpha_m$; it is not hard to show that this can be done without loss of generality.

Theorem 4.3. *A voting rule $\mathcal{M}\text{-}\mathcal{R}_\alpha$ with a normalized scoring vector $\alpha = (\alpha_1, \dots, \alpha_m)$ that satisfies $\alpha_1 \leq \dots \leq \alpha_m$ is homogeneous if and only if $\alpha_m = 1$ or $\alpha_{\lceil \frac{m}{2} \rceil} = 0$.*

Proof. Suppose first that $\alpha_m = 1$. Then there exists some k , $1 \leq k < m$, such that $\alpha_i = 0$ for $i \leq k$, $\alpha_i = 1$ for $i > k$. Consider an election $E = (C, V)$ with $V = (v_1, \dots, v_n)$ and fix an integer $s > 1$. If there are candidates ranked in top k positions by a majority of voters, these candidates form the set of winners both in E and in sE . Otherwise, each candidate has a strictly positive score under $\mathcal{M}\text{-}\mathcal{R}_\alpha$. Moreover, in this case the $\mathcal{M}\text{-}\mathcal{R}_\alpha$ -score of each $c \in C$ is simply the difference between $\lfloor \frac{m}{2} \rfloor + 1$ and the number of voters that rank c in top k positions. Hence the winners in both E and sE are the candidates that are ranked in top k positions by the maximum number of voters.

Now, set $h = \lceil \frac{m}{2} \rceil$ and suppose that $\alpha_h = 0$. Again, consider an election $E = (C, V)$ with $V = (v_1, \dots, v_n)$ and an integer $s > 1$. If m is odd or $\alpha_{h+1} = 0$, then by the pigeonhole principle there is at least one candidate $c \in C$ that is ranked in top h positions by a majority of voters. In this case, the sets of winners in both E and sE consist of all such candidates. It remains to consider the case $m = 2h$, $\alpha_{h+1} = 1$. If there exists a candidate $c \in C$ that is ranked among the top h positions by more than half of the voters, then the same argument as in the previous case shows that $\mathcal{M}\text{-}\mathcal{R}_\alpha(E) = \mathcal{M}\text{-}\mathcal{R}_\alpha(sE)$. On the other hand, if no candidate is ranked among the top h positions by more than half of the voters, then we see—again by the pigeonhole principle—that each candidate is ranked among the top h positions by exactly $\frac{n}{2}$ voters (note that this case is possible only if n is even). Thus, the \mathcal{M} -score of each candidate is of the form α_j , $j > h$. Further, each candidate's score remains the same in E and in sE . Thus, E and sE have the same sets of winners under $\mathcal{M}\text{-}\mathcal{R}_\alpha$.

It remains to argue that in all other cases, i.e., if $\alpha_m > 1$ and $\alpha_h > 0$, the rule $\mathcal{M}\text{-}\mathcal{R}_\alpha$ is not homogeneous. For readability, we will first consider the case $\alpha_3 > 1$ (note that this implies $\alpha_2 = 1$). This will be done in the following lemma. Later, we will show how to use ideas from this proof for the general case.

Lemma 4.4. *If $\alpha_3 > 1$ then the rule $\mathcal{M}\text{-}\mathcal{R}_\alpha$ is not homogeneous.*

Proof. Recall that we have $\alpha_1 = 0$, $\alpha_2 = 1$. Set $\alpha = \alpha_3$. We start by considering the case $m = 3$; later, we will generalize our construction to arbitrary values of m . Suppose first that α is a rational number, i.e., $\alpha = \frac{p}{q}$, where p and q are relatively prime. We construct an election $E = (C, V)$, where $C = \{a, b, c\}$ and V consists of the following votes:

1. $2p + q + 1$ votes $a \succ b \succ c$,
2. $2q + p + 1$ votes $b \succ c \succ a$, and

3. $p + q - 2$ votes $c \succ b \succ a$.

We observe that $|V| = 4(p + q)$, each of the candidates a and b gets p points as their \mathcal{M} -scores, and c gets $p + q + 3$ points (we use the fact that $q \geq 2$). Thus, the \mathcal{M} -score of candidate c is higher than that of a and b , and hence both a and b are winners of E .

The reader can verify that if we consider the election $2E = (C, 2V)$, then the \mathcal{M} -scores of candidates a and b are, respectively, $(2q - 1)\alpha = 2p - \alpha$ and $2p - 1$. Since $\alpha > 1$, it cannot be the case that both a and b are winners of $2E$. Thus, in this case $\mathcal{M}\text{-}\mathcal{R}_\alpha$ is not homogeneous.

Now, if α is irrational, consider its continued fraction expansion $\alpha = (a_0, a_1, \dots)$, and the successive convergents $\frac{h_i}{k_i}$, $i = 0, 1, \dots$, where $h_0 = a_0$, $k_0 = 1$, $h_1 = a_1 h_0 + 1$, $k_1 = a_1$, and $h_i = a_i h_{i-1} + h_{i-2}$, $k_i = a_i k_{i-1} + k_{i-2}$ for $i \geq 2$. We know that for even values of i we have $\frac{h_i}{k_i} < \alpha$ and $|\alpha - \frac{h_i}{k_i}| < \frac{1}{k_i k_{i+1}}$. Also, it is not hard to show that for any $N > 0$ there exists an even value of i such that $k_{i+1} > N$. Thus, we pick an even i such that $k_{i+1} > \frac{2}{\alpha - 1}$ (recall that $\alpha > 1$). We obtain

$$0 < \alpha - \frac{h_i}{k_i} < \frac{1}{k_i k_{i+1}} < \frac{\alpha - 1}{2k_i}.$$

Now, set $p = h_i$, $q = k_i$, let $\varepsilon = \alpha - \frac{p}{q}$, and use the same construction as above. In E , the \mathcal{M} -score of a is $q\alpha$, the \mathcal{M} -score of b is $p < q\alpha$, and the \mathcal{M} -score of c exceeds that of a and b , so b is the unique winner. On the other hand, in $2E$ the \mathcal{M} -score of a is $(2q - 1)\alpha = 2p + 2q\varepsilon - \alpha$, while the \mathcal{M} -score of b is $2p - 1$. We have $\varepsilon < \frac{\alpha - 1}{2q}$, so a has a lower \mathcal{M} -score than b , and hence b cannot be the winner of $2E$. Thus, in this case, too, our rule is not homogeneous.

