Heuristic Rating Estimation approach to the pairwise comparisons method

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Outline

- Pairwise comparisons – motivation
- Classical approach
- Heuristic Rating Estimation (HRE) approach – motivation
- HRE method
- Possible areas of application
- Extensions
- Bibliography
Pairwise comparison
motivation

- FED Museum – Atlanta

  http://www.frbatlanta.org/about/tours/virtual/

  To start the story from the very beginning
Pairwise comparison
motivation

- Before money - barter
  - (see FED Museum: http://www.frbatlanta.org/about/tours/virtual/money/)

- History of trade is in fact history of pairwise comparisons
Pairwise comparison

motivation

- Barter – comparing incomparable
  - It’s hard to judge the actual value of things, hence the judgment is always subjective
Pairwise comparison
motivation

- Experts judgment implies relative value of goods:

\[
\approx \frac{1}{2}
\]
Pairwise comparison
motivation

- More comparisons:
  \[
  a \approx b \\
  a \approx c \\
  c \approx d \\
  ? \approx d \\
  \]

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Pairwise comparison

motivation

The need for methods of synthesis of partial results

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Pairwise comparison motivation

- Problems with expert judgments
  - Reciprocity
  
  ![Reciprocity Image]

  \[ ? \]

  - Consistency

  ![Consistency Image]

  \[ = a, \quad \quad = b \quad \Rightarrow \quad = ab \]

(See: *A new definition of consistency of pairwise comparisons*, Koczkodaj 1993),

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Pairwise comparison

Method
Pairwise comparison (PC) method
Result synthesis

- Pairwise comparisons matrix

\[
M = \begin{bmatrix}
1 & m_{12} & m_{13} & m_{14} & m_{15} \\
 m_{21} & 1 & m_{23} & m_{24} & m_{25} \\
 m_{31} & m_{32} & 1 & m_{34} & m_{35} \\
 m_{41} & m_{42} & m_{43} & 1 & m_{45} \\
 m_{51} & m_{52} & m_{53} & m_{54} & 1 \\
\end{bmatrix}
\]

where \( m_{ij} \in \mathbb{R}_+ \)
PC method
Classical approach

- Classical approach by Thomas Saaty
  - Assumption – matrix must be reciprocal i.e.
    \[ m_{ij} = \frac{1}{m_{ji}} \]
  - Value quantification:
    \[ m_{ij} \in \{a/b : a,b \in \{1,\ldots,10\}\} \]
  - Method of synthesis:
    - Finding principal eigenvector \( x \)
      \[ Mx = \lambda_{\text{max}} x \]

PC method
Classical approach

- Relative value of $\pi$, $\rho$, $\omega$, $\lambda$, $\kappa$ is given as:

$$x = \begin{bmatrix} x_1, x_2, \ldots, x_5 \end{bmatrix}$$

or even better, as normalized principal eigenvector:

$$\hat{x} = \begin{bmatrix} \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_5 \end{bmatrix}$$

where:

$$\hat{x}_i = \frac{x_i}{\sum_{j=1}^{5} x_j}$$
PC method
Classical approach

It is possible to determine the relative value of goods
PC method

Problem

- What happen when some goods are exchanged for money?
  - E.g.:
PC method

Problem

- What happen when some goods are exchanged for money?
  - E.g.:
    - Then:
      
      \[
      \frac{\text{Rabbit}}{\text{Fox}} = \frac{\$10}{\$20} = \frac{1}{2}
      \]
PC method

Problem

- Following Prof. Słowiński:
  Si l’ordre apparaît quelque part dans la qualité, pourquoi chercherions-nous à passer par l’intermédiaire du nombre?” (G. Bachelard 1934)


- Do not introduce the order when it is given…
PC method
Heuristic Rating Estimation (HRE) approach

- Let us do not (re) introduce the order where it is given...
  - (here the order is introduced by value in Dollars)

but

- Adopt the given values as reference and estimate the others
PC method
HRE approach

- Set of concepts:
  \[ C = C_K \cup C_U \]

