Graphs and hypergraphs with no path or cycle of given length

Ervin Győri

Alfred Renyi Institute of Mathematics, Realtanoda str. 13-15. H-1053 Budapest, Hungary

After a short historical survey, we plan to cover extremal graph and hypergraph problems about paths and cycles (partly joint results with Hao Li, Nathan Lemons and Gyula Katona, Jr.). Some of the important results are as follows.

Considering the natural converse of a famous conjecture of Erdős, we studied the number of triangles in graphs not containing odd cycle of given length. We proved

Theorem 1. For any integer $k \geq 2$, let G be any C_{2k+1}-free simple graph and let $t(G)$ denote the number of the triangles in G. Then

$$t(G) \leq \frac{(2k-2)(16k-8)}{3} \ ex(n, C_{2k}).$$

Erdős and Gallai proved the following classical extremal result about paths:

Theorem 2. Let G be a graph on n vertices containing no path of length k. Then $e(G) \leq \frac{1}{2}(k-1)n$. Equality holds iff G is the disjoint union of complete graphs on k vertices.

We studied several generalizations of this theorem for hypergraphs. We proved

Theorem 3. Fix $r > 2$, let $k > r$ and let $\mathbf{H}$ be a hypergraph containing no Berge path of length k. Then $e(\mathbf{H}) \leq \frac{\mathbf{n}}{\mathbf{k}}\binom{\mathbf{k}}{\mathbf{r}}$.

On the other hand, if $k \leq r$, we have a different theorem.

We proved somewhat surprising results for Berge cycles. Our main theorem for 3-uniform hypergraphs is the following.

Theorem 4. Let $\mathbf{H}$ be a 3-uniform hypergraph containing no cycle of length $2k+1$. Then $e(\mathbf{H}) < 4k^4 n^{(k+1)/k} + 15k^4 n + 10k^2 n = c(k) n^{(k+1)/k} + O(n)$.

The theorem above has generalizations to r-uniform and non-uniform hypergraphs too.