Finally, it is easy to see that for the case of $m > 3$ it suffices to modify the above construction by adding $m - 3$ dummy candidates that each voter ranks last (in some arbitrary order). \square

We will now consider the general case. Since we have $\alpha_m > 1$, the scoring vector can be written as

$$\underbrace{(0, \dots, 0)}_x, \underbrace{(1, \dots, 1)}_y, \underbrace{(\alpha, \dots, \alpha)}_z, \alpha_{x+y+z+1}, \dots, \alpha_m$$

for some $\alpha > 1$ and $x, y, z \geq 1$. If $x = y = 1$, the condition of Lemma 4.4 is satisfied, so we can assume that this is not the case. Also, since $\alpha_h \neq 0$, we have $x < h$.

We will now modify the construction of the election $E = (C, V)$ from the proof of Lemma 4.4 as follows. We set $C = \{a, b, c\} \cup D$, where $D = \{d_1, \dots, d_{m-3}\}$. If α is a rational number, we set $\alpha = \frac{p}{q}$, where p and q are relatively prime; otherwise, we construct p and q as in the proof of Lemma 4.4.

We replace each voter in V with a voter that grants the same number of points to a , b , and c as the replaced voter. Thus, we construct

1. $2p + q + 1$ voters that rank a in the 1-st position, b in the $(x + 1)$ -st position and c in the $(x + y + 1)$ -st position,

2. $2q + p + 1$ voters that rank b in the 1-st position, c in the $(x + 1)$ -st position and a in the $(x + y + 1)$ -st position, and
3. $p + q - 2$ voters that rank c in the 1-st position, b in the $(x + 1)$ -st position and a in the $(x + y + 1)$ -st position.

The candidates in D are ranked in an arbitrary order among the remaining positions in these votes.

We also construct additional voters so as to ensure that the candidates in D are not among the winners of E . Set $s = 3p + 2q$. For each candidate $d \in D$ we create s pairs of voters with the following preferences. In each pair, the first voter ranks a in the 1-st position and b in the $(x + y + 1)$ -st position, and the second voter ranks b in the 1-st position and a in the $(x + y + 1)$ -st position. Both of these voters rank d in the $(x + 1)$ -st position. Finally, the voters in each pair rank the candidates in $(D \setminus \{d\}) \cup \{c\}$ in each of the votes in the opposite order in the remaining positions. Since $x < h$, this ensures that no voter in $D \cup \{c\}$ is ranked in the top x positions by both voters in the pair. Altogether, we have $4p + 4q + 2s(m - 3)$ voters.

Since both a and b are ranked in the first position by exactly one voter in each newly constructed pair, these new votes do not affect the \mathcal{M} -scores of a and b . Indeed, it is easy to see that a has $q\alpha$ points and b has p points. Similarly, the \mathcal{M} -score of c is still at least $p + q + 3$. Finally, the \mathcal{M} -score of any $d \in D$ is at least $s + 1 - 2(p + q) > p$ by our choice of s . Thus, for the resulting election E we have $\mathcal{M}\text{-}\mathcal{R}_\alpha(E) = \{a, b\}$ if α is rational and $\mathcal{M}\text{-}\mathcal{R}_\alpha(E) = \{b\}$ if α is irrational. On the other hand, as in the proof of Lemma 4.4, in $2E$ the \mathcal{M} -score of a is $(2q - 1)\alpha$, the \mathcal{M} -score of b is $2p - 1$, and $(2q - 1)\alpha < 2p - 1$, so $\mathcal{M}\text{-}\mathcal{R}_\alpha(2E) \neq \mathcal{M}\text{-}\mathcal{R}_\alpha(E)$. \square

We have demonstrated that many voting rules that are ℓ_1 -votewise distance rationalizable with respect to \mathcal{M} are not homogeneous. However, homogeneity appears to be easier to achieve if we use the ℓ_∞ -norm instead of ℓ_1 . For example, Simplified Bucklin has been shown to be $(\mathcal{M}, \widehat{d_{\text{ser}}^\infty})$ -rationalizable [12] and it can be shown to be homogeneous. Indeed, this follows from a more general result stating that ℓ_∞ -votewise rules are homogeneous as long as they are rationalized via a consensus class that satisfies a fairly weak requirement.

Definition 4.5. *A consensus class \mathcal{K} is split-homogeneous if the following two conditions hold:*

- (a) *If U is a \mathcal{K} -consensus then for every positive integer s it holds that sU is a \mathcal{K} -consensus with the same winner;*
- (b) *If U and W are two profiles, with n votes each, such that $U + W$ is a \mathcal{K} -consensus, then at least one of U and W is a \mathcal{K} -consensus with the same winner as $U + W$.*

It turns out that combining a split-homogeneous consensus class with an ℓ_∞ -votewise distance produces a homogeneous rule.

Theorem 4.6. *For any split-homogeneous consensus class \mathcal{K} and any pseudodistance d on votes, the voting rule that is rationalized via \mathcal{K} and $\widehat{d^\infty}$ is homogeneous.*

Proof. We will prove that for any election $E = (C, V)$ we have $\mathcal{R}(E) = \mathcal{R}(2E)$; the general case is similar. Let c be a winner of E and let U be the consensus profile that witnesses this. Then for each $U' \in \mathcal{K}$ we have

$$k = \widehat{d^\infty}(V, U) \leq \widehat{d^\infty}(V, U'). \quad (1)$$

Due to the nature of ℓ_∞ -metric we have

$$\widehat{d^\infty}(2V, 2U) = \widehat{d^\infty}(V, U) = k, \quad (2)$$

and $2U$ is a consensus profile by condition (a) of Definition 4.5. Suppose that $c \in \mathcal{R}(U) = \mathcal{R}(2U)$ is not a winner of $2E$. Then there exist a profile $X + Y \in \mathcal{K}$, $|X| = |Y| = n$, such that $\widehat{d^\infty}(2V, X + Y) < k$. Since our distance is an ℓ_∞ one, we have

$$\widehat{d^\infty}(V, X) < k \quad \text{and} \quad \widehat{d^\infty}(V, Y) < k.$$

However by condition (b) either $X \in \mathcal{K}$ or $Y \in \mathcal{K}$ which contradicts (1) and (2). \square

It is not hard to see that the consensus classes \mathcal{S} , \mathcal{U} and \mathcal{M} are split-homogeneous. Thus, we obtain the following corollary.

