COMPUTATIONAL INTELLIGENCE

Fuzzy Logic and Neuro-Fuzzy Systems

AGH

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FUZZY SYSTEMS

Fuzzy values are a natural way to identify certain features by people in a uncertain, precarious, imprecise, random, or incomplete environment:

- very small, small, medium, large, very large, huge, high, low,
- good, bad, not good, average, very well,
- weak, strong, moderate,
- nice, ugly, tasteful, luxurious,
- cold, lukewarm, warm, hot,
- old, young, contemporary,
- recently, early, late, in the future etc.

In the case of use such an imprecise, fuzzy linguistic (symbolic) terms in the algorithms, we come across a difficult **problem of encoding** them and operate on them because we usually cannot simply transform these terms into sharp, precise, and certain values for further computations!

We need to operate on **fuzzy values** and **fuzzy ranges** which have not **sharp bounds** and the transition between various states is **blurred**!

FUZZY SYSTEMS

We can use fuzzy logic and fuzzy systems that operate on fuzzy values and linguistic terms, namely they allow to:

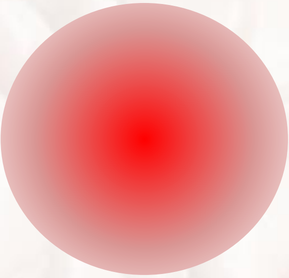
- ✓ represent linguistic and symbolic terms using fuzzy values
- ✓ assign linguistic terms the numerical fuzzy values
- ✓ fuzzify (make blur) terms and defuzzify (sharpen) values
- ✓ operate on fuzzy variables and fuzzy sets
- ✓ use unprecise symbolic data for inference
- ✓ describe and manipulate on non-precise, fuzzy, or ambiguous values
- ✓ create fuzzy rules and fuzzy ways of inference
- ✓ create and adapt fuzzy decision and classification systems
- ✓ connect fuzzy systems with neural systems and genetic approaches

FUZZY SETS

The fuzzy set is defined as a set of elements on which **the membership function $\mu_A(x)$** is defined. This function assigns each term **x** from the given space **X** a fuzzy value in the range **[0,1]**:

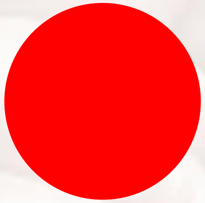
$$A = \{ (x, \mu_A(x)): x \in X\}$$

FUZZY



$$\mu_A(x): X \rightarrow [0,1]$$

SHARP



$$\mu_S(x): X \rightarrow \{0,1\}$$

There is used a **multivalued logic** to describe operations on fuzzy sets.

We use **fuzzy decision rules** operating on such sets.

Linguistic (symbolic) values are assigned **specific numerical variables**, for which functions describing the **range of variability** of parameters can be determined, e.g. „hot water" is a water which temperature is between 36 and 100 degrees Celsius.

ZADEH'S NOTATION

Let X be a space with a finite number of elements: $X = \{x_1, \ldots, x_n\}$:

$$A = \frac{\mu_A(x_1)}{x_1} + \cdots + \frac{\mu_A(x_n)}{x_n} = \sum_{i=1}^{n} \frac{\mu_A(x_i)}{x_i}$$

Here, let X = N be a set of natural numbers.

Let's try to define a fuzzy set $A \subseteq X$ of natural numbers „close to 7":

$$A = \frac{0{,}2}{4} + \frac{0{,}5}{5} + \frac{0{,}8}{6} + \frac{1}{7} + \frac{0{,}8}{8} + \frac{0{,}5}{9} + \frac{0{,}2}{10}$$

The membership function assigns various natural numbers their fuzzy values which define their fuzzy membership in the set A.

ZADEH'S NOTATION

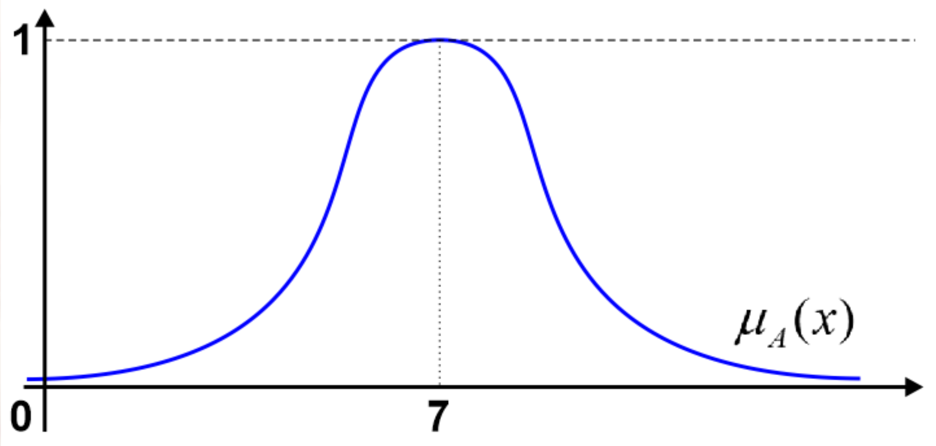
When X is an Infinite space of elements, e.g. when X=R real numbers, then we define the fuzzy set A as follows:

$$A = \int_X \frac{\mu_A(x)}{x}$$

Here, let X = R be a set of real numbers.