- where
  - means \( C_K \) concepts for which the value \( \mu \) is initially known \( \mu(c_i) \in C_K \)
  - means \( C_U \) concepts for which \( \mu \) need to be determined
PC method

HRE approach

- HRE Input
  - Known concepts:
    \[ C_K = \{ \text{rabbit, fox} \} \]
    where
    \[ \mu(\text{rabbit}) = 10, \; \mu(\text{fox}) = 20 \]
  - Unknown concepts:
    \[ C_U = \{ \text{fish, salmon, crab} \} \]
PC method
HRE approach – result synthesis

- Pairwise comparisons matrix

\[ M = \begin{bmatrix}
1 & m_{12} & m_{13} & m_{14} & m_{15} \\
 m_{21} & 1 & m_{23} & m_{24} & m_{25} \\
 m_{31} & m_{32} & 1 & 2 & m_{35} \\
 m_{41} & m_{42} & 1/2 & 1 & m_{45} \\
 m_{51} & m_{52} & m_{53} & m_{54} & 1
\end{bmatrix} \]

where \( m_{ij} \in \mathbb{R}_+ \)
PC method
HRE approach – result synthesis algorithm

- **Input data:**

\[ \mu(c_3) = 20 \]
\[ 2 \]
\[ \mu(c_4) = 10 \]
PC method
HRE approach – result synthesis algorithm

- First step

We may expect that:

\[ \mu(c_1) = \frac{1}{2}(m_{13}\mu(c_3) + m_{14}\mu(c_4)) \]

\[ \mu(c_2) = \frac{1}{2}(m_{23}\mu(c_3) + m_{24}\mu(c_4)) \]

\[ \mu(c_5) = \frac{1}{2}(m_{53}\mu(c_3) + m_{54}\mu(c_4)) \]
PC method
HRE approach – result synthesis algorithm

- Second step and further
  - The values of \( c_1, \ldots, c_5 \) are known (defined)
    \[
    \mu(c_1) = \frac{1}{4} \sum_{i=2}^{5} m_{1i} \mu(c_i)
    \]
  - and in general:
    \[
    \mu(c_j) = \frac{1}{4} \sum_{i \in \{1, \ldots, 5\} \setminus \{j\}} m_{ji} \mu(c_i)
    \]
PC method
HRE approach – result synthesis algorithm

- Update equation (general form):

\[ \mu_r(c_j) = \frac{1}{\left| C_j^{r-1} \right|} \sum_{c_i \in C_j^{r-1}} m_{ji} \mu_{r-1}(c_i) \]

where:

\[ C_j^{r-1} = \{ c \in C : \mu_{r-1}(c) \text{ is defined and } c \neq c_j \}, \text{ and } C_j^0 = C_K \]
PC method

HRE approach – result synthesis algorithm

- Relative value of concepts is given as:
  \[ \mu = \left[ \mu(c_1), \ldots, \mu(c_k), \mu(c_{k+1}), \ldots, \mu(c_n) \right] \]

  HRE procedure \quad \text{a priori given} \quad (\text{assume } C_K = \{c_{k+1}, \ldots, c_n\})

- Normalization: \[ \hat{\mu} = \left[ \hat{\mu}(c_1), \ldots, \hat{\mu}(c_n) \right] \]
  \[ \hat{\mu}(c_i) = \frac{\mu(c_i)}{\sum_{j=1}^{n} \mu(c_j)} \]
What is $\mu$ when the number of iterations tends to $\infty$?
PC method
From algorithm to linear equation

- The algorithm as shown above follows the Jacobi iterative method for the linear equation system in form:

\[ A\mu = b \]

where

- \( A \) matrix of \( r \times r \) where \( r = |C| - |C_K| = |C_U| \)
- \( b \) vector of constant terms
- \( \mu \) vector of values

\[ \mu^T = [\mu(c_1), \ldots, \mu(c_k)] \]
PC method
From algorithm to linear equation

- Assuming (for the sake of simplicity) that
  \[ C = \{c_1, \ldots, c_n\} \quad \text