Corollary 4.7. *For any $\mathcal{K} \in \{\mathcal{S}, \mathcal{U}, \mathcal{M}\}$ and any pseudodistance d on votes, the voting rule that is rationalized via \mathcal{K} and $\widehat{d^\infty}$ is homogeneous.*

In contrast, the Condorcet consensus is not split-homogeneous.

Example 4.8. Consider the following election $E = (C, V)$ with $C = \{a, b, c, d, e\}$ and $V = (v_1, \dots, v_{12})$:

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}
a	b	c	d	e	c	e	a	b	c	d	c
b	c	d	e	a	a	d	e	a	b	c	a
c	d	e	a	b	b	c	d	e	a	b	b
d	e	a	b	c	d	b	c	d	e	a	d
e	a	b	c	d	e	a	b	c	d	e	e

Voters v_1, \dots, v_5 form a Condorcet cycle, and voters v_7, \dots, v_{11} are obtained from voters v_1, \dots, v_5 by reversing their preferences. Voters v_6 and v_{12} are identical and rank c first. It is not hard to verify that c is the Condorcet winner in E . On the other hand, in elections $E_1 = (C, V_1)$ and $E_2 = (C, V_2)$, where $V_1 = (v_1, \dots, v_6)$ and $V_2 = (v_7, \dots, v_{12})$, c is not a Condorcet winner both in E_1 and in E_2 .

Indeed, we can construct an ℓ_∞ -votewise distance that combined with \mathcal{C} yields a non-homogeneous rule.

v_1	v_2	v_3	v_4	v_5	v_6
c	a	c	b	a	c
↓	b	a	↓	↓	↓
b	c	b	a	b	a
↓	↓	↓	↓	↓	↓
a	↓	↓	c	c	b
↓	↓	↓	↓	↓	↓

Table 1: Election $E = (C, V)$ from the proof of Proposition 4.9.

Proposition 4.9. *There exists a distance d on votes such that the rule rationalized by \mathcal{C} and $\widehat{d^\infty}$ is not homogeneous.*

Proof. We first define two additional types of swap operations for preference orders. A *forward distance-two swap* of candidate c transforms this preference order as follows: the candidate ranked two positions higher than c , is moved from his current position and placed directly below c . If c were ranked first or second, a forward distance-two swap is not defined. For example, if $C = \{a, b, c, d, e\}$ and the preference order is $a \succ b \succ c \succ d \succ e$, then the result of a forward distance-two swap of candidate c will be $b \succ c \succ a \succ d \succ e$. A *backward distance-two swap* is defined similarly.

It is easy to see that a single forward distance-two swap can be reversed by applying a single backward distance-two swap and the other way round.

We can now define our distance d . Let us fix some candidate set $C = \{c_1, \dots, c_m\}$. For each two preference orders u and v over C we define $d(u, v)$ to be the minimal number of swaps of adjacent candidates and distance-two swaps of candidates needed to transform vote u into vote v . It is easy to see that d indeed is a distance because it counts the number of reversible operations that transform one preference order into the other. As before, $\widehat{d^\infty}$ is the ℓ_∞ -votewise extension of d to a distance over elections.

Let \mathcal{R} be a voting rule that is $(\mathcal{C}, \widehat{d^\infty})$ -rationalized. We will now build an election $E = (C, V)$ such that $\mathcal{R}(E) \neq \mathcal{R}(2E)$. We set $C = \{a, b, c, x_1, \dots, x_t\}$ where t is a sufficiently large integer. (After reading our description of the votes in V it will become clear what we mean by *sufficiently large*.) The set of voters V will contain six voters, v_1, \dots, v_6 , whose preference orders are presented in Table 1. Note that in this table we only showed how candidates in $\{a, b, c\}$ are ranked. Remaining candidates are ranked in the places of arrows, in such a way that (a) each candidate in $\{a, b, c\}$ is preferred to each candidate x_i , $1 \leq i \leq t$, by a majority of voters, and (b) one needs at least three swaps or distance-two swaps to change the relative order of two candidates from $\{a, b, c\}$ that are separated by an arrow.

We have the following results of head-to-head contests in E : four voters prefer a to b , a and c are tied, and b and c are tied. Thus, a single swap of a and c in vote v_3 makes a a Condorcet winner of the election. On the other hand, it is easy to see that being allowed one (possibly distance-two) swap per vote, it is impossible to make either b or c the Condorcet winner. Thus, a is the unique \mathcal{R} -winner of E .

In $2E$, similarly, a single swap (within one of the copies of v_3) suffices to make a the Condorcet winner. However, now also a single swap per vote suffices to make c a Condorcet winner. Indeed, in one copy of v_2 we transform $a \succ b \succ c$ into $a \succ c \succ b$ and in the other into $b \succ c \succ a$. These two transformations allow c to break a tie with both a and b , and become the Condorcet winner. \square

The combination of \mathcal{C} and an ℓ_1 -votewise distance does not necessarily lead to a homogeneous rule either: it is well known that the Dodgson rule is not homogeneous (see, e.g., [4] for a recent survey of Dodgson rule deficiencies), yet it is $(\mathcal{C}, \widehat{d}_{\text{swap}})$ -rationalizable. In fact, we are not aware of any homogeneous voting rule that is ℓ_1 -votewise distance-rationalizable with respect to \mathcal{C} . In contrast, we can construct a homogeneous rule that is ℓ_∞ -votewise distance-rationalizable with respect to \mathcal{C} by replacing ℓ_1 with ℓ_∞ in the rationalization of the Dodgson rule. We will call the resulting rule Dodgson^∞ ; the next section will explain the name of the rule. To prove that Dodgson^∞ is homogeneous, we will first explain how to determine the winners under this rule. It turns out that, in contrast to the Dodgson rule itself, Dodgson^∞ admits a polynomial-time winner determination algorithm.