Let's try to define a fuzzy set A ⊆ X of real numbers „close to 7":

$$\mu_A(x) = \frac{1}{1+(x-7)^2}$$



MEMBERSHIP FUNCTIONS

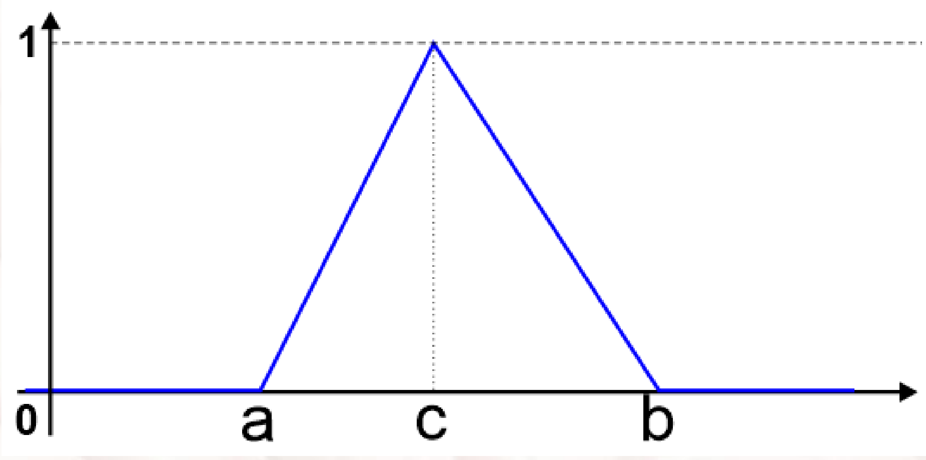
As mentioned before, the membership function $\mu_A(x)$ defines the degree of belonging of the value x to a fuzzy set A.

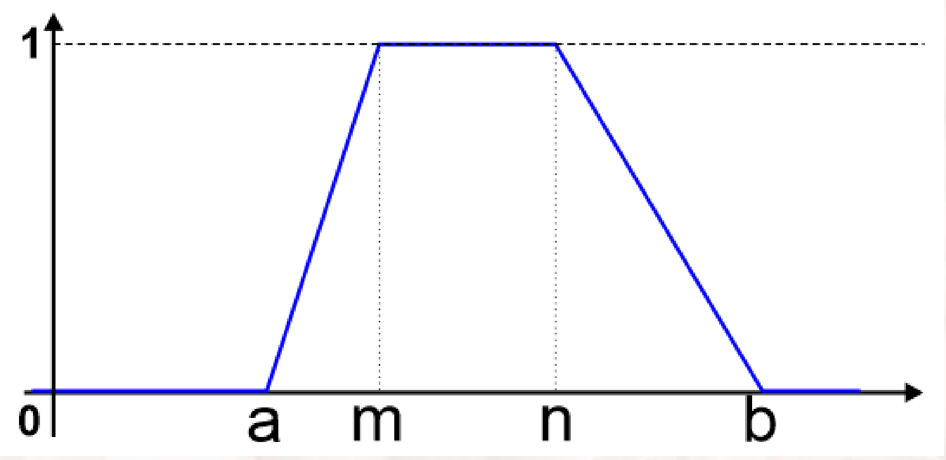
The most commonly used membership functions are:

triangular function and **trapezoidal function**:

$$\mu(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{c-a} & \text{if } x \in [a, c] \\ \frac{b-x}{b-c} & \text{if } x \in [c, b] \\ 0 & \text{if } x > b \end{cases}$$

$$\mu(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{m-a} & \text{if } x \in [a, m] \\ 1 & \text{if } x \in [m, n] \\ \frac{b-x}{b-n} & \text{if } x \in [n, b] \\ 0 & \text{if } x > b \end{cases}$$

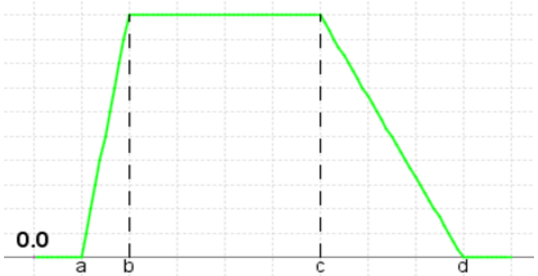




MEMBERSHIP FUNCTIONS

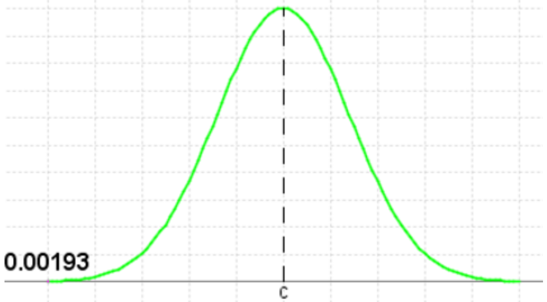
TRAPEZOIDAL MEMBERSHIP FUNCTION

$$trapez\,(x; a, b, c, d) = 0 \vee \left(1 \wedge \frac{x-a}{b-a} \wedge \frac{d-x}{d-c}\right), \qquad a < b \leqslant c < d, \quad x \in \mathbb{X}$$



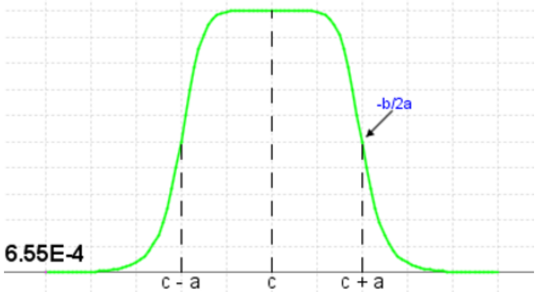
GAUSS MEMBERSHIP FUNCTION

$$gauss\,(x; a, b) = \exp\left[-\left(\frac{x-b}{a}\right)^2\right]$$



BELL MEMBERSHIP FUNCTION

$$bell\,(x; a, b, c) = \frac{1}{1 + \left|\frac{x-c}{a}\right|^{2b}}$$



PROPERTIES OF FUZZY SETS

The fuzzy set does not specify the sharp boundary between the elements that belong to a given set and those that do not.

The boundary is fuzzy and contains many values, and the membership of each element to the set is defined as a number in the range [0,1].

The variable x can be assigned a specific degree of belonging [0,1] the fuzzy set A on the basis of a certain membership function μ_A that must be defined:

$\mu_A(x) = 0$ - denotes that x does not belong to the fuzzy set A

$\mu_A(x) = 1$ - denotes full affiliation of x to the fuzzy set A

$\mu_A(x) \in (0,1)$ - means partial affiliation of x to the fuzzy set A

DEFINITIONS AND PROPERTIES

The element set $A \subseteq X$, for which $\mu_A(x) > 0$,
is called a suport of the fuzzy set A (denoted as supp A):

$$\text{supp A} = \{x \in \mathbf{X}; \mu_A(x) > 0\}$$

The height of the fuzzy set A (denoted as h(A)) is defined as:

$$h(\text{A}) = \sup_{x \in A} \mu_A(x)$$

The fuzzy set A is called normal if and only if h(A) = 1.

If the fuzzy set A is not normal then it can be normalized using the following transformation:

$$\mu_{A_N}(x) = \frac{\mu_A(x)}{h(\text{A})} \qquad \forall_{x \in \text{X}}$$

DEFINITIONS AND PROPERTIES

The fuzzy set A is empty (denoted as $A = \emptyset$)
if and only if $\mu_A(x) = 0$ for each $x \in X$.

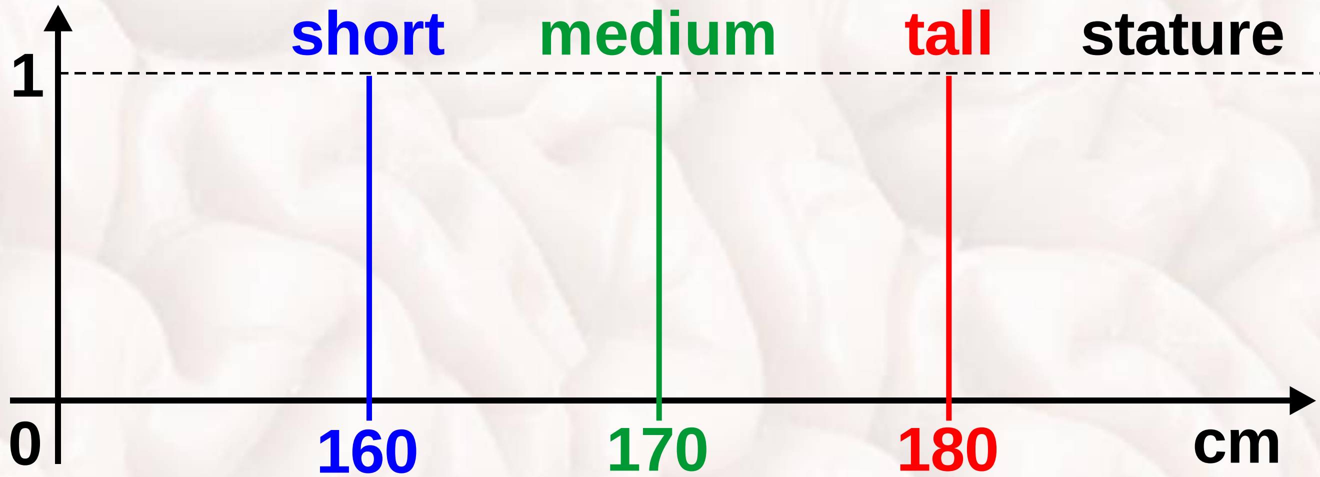
The fuzzy set A is contained in the fuzzy set B, (denoted as $A \subset B$)
if and only if $\mu_A(x) \leq \mu_B(x)$ for each $x \in X$.

The fuzzy set A is equal to the fuzzy set B, (denoted as $A = B$)
if and only if $\mu_A(x) = \mu_B(x)$ for each $x \in X$.