Proposition 4.10. *Given an election $E = (C, V)$, the problem of computing the $(\mathcal{C}, \widehat{d}_{\text{swap}}^\infty)$ -score of a given candidate $c \in C$ is in P.*

Proof. Consider the following algorithm:

1. Set $k = 0$.
2. If c is a Condorcet winner of E then return k .
3. For each vote where c is not ranked first, swap c and its predecessor.
4. Increase k by 1.
5. Go to Step 2.

Suppose that c 's $(\mathcal{C}, \widehat{d}_{\text{swap}}^\infty)$ -score is k . Since our algorithm does not stop until it finds a Condorcet consensus, it will not stop before step k . On the other hand, there exists a Condorcet consensus U with winner c such that $\widehat{d}_{\text{swap}}^\infty(E, U) = k$. Note that we can assume that U has been obtained from E by shifting c upwards, and, moreover, c has been shifted by k positions in at least one vote, and by at most k positions in all remaining votes. Now, consider an election U' in which c has been shifted upwards by exactly k positions in each vote or moved to the top position if its rank is smaller than k . Clearly, U' is also a Condorcet consensus, and $\widehat{d}_{\text{swap}}^\infty(E, U') = k$. Moreover, U' will be discovered at the k -th step of our algorithm.

Since the algorithm terminates after at most $|C|$ iterations, it is easy to see that it runs in polynomial time. This completes the proof. \square

Using the algorithm given in the proof of Proposition 4.10, it is not hard to show that Dodgson^∞ is homogeneous.

Proposition 4.11. *Dodgson[∞] is homogeneous.*

Proof. Let $E = (C, V)$ be an election where $V = (v_1, \dots, v_n)$ and let c be a candidate in C . The algorithm in the proof of Proposition 4.10 finds the smallest value of k such that after shifting a given candidate upwards by k positions in each vote, this candidate becomes the Condorcet winner. Therefore, if two votes are identical before running the algorithm, these votes remain identical in the resulting Condorcet consensus. This shows that Dodgson[∞]-score of c is the same in E and in kE . \square

4.1 ℓ_1 -Votewise Rules versus ℓ_∞ -Votewise Rules

Inspired by Proposition 4.11, in this section we take a brief detour from the discussion of homogeneity and monotonicity in votewise rules, and discuss the relationship between ℓ_1 -votewise rules and ℓ_∞ -votewise rules. It turns out that in a certain weak sense, ℓ_∞ -votewise rules are approximations of the corresponding ℓ_1 -votewise rules. The next theorem expresses this “weak sense” precisely.

Theorem 4.12. *For any consensus class $\mathcal{K} \in \{\mathcal{S}, \mathcal{U}, \mathcal{M}, \mathcal{C}\}$ and any distance d on votes, let \mathcal{R} and \mathcal{R}^∞ be the voting rules rationalized via \mathcal{K} and \widehat{d} and \widehat{d}^∞ , respectively. Let $\text{score}_E^{\mathcal{R}}(c)$ (respectively, $\text{score}_E^{\mathcal{R}^\infty}(c)$) denote the $(\mathcal{K}, \widehat{d})$ -score (respectively, $(\mathcal{K}, \widehat{d}^\infty)$ -score) of a candidate c in an election $E = (C, V)$. Then for each election $E = (C, V)$ and each candidate $c \in C$ we have*

$$\text{score}_E^{\mathcal{R}^\infty}(c) \leq \text{score}_E^{\mathcal{R}}(c) \leq |V| \cdot \text{score}_E^{\mathcal{R}^\infty}(c).$$

Proof. Consider an election $E = (C, V)$ with $C = \{c_1, \dots, c_m\}$ and $V = (v_1, \dots, v_n)$, and fix a candidate $c \in C$.

We first claim that $\text{score}_E^{\mathcal{R}}(c) \leq |V| \cdot \text{score}_E^{\mathcal{R}^\infty}(c)$. Let (C, W) , where $W = (w_1, \dots, w_n)$, be a \mathcal{K} -consensus where i is a winner and such that $\text{score}_E^{\mathcal{R}^\infty}(c) = \widehat{d}^\infty(V, W)$. By definition, we have $\text{score}_E^{\mathcal{R}}(c) \leq \widehat{d}(V, W) = \sum_{i=1}^n d(v_i, w_i) \leq n \max\{d(v_i, w_i) \mid 1 \leq i \leq n\} = n\widehat{d}^\infty(V, W) = |V| \cdot \text{score}_E^{\mathcal{R}^\infty}(c)$, which proves the claim.

On the other hand, let (C, U) , where $U = (u_1, \dots, u_n)$, be a \mathcal{K} -consensus where c is the winner and for which $\text{score}_E^{\mathcal{R}}(c) = \widehat{d}(V, U)$. By definition, it holds that $\text{score}_E^{\mathcal{R}}(c) = \widehat{d}(V, U) = \sum_{i=1}^n d(v_i, u_i) \geq \max\{d(v_i, u_i) \mid 1 \leq i \leq n\} = \widehat{d}^\infty(V, U) \geq \text{score}_E^{\mathcal{R}^\infty}(c)$, and so $\text{score}_E^{\mathcal{R}^\infty}(c) \leq \text{score}_E^{\mathcal{R}}(c)$. This completes the proof. \square

In other words, any ℓ_∞ -votewise rule is a $|V|$ -approximation of a corresponding ℓ_1 -votewise rule in the sense of Caragiannis et al. [5, 6]. For the majority consensus we can strengthen the approximation guarantee from $|V|$ to $\lceil \frac{|V|}{2} + 1 \rceil$ using the fact that we only need the majority of the voters to rank a candidate first for him to be the \mathcal{M} -winner.