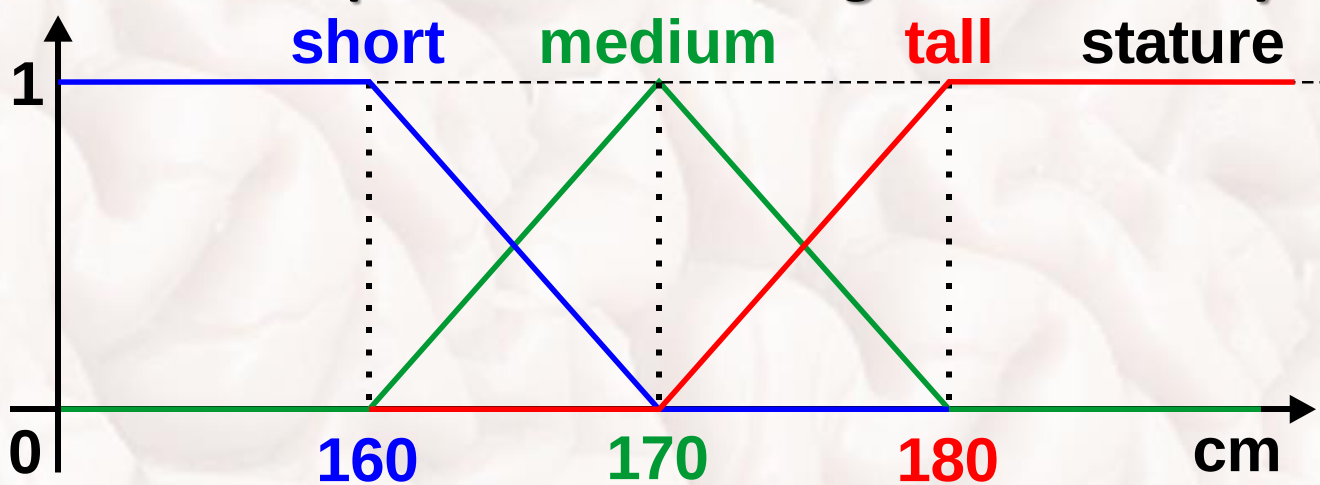
There are many other properties of fuzzy sets that can be distinguished!

FUZZY ALGEBRA

The Boolean algebra uniquely assigns data a set, specifying their membership as one value from the set {0,1}:



The fuzzy algebra assigns data a set using values in the range [0,1] to determine their membership with different degree of certainty or probability:



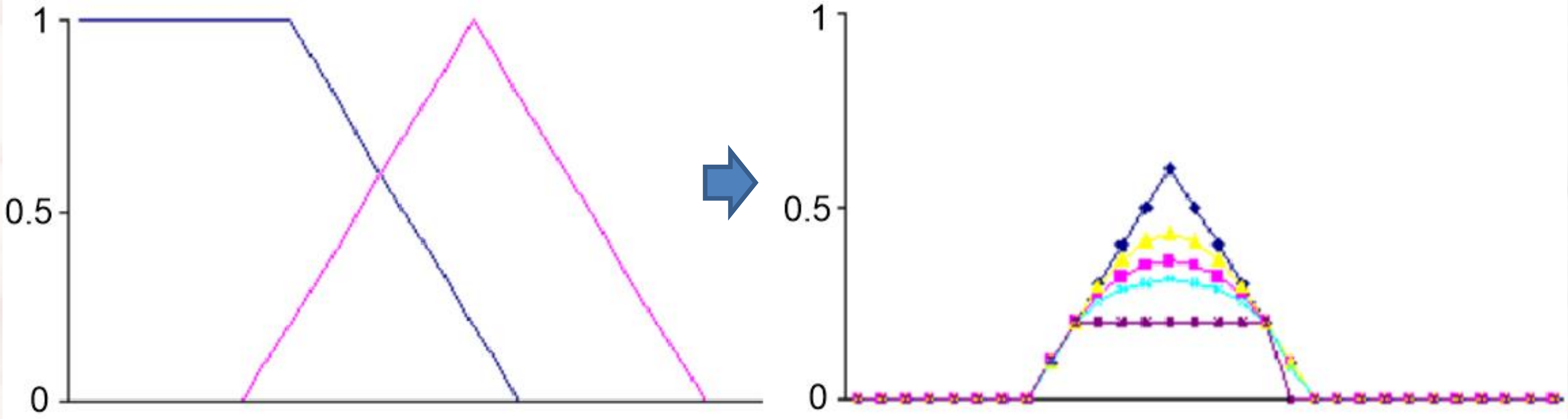
FUZZY OPERATORS

In fuzzy logic, we use fuzzy operators, which enable us to perform fuzzy logic operations on fuzzy sets:

- **Fuzzy AND called T-norm (commonly modeled as MIN),**
- **Fuzzy OR called S-norm (commonly modeled as MAX):**