Of course, these approximations are very weak as they depend linearly on the number of voters; their appeal is in their generality. Further, since for the Dodgson rule its ℓ_∞ -variant is homogeneous and polynomial-time computable, an appealing conjecture is that replacing ℓ_1 with ℓ_∞ in the rationalization of a voting rule is a general recipe for designing

voting rules that are homogeneous and admit an efficient winner determination algorithm. It is unlikely that this conjecture holds unconditionally, but it would be very interesting to identify sufficient conditions for it to hold. In particular, it would be interesting to determine whether the $(\mathcal{S}, \widehat{d}_{\text{swap}}^\infty)$ -rationalizable rule, i.e. Kemeny^∞ admits a polynomial-time winner determination algorithm (briefly put, Kemeny is the rule that is rationalized by \mathcal{S} and $\widehat{d}_{\text{swap}}$). Of course, there are much better approximation algorithms known for Kemeny [1, 9, 14] and the value of resolving the above question lies in it enhancing our understanding of Kemeny and relations between ℓ_1 - and ℓ_∞ -votewise rules.

5 Monotonicity

Monotonicity is a very desirable property of voting rules: it stipulates that campaigning in favor of a candidate should not hurt him. While homogeneity seems to be essentially a function of the norm and the consensus class (as illustrated by Theorem 4.1 and Theorem 4.6, which hold for any distance d on votes), monotonicity seems to be most closely related to the properties of the distance on votes. Therefore, in this section we propose several notions of monotonicity for distances on votes that, combined with appropriate norms and consensus classes, produce a monotone rule. We do not consider the Condorcet consensus in this section: even a very well-behaved distance such as $\widehat{d}_{\text{swap}}$ may produce a non-monotone rule when combined with \mathcal{C} (recall that the resulting rule is Dodgson , which is known to be non-monotone (see, e.g., [4]). Also, for simplicity, we focus on ℓ_1 -votewise rules and ℓ_∞ -votewise rules.

Let C be a set of candidates and let d be a distance on votes. How can we specify a condition on d so that voting rules rationalized using this distance are monotone? Consider an election with a winner c , a vote y , a vote $x \in \mathcal{P}(C, c)$ and a vote $z \in \mathcal{P}(C, a)$ for some $a \neq c$. It is tempting to require that for any vote y' obtained from y by pushing c forward it holds that $d(y', x) \leq d(y, x)$ and $d(y', z) \geq d(y, z)$. However, this condition turns out to be so strong that no reasonable distance can satisfy it. Indeed, suppose that y ranks c in position three or lower, and y' is obtained from y by shifting c by one position. Then y does not rank c in the first position, and our condition should hold for $z = y'$, implying $d(y, y') \leq 0$, which is clearly impossible.

Thus, we need to relax the condition above. There are two ways of doing so. First, we can require that when we move c forward in the vote, the distance to x declines faster than the distance to z . Alternatively, instead of imposing this condition for all $x \in \mathcal{P}(C, c)$ and $z \in \mathcal{P}(C, a)$, we can require that it holds for the closest vote that ranks c first, and the closest vote that ranks a first, respectively. We will now show that both relaxations, which we call, respectively, *relative monotonicity* and *min-monotonicity*, lead to meaningful conditions that are satisfied by some natural distances, and, combined with appropriate consensus classes, result in monotone voting rules. We consider relative monotonicity first.

Definition 5.1. *Let C be a set of candidates. We say that a distance d on $\mathcal{P}(C)$ is relatively monotone if for each $c \in C$, every two preference orders y and y' such that y' is identical*

to y except that y' ranks c higher than y , and every two preference orders x and z such that x ranks c first and z does not, it holds that

$$d(x, y) - d(x, y') \geq d(z, y) - d(z, y').$$

As a quick sanity check, we note that the swap distance, d_{swap} , satisfies the relative monotonicity condition. Indeed, let $d = d_{\text{swap}}$ and let C be a set of candidates, c be a candidate in C , and let y, y', x , and z be as in the definition of relative monotonicity. In addition, let k be a positive integer such that y' is identical to y except in y' candidate c is ranked k positions higher. We need k swaps to transform y into y' so $d(y, y') = k$. We first note that $d(x, y) - d(x, y') = k$. This is so because the swap distance measures the number of inverses between two preference orders. As x ranks c on top and y' ranks it k positions higher than y does (without any other changes), the number of inverses between x and y' is the same as that between x and y less k . By the triangle inequality $d(z, y) \leq d(z, y') + d(y', y) = d(z, y') + k$, hence $d(z, y) - d(z, y') \leq k$ and this completes the proof.

Relative monotonicity of a distance on votes naturally translates to the monotonicity of the resulting voting rule, provided we use ℓ_1 as a norm and either \mathcal{S} or \mathcal{U} as a consensus.

Theorem 5.2. *Let \mathcal{R} be a voting rule rationalized by $(\mathcal{K}, \widehat{d})$, where $\mathcal{K} \in \{\mathcal{S}, \mathcal{U}\}$ and d is a relatively monotone distance on votes. Then \mathcal{R} is monotone.*

Proof. Let $E = (C, V)$ be an election, where $V = (v_1, \dots, v_n)$, and $c \in C$ be a candidate such that $c \in \mathcal{R}(E)$. Let $E' = (C, V')$, where $V' = (v'_1, \dots, v'_n)$, be an arbitrary election that is identical to E except one voter, say v'_1 , ranks c higher ceteris paribus. It suffices to show that $c \in \mathcal{R}(E')$.

To show this, we give a proof by contradiction. Let $(C, U) \in \mathcal{K}$, where $U = (u_1, \dots, u_n)$, be a consensus witnessing that $c \in \mathcal{R}(E)$, and let (C, W) , where $W = (w_1, \dots, w_n)$, be any consensus in \mathcal{K} such that c is not a consensus winner of (C, W) . For the sake of contradiction, let us assume that $\widehat{d}(U, V') > \widehat{d}(W, V')$. If \mathcal{K} is either \mathcal{U} or \mathcal{S} , then we know that u_1 ranks c first and that w_1 does not rank c first. By relative monotonicity, this means that

$$d(u_1, v_1) - d(u_1, v'_1) \geq d(w_1, v_1) - d(w_1, v'_1). \quad (3)$$

However, since $\widehat{d}(U, V') > \widehat{d}(W, V')$ and V differs from V' only on the first voter, it holds that

$$d(u_1, v'_1) + \sum_{i=2}^n d(u_i, v_i) > d(w_1, v'_1) + \sum_{i=2}^n d(w_i, v_i). \quad (4)$$

If we add inequality (3) sideways to inequality (4), we obtain

$$d(u_1, v_1) + \sum_{i=2}^n d(u_i, v_i) > d(w_1, v_1) + \sum_{i=2}^n d(w_i, v_i).$$

That is, $\widehat{d}(U, V) > \widehat{d}(W, V)$, which is a contradiction by our choice of U . \square

However, relative monotonicity is a remarkably strong condition, not satisfied even by very natural distances that are, intuitively, monotone.

Example 5.3. Consider a scoring vector $\alpha = (0, 1, 2, 3, 4, 5)$ that corresponds to the 6-candidate Borda rule and a candidate set $C = \{c, d, x_1, x_2, x_3, x_4\}$. Consider the following four votes:

$$\begin{aligned} x &: c > d > x_1 > x_2 > x_3 > x_4, \\ z &: x_1 > c > x_2 > x_3 > x_4 > d, \\ y &: x_1 > x_2 > d > c > x_3 > x_4, \\ y' &: x_1 > x_2 > c > d > x_3 > x_4. \end{aligned}$$

Note that y and y' are identical except that in y' candidate c is ranked one position higher, and that c is ranked on top in x and is not ranked on top in z . We verify that $d_\alpha(x, y) - d_\alpha(x, y') = 0$ but $d_\alpha(z, y) - d_\alpha(z, y') = 2$. Thus, d_α is not relatively monotone.

Our second approach to monotone distances, i.e., min-monotonicity, captures the intuition that d_α in the example above should be classified as monotone. We first define min-monotonicity formally.

Definition 5.4. Let C be a set of candidates. We say that a distance d on $\mathcal{P}(C)$ is min-monotone if for every candidate $c \in C$ and every two preference orders y and y' such that y' is the same as y except that it ranks c higher, for each $a \in C \setminus \{c\}$ we have:

$$\begin{aligned} \min_{x \in \mathcal{P}(C, c)} d(x, y) &\geq \min_{x' \in \mathcal{P}(C, c)} d(x', y'), \\ \min_{z \in \mathcal{P}(C, a)} d(z, y) &\leq \min_{z' \in \mathcal{P}(C, a)} d(z', y'). \end{aligned}$$

We will now argue that for any non-decreasing scoring vector α the distance d_α is min-monotone.

Proposition 5.5. Let $\alpha = (\alpha_1, \dots, \alpha_m)$ be a normalized scoring vector. (Pseudo)distance d_α is min-monotone if and only if α is nondecreasing.

Proof. Let us fix some two distinct candidates $c = c_i$ and $e = c_j$ in C . Let y and y' be two votes that are identical, except that c is ranked on some position k in y and in y' candidate c is shifted to position k' , where $k' < k$. By Lemma 2.2, it holds that $\min_{x \in \mathcal{P}(C, c_i)} d(x, y) = 2|\alpha_k - \alpha_1|$ and $\min_{x' \in \mathcal{P}(C, c_i)} d(x', y') = 2|\alpha_{k'} - \alpha_1|$. If α is nondecreasing then $2|\alpha_k - \alpha_1| = 2\alpha_k \geq 2\alpha_{k'} = 2|\alpha_{k'} - \alpha_1|$. On the other hand, if α is not nondecreasing, then it is possible to choose k and k' , where $k' < k$, such that $2|\alpha_k - \alpha_1| < 2|\alpha_{k'} - \alpha_1|$. Thus, the first inequality from the definition of min-monotonicity is satisfied if and only if α is nondecreasing. One can analogously show that the same holds for the second inequality (in essence, the proof works by arguing that either $\text{rank}(y, e) = \text{rank}(y', e)$ or $\text{rank}(y, e) = \text{rank}(y', e) - 1$, and then showing that pushing a candidate back does not decrease his distance from being ranked first if and only if α is nondecreasing). \square

Proposition 5.5, combined with the proof of Theorem 4.9 of [12] gives the next corollary.

Corollary 5.6. *A voting rule \mathcal{R} is (U, \widehat{d}) -rationalizable for some min-monotone neutral pseudodistance d on votes if and only if \mathcal{R} can be defined via a family of nondecreasing scoring vectors (one for each number of candidates).*

In essence, Proposition 5.5 ensures that for every nondecreasing scoring vector α , \mathcal{R}_α is ℓ_1 -votewise rationalizable with respect to \mathcal{U} via a min-monotone distance over votes, and the definition of min-monotonicity ensures that the scoring vector derived in the proof of Theorem 4.9 of [12] is nondecreasing.

Min-monotonicity is also useful in the context of the majority consensus: for \mathcal{M} , we can show an analogue of Theorem 5.2 both for ℓ_1 -votewise rules and for ℓ_∞ -votewise rules.

Theorem 5.7. *Let d be a min-monotone distance on votes, and let \mathcal{R} be the voting rule rationalized by $(\mathcal{M}, \widehat{d}^{\mathcal{N}})$, where $\mathcal{N} \in \{\ell_1, \ell_\infty\}$. Then \mathcal{R} is monotone.*

Proof. Let \mathcal{R} and \widehat{d} be as in the statement of the theorem. Let $E = (C, V)$ be an election where $V = (v_1, \dots, v_n)$ and let $c \in \mathcal{R}(E)$ be one of the winners of E . Let (C, U) be a majority consensus witnessing that c is an \mathcal{R} winner of E . Let $E' = (C, V')$, where $V' = (v'_1, v'_2, \dots, v'_n)$, be an election where v'_1 is identical to v_1 except that it ranks c higher and for each i , $2 \leq i \leq n$, $v'_i = v_i$.