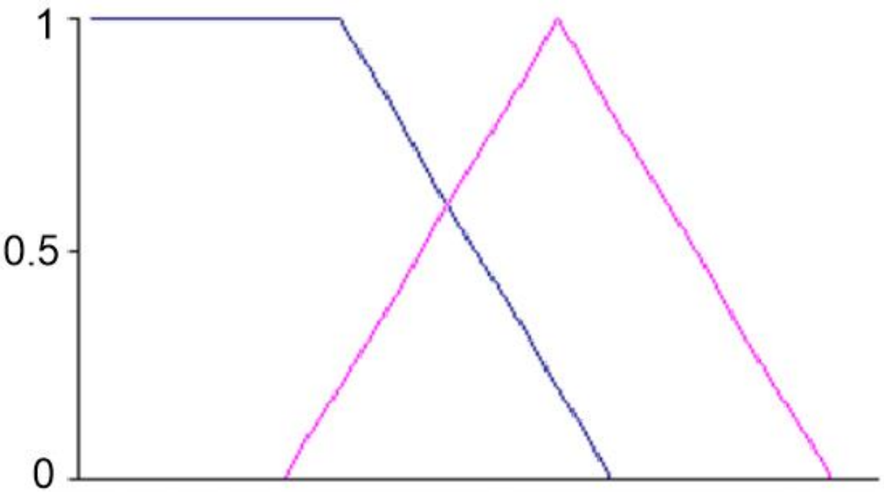
AND T-Norm $T(\mu_A(x), \mu_B(x))$	OR S-Norm $S(\mu_A(x), \mu_B(x))$
Minimum $\mathrm{MIN}(\mu_A(x), \mu_B(x))$	Maximum $\mathrm{MAX}(\mu_A(x), \mu_B(x))$
Algebraic product $\mu_A(x)\mu_B(x)$	Algebraic sum $\mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x)$
Drastic product $\mathrm{MIN}(\mu_A(x), \mu_B(x))$ if $\mathrm{MAX}(\mu_A(x), \mu_B(x)) = 1$ 0 otherwise	Drastic sum $\mathrm{MAX}(\mu_A(x), \mu_B(x))$ if $\mathrm{MIN}(\mu_A(x), \mu_B(x)) = 0$ 1 otherwise
Lukasiewicz AND (Bounded Difference) $\mathrm{MAX}(0, \mu_A(x) + \mu_B(x) - 1)$	Lukasiewicz OR (Bounded Sum) $\mathrm{MIN}(1, \mu_A(x) + \mu_B(x))$
Einstein product $\mu_A(x)\mu_B(x)/(2 - (\mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x)))$	Einstein sum $(\mu_A(x) + \mu_B(x))/(1 + \mu_A(x)\mu_B(x))$
Hamacher product $\mu_A(x)\mu_B(x)/(\mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x))$	Hamacher sum $(\mu_A(x) + \mu_B(x) - 2\mu_A(x)\mu_B(x))/(1 - \mu_A(x)\mu_B(x))$
Yager operator $1 - \mathrm{MIN}(1, ((1 - \mu_A(x))^b + (1 - \mu_B(x)^b)^{1/b}))$	Yager operator $\mathrm{MIN}(1, (\mu_A(x)^b + \mu_B(x)^b)^{1/b})$

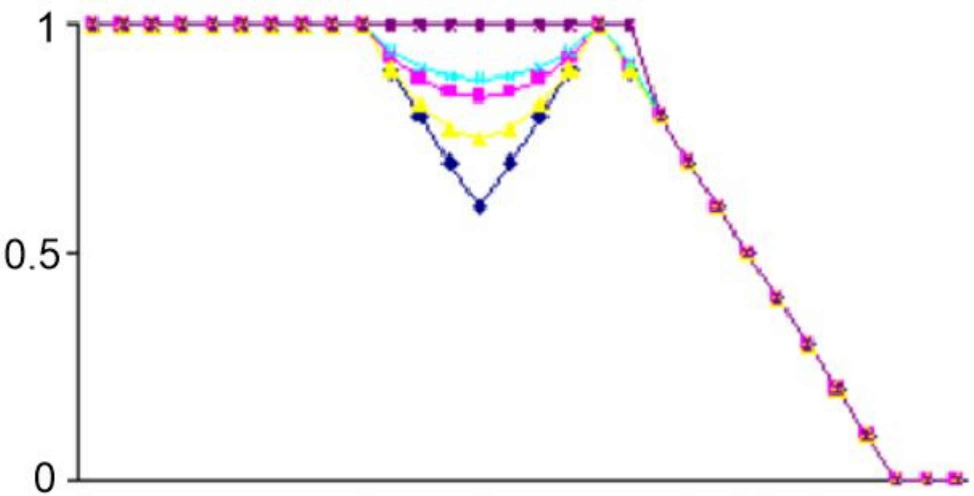
T-NORM OPERATIONS



Operator Name	Formula
Minimum (MIN)	$\mu_{A\cap B}(x) = MIN(\mu_A(x), \mu_B(x))$
Product (PROD)	$\mu_{A\cap B}(x) = \mu_A(x) \cdot \mu_B(x)$
Hamacher Product	$\mu_{A\cap B}(x) = \dfrac{\mu_A(x) \cdot \mu_B(x)}{\mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)}$
Einstein Product	$\mu_{A\cap B}(x) = \dfrac{\mu_A(x) \cdot \mu_B(x)}{2 - (\mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x))}$
Drastic Product	$\mu_{A\cap B}(x) = \begin{cases} MIN(\mu_A(x), \mu_B(x)) & \text{for } MAX(\mu_A(x), \mu_B(x)) = 1 \\ 0 & \text{in the other cases} \end{cases}$
Bounded Difference	$\mu_{A\cap B}(x) = MAX(0, \mu_A(x) + \mu_B(x) - 1)$

S-NORM OPERATIONS





Operator Name	Formula
Maximum (MAX)	$\mu_{A\cup B}(x) = MAX\,[\mu_A(x), \mu_B(x)]$
Sum	$\mu_{A\cup B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x)\cdot\mu_B(x)$
Hamacher Sum	$\mu_{A\cup B}(x) = \dfrac{\mu_A(x) + \mu_B(x) - 2\mu_A(x)\cdot\mu_B(x)}{1 - \mu_A(x)\cdot\mu_B(x)}$
Einstein Sum	$\mu_{A\cup B}(x) = \dfrac{\mu_A(x) + \mu_B(x)}{1 + \mu_A(x)\cdot\mu_B(x)}$
Drastic Sum	$\mu_{A\cup B}(x) = \begin{cases} MAX[\mu_A(x), \mu_B(x)] & \text{for } MIN(\mu_A, \mu_B) = 0 \\ 1 & \text{in the other cases} \end{cases}$
Bounded Sum	$\mu_{A\cup B}(x) = MIN\,[1, \mu_A(x) + \mu_B(x)]$

EXAMPLE: NOT, MAX, and MIN OPERATORS

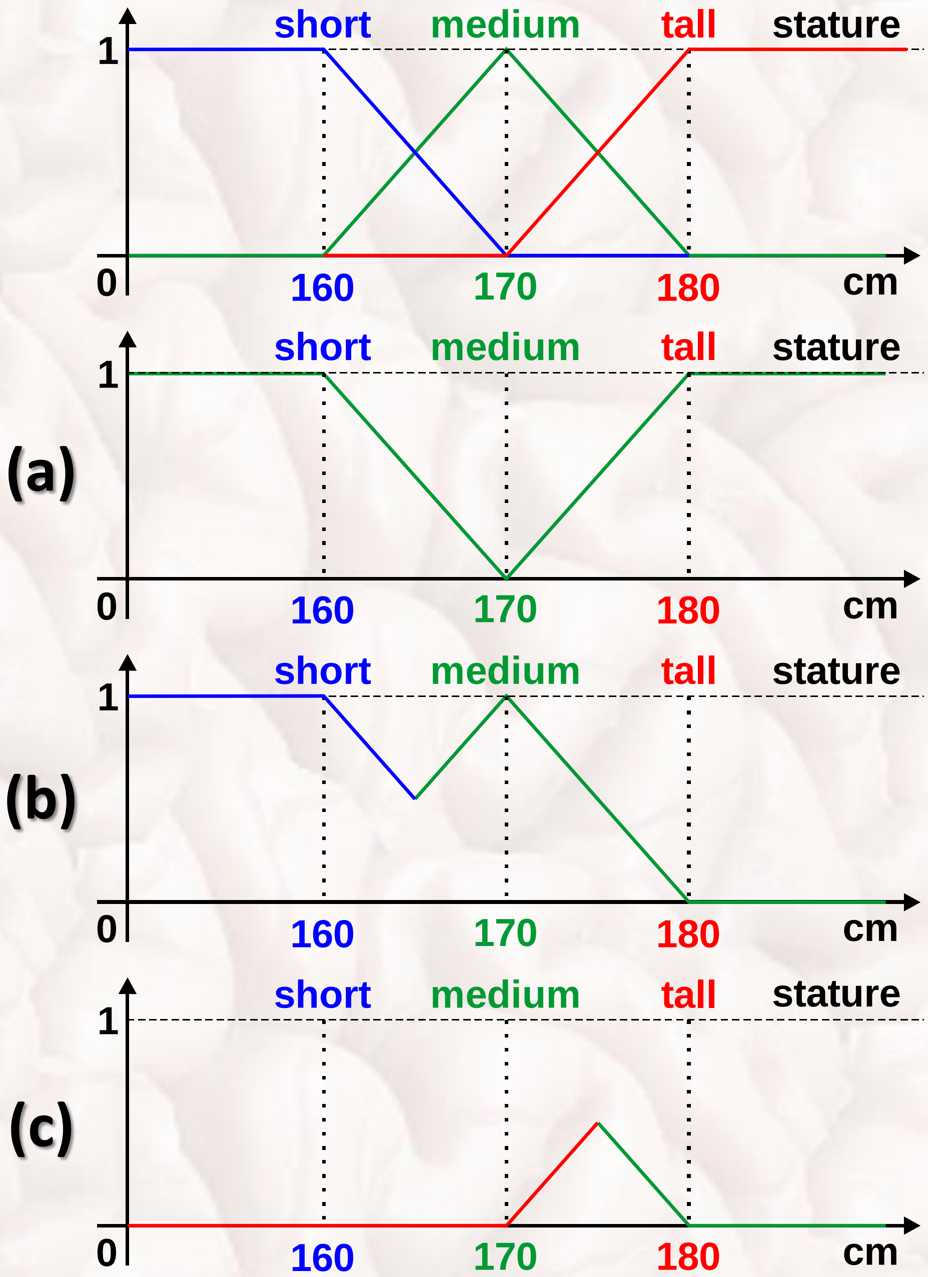
Suppose we have three fuzzy sets describing the statures of people as: short, medium, or tall.