For the sake of contradiction, we assume that c is not an \mathcal{R} winner of E' , but that some candidate $e \in C \setminus \{c\}$ is. Let (C, W') , where $W' = (w'_1, \dots, w'_n)$ be a majority consensus witnessing that e is a winner of E' . Let us form two new \mathcal{M} consensuses, U' and W .

1. $U' = (u'_1, \dots, u'_n)$. For each i , $2 \leq i \leq n$, $u'_i = u_i$. If u_1 ranks c first then $u'_1 \in \arg \min_{x' \in \mathcal{P}(C, c)} d(x', v'_1)$, and otherwise $u'_1 = v'_1$.
2. $W = (w_1, \dots, w_n)$. For each i , $2 \leq i \leq n$, $w_i = w'_i$. If w'_1 ranks e first then $w_1 \in \arg \min_{z \in \mathcal{P}(C, e)} d(z, v_1)$, and otherwise $w_1 = v_1$.

Thus, by Lemma 3.5 and min-monotonicity of d it is easy to see that:

$$d(u'_1, v'_1) \leq d(u_1, v_1), \quad (5)$$

$$d(w'_1, v'_1) \geq d(w_1, v_1). \quad (6)$$

Now, using the fact that V and V' agree on all voters but the first one, our choice of W , and the two above inequalities, we can see that the following inequality holds:

$$\begin{aligned} \widehat{d}(U, V) &= d(u_1, v_1) + \sum_{i=2}^n d(u_i, v_i) \geq d(u'_1, v'_1) + \sum_{i=2}^n d(u_i, v_i) \\ &> d(w'_1, v'_1) + \sum_{i=2}^n d(u_i, v_i) \geq d(w_1, v_1) + \sum_{i=2}^n d(u_i, v_i) = \widehat{d}(W, V). \end{aligned}$$

However, this is a contradiction because by our choice of U , $\widehat{d}(U, V)$ is a minimal distance between V and any majority consensus with n voters. \square

However, it is not clear how to apply the notion of min-monotonicity in the context of the strong unanimity consensus. The reason is that given a profile V of voters over some candidate set C , finding an \mathcal{S} -consensus closest to V requires finding a single preference order u that minimizes the aggregated distance from V to this order. However, it need not be the case that u is a preference order that minimizes the distance from some vote $v \in V$ to a preference order that ranks $\text{top}(u)$ first.

Finally, we remark that we can combine both relaxations considered in this section, obtaining a class of distances that includes both relatively monotone distances and min-monotone distances.

Definition 5.8. *Let C be a set of candidates. We say that a distance d on $\mathcal{P}(C)$ is relatively min-monotone if for each candidate $c \in C$ and each two preference orders y and y' such that y' is identical to y except that y' ranks c higher than y , for each candidate $a \in C \setminus \{c\}$ it holds that*

$$\begin{aligned} \min_{x \in \mathcal{P}(C,c)} d(x, y) - \min_{x' \in \mathcal{P}(C,c)} d(x', y') &\geq \\ \min_{z \in \mathcal{P}(C,a)} d(z, y) - \min_{z' \in \mathcal{P}(C,a)} d(z', y'). \end{aligned}$$

Proposition 5.9. *Each distance on votes that is relatively monotone or min-monotone is relatively min-monotone.*

Proof. We show that each relatively monotone distance d is relatively min-monotone. Let C be a set of candidates, $c, a \in C$, and let $y, y' \in \mathcal{P}(C)$ be identical, except y' ranks c higher than y . Pick $\hat{x} \in \arg \min_{x' \in \mathcal{P}(C,c)} d(x', y)$, $\hat{z} \in \arg \min_{z' \in \mathcal{P}(C,a)} d(z', y')$. Then

$$\begin{aligned} \min_{x \in \mathcal{P}(C,c)} d(x, y) - \min_{x' \in \mathcal{P}(C,c)} d(x', y') &\geq d(\hat{x}, y) - d(\hat{x}, y') \geq \\ d(\hat{z}, y) - d(\hat{z}, y') &\geq \min_{z \in \mathcal{P}(C,a)} d(z, y) - \min_{z' \in \mathcal{P}(C,a)} d(z', y'). \end{aligned}$$

Thus d is relatively min-monotone. We omit the easy second part of the proof due to space. \square

For \mathcal{U} the proof of Theorem 5.2 extends to relatively min-monotone distances (and hence to min-monotone distances).

Corollary 5.10. *Any voting rule rationalized by \mathcal{U} and \hat{d} , where d is relatively min-monotone distance on votes, is monotone.*

6 Conclusions

We have discussed homogeneity and monotonicity of voting rules that are distance-rationalizable via votewise distances, focusing on ℓ_p -votewise rules, $p \in \mathbb{N} \cup \{+\infty\}$. A quick summary of our results is given in Tables 2 and 3.

	\mathcal{S}	\mathcal{U}	\mathcal{M}	\mathcal{C}
ℓ_1	Y (Th. 4.1)	Y (Th. 4.1)	Y/N (Th. 4.3)	n (Dodgson)
ℓ_∞	Y (Th. 4.6)	Y (Th. 4.6)	Y (Th. 4.6)	y (Prop. 4.11)/ n (Prop. 4.9)

Table 2: (Homogeneity) Y at the intersection of column \mathcal{K} and row \mathcal{N} indicates that for any distance d on votes the $(\mathcal{K}, \widehat{d^{\mathcal{N}}})$ -rationalizable rule is homogeneous. Y/N refers to a dichotomy result, and y/n refer to examples of homogeneous/non-homogeneous rules.