Designate fuzzy sets which describe:

a) people which are **not** medium

b) short **or** medium people

c) medium **and** tall people



INTUITIVENESS OF FUZZY RULES

In comparison to the artificial neuronal systems, fuzzy sets and fuzzy rules are intuitive for interpretation because they use easy to understand rules:

IF [premise] THEN [conclusion]

IF x is A THEN y is B

IF x_1 is A_1 AND x_2 is A_2 AND … THEN y is B

IF x_1 is A_1 OR x_2 is A_2 OR … THEN y is B

where A, B are linguistic values defined as fuzzy sets in the space X and Y appropriately, x is an input variable and y is an output variable.

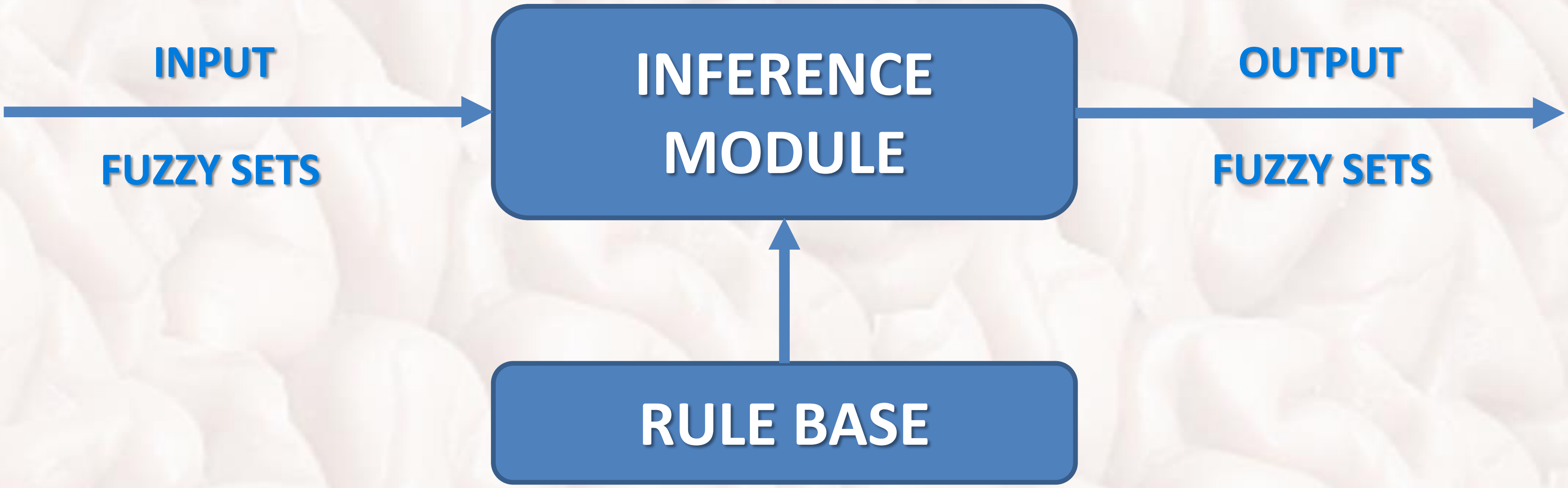
Fuzzy rules can be defined:

- **by an expert who defines fuzzy inference rules,**
- **by the learning system which is based on the learning data set for which fuzzy values are defined and the fuzzy inference rules and membership functions are determined during learning (adaptation) process of the selected model.**

FUZZY SYSTEMS

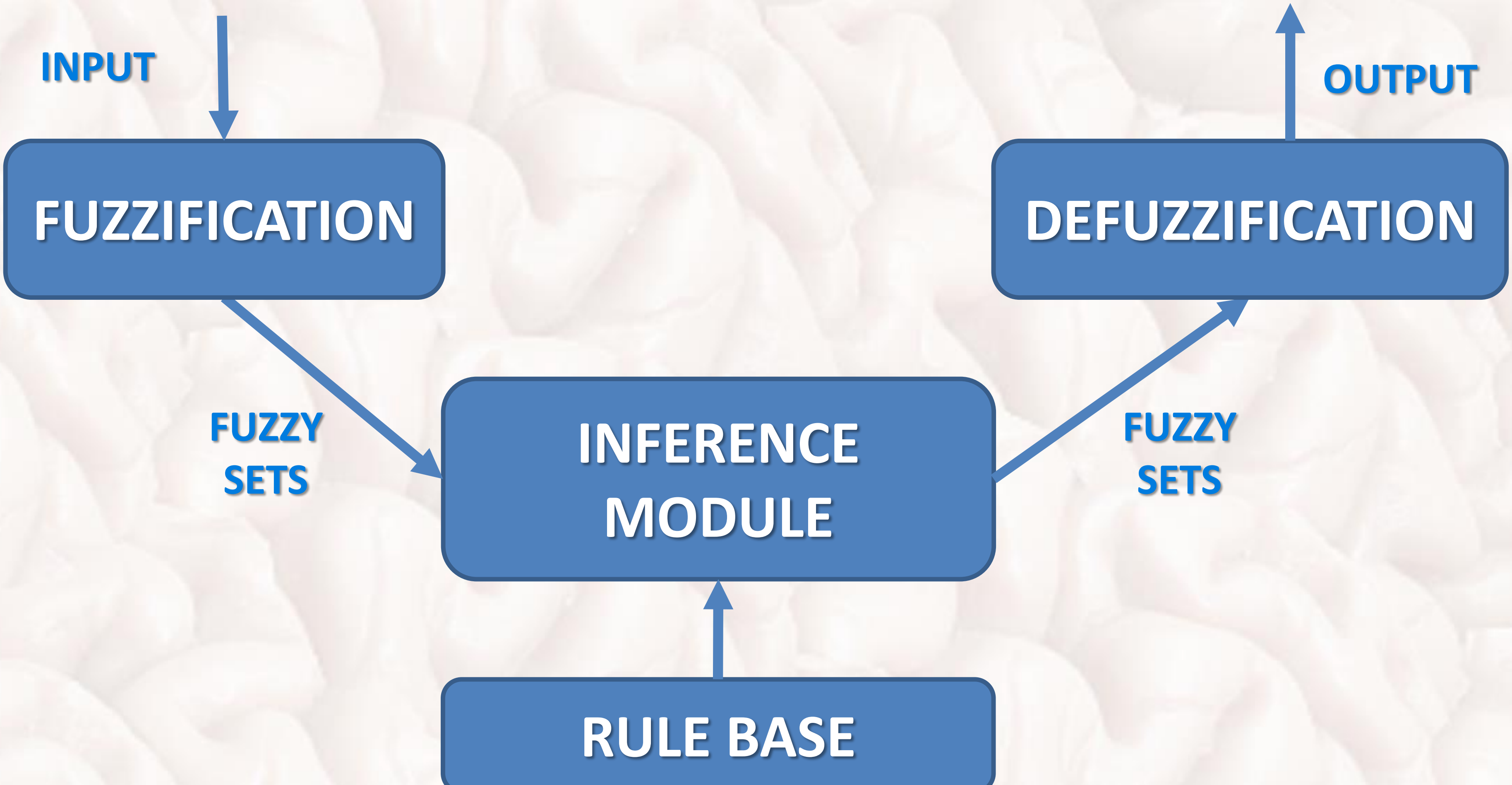
Each fuzzy system contains at least two basic modules:

- the inference module
- the rule base



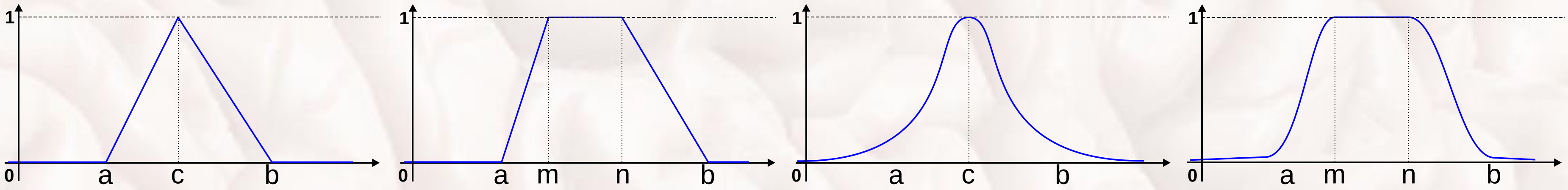
FUZZY SYSTEMS

We usually use two other modules to **fuzzify** input data and **defuzzify** output data. In these module we proceed **fuzzification** of the input data and **defuzzification** of the fuzzy sets after inference:



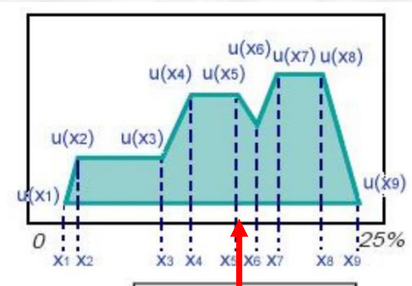
FUZZIFICATION AND DEFUZZIFICATION

Fuzzificator transforms the n-dimensional input vector $x = [x_1, x_2, ..., x_N]$ into the fuzzy set F using the defined membership function:



Defuzzificator transforms the resultant fuzzy set into the sharp value y on the basis of the center of gravity of this fuzzy set or using weighted average centers taking into account the shape of this fuzzy set:

DEFUZZIFICATION USING THE CENTROID



$$g = \frac{\sum_{i=1}^{9} x_i \cdot u(x_i)}{\sum_{i=1}^{9} u(x_i)} = 16{,}7$$

tip= 16,7%

THE RESULT OF DEFUZZIFICATION

Fuzzy functions can represent and approximate any continuous function.

FUZZY KNOWLEDGE BASE

Fuzzy rules create **a fuzzy knowledge base** that is used for inference processes for a given fuzzy system.

Such a system using the fuzzy knowledge base we call **a fuzzy inference system**.

The inference rules can be automatically determined on the basis of various **learning methods** and **training data sets**.

Fuzzy systems can be even transformed into the form of **fuzzy network** that resembles an artificial neural network.

GENETIC NEURO-FUZZY SYSTEMS

Intelligent computing systems often merge the advantages of different computational intelligence methods:

- ✓ Using training abilities of neural networks (e.g. backpropagation),
- ✓ Having clear interpretation of actions based on the rule-based representation of knowledge,
- ✓ Using symbolic and linguistic variables,
- ✓ Performing global optimization of parameters using evolutional or genetic approaches,
- ✓ Using the Mamdami-type inference or the Takagi-Sugeno scheme known from fuzzy systems.

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