	\mathcal{S}	\mathcal{U}	\mathcal{M}
ℓ_1	rel-mon (Th. 5.2)	rel-min-mon (Cor. 5.10)	min-mon (Th. 5.7)
ℓ_∞	?	?	min-mon (Th. 5.7)

Table 3: (Monotonicity) At the intersection of column \mathcal{K} and row \mathcal{N} , we indicate a sufficient condition on d (relative monotonicity, min-monotonicity, relative min-monotonicity) for the $(\mathcal{K}, \widehat{d^{\mathcal{N}}})$ -rationalizable rule to be monotone.

Motivated by our goal, we obtained a number of results, that, while not directly related to the primary topic of our study, contribute to the general understanding of votewise rationalizable rules. In particular, we identified a natural family of voting rules, which we called \mathcal{M} -scoring rules. These rules constitute a (provably distinct) variant of scoring rules that, when counting points for a given candidate, ignore the less favorable half of the votes. We have shown that \mathcal{M} -scoring rules have a natural interpretation in the context of distance rationalizability. By establishing a relationship between rules that are rationalizable with respect to \mathcal{U} and \mathcal{M} , we resolved (in the negative) an open question about votewise rationalizability of STV posed in [11]. Also, our study of monotonicity allowed us to refine a result of [12] characterizing the class of scoring rules in terms of distance-rationalizability (our Corollary 5.6).

Our work leads to several open problems. First, we are far from having a complete understanding of homogeneity of the rules that are votewise distance-rationalizable with respect to the Condorcet consensus; even less is known about the monotonicity of such rules. Also, it would be interesting to know whether there are distances $d \neq d_{\text{swap}}$ for which the winner determination for the $(\mathcal{C}, \widehat{d^\infty})$ -rationalizable rule is easier than for the $(\mathcal{C}, \widehat{d})$ -rationalizable rule; the same question can be asked for the consensus class \mathcal{S} . We are also very much interested in finding less demanding, yet practically useful, conditions on distances that lead to monotone rules.

References

- [1] N. Ailon, M. Charikar, and A. Newman. Aggregating inconsistent information: Ranking and clustering. *Journal of the ACM*, 55(5):Article 23, 2008.
- [2] N. Baigent. Metric rationalisation of social choice functions according to principles of social choice. *Mathematical Social Sciences*, 13(1):59–65, 1987.
- [3] F. Bauer, J. Stoer, and C. Witzgall. Absolute and monotonic norms. *Numerische Matematic*, 3:257–264, 1961.
- [4] F. Brandt. Some remarks on Dodgson’s voting rule. *Mathematical Logic Quarterly*, 55(4):460–463, 2009.
- [5] I. Caragiannis, J. Covey, M. Feldman, C. Homan, C. Kaklamanis, N. Karanikolas, A. Procaccia, and J. Rosenschein. On the approximability of Dodgson and Young elections. In *Proceedings of the 20th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1058–1067. Society for Industrial and Applied Mathematics, Jan. 2009.
- [6] I. Caragiannis, C. Kaklamanis, N. Karanikolas, and A. Procaccia. Socially desirable approximations for Dodgson’s voting rule. In *Proceedings of the 11th ACM Conference on Electronic Commerce*, pages 253–262. ACM Press, June 2010.
- [7] V. Conitzer, M. Rognlie, and L. Xia. Preference functions that score rankings and maximum likelihood estimation. In *Proceedings of the 21st International Joint Conference on Artificial Intelligence*, pages 109–115. AAAI Press, July 2009.
- [8] V. Conitzer and T. Sandholm. Common voting rules as maximum likelihood estimators. In *Proceedings of the 21st Conference in Uncertainty in Artificial Intelligence*, pages 145–152. AUAI Press, July 2005.
- [9] D. Coppersmith, L. Fleisher, and A. Rurda. Ordering by weighted number of wins gives a good ranking for weighted tournaments. *ACM Transactions on Algorithms*, 6(3):Article 55, 2010.
- [10] E. Elkind, P. Faliszewski, and A. Slinko. On distance rationalizability of some voting rules. In *Proceedings of the 12th Conference on Theoretical Aspects of Rationality and Knowledge*. ACM Press, July 2009.
- [11] E. Elkind, P. Faliszewski, and A. Slinko. Good rationalizations of voting rules. In *Proceedings of the 24th AAAI Conference on Artificial Intelligence*. AAAI Press, 2010. To appear.
- [12] E. Elkind, P. Faliszewski, and A. Slinko. On the role of distances in defining voting rules. In *Proceedings of the 9th International Conference on Autonomous Agents and Multiagent Systems*, pages 375–382. IFAAMAS, 2010.

- [13] E. Ephrati and J. Rosenschein. A heuristic technique for multi-agent planning. *Annals of Mathematics and Artificial Intelligence*, 20(1–4):13–67, 1997.
- [14] C. Kenyon-Mathieu and W. Schudy. How to rank with few errors. In *Proceedings of the 39th ACM Symposium on Theory of Computing*, pages 95–103. ACM Press, June 2007.
- [15] C. Klamler. Borda and Condorcet: Some distance results. *Theory and Decision*, 59(2):97–109, 2005.
- [16] C. Klamler. The Copeland rule and Condorcet’s principle. *Economic Theory*, 25(3):745–749, 2005.
- [17] T. Meskanen and H. Nurmi. Closeness counts in social choice. In M. Braham and F. Steffen, editors, *Power, Freedom, and Voting*. Springer-Verlag, 2008.
- [18] S. Nitzan. Some measures of closeness to unanimity and their implications. *Theory and Decision*, 13(2):129–138, 1981.
- [19] L. Xia, V. Conitzer, and J. Lang. Aggregating preferences in multi-issue domains by using maximum likelihood estimators. In *Proceedings of the 9th International Conference on Autonomous Agents and Multiagent Systems*, pages 399–406. IFAAMAS, May 2010.
- [20] H. Young. Social choice scoring functions. *SIAM Journal on Applied Mathematics*, 28(4):824–838, 1975.
- [21] H. Young. Extending Condorcet’s rule. *Journal of Economic Theory*, 16(2):335–353, 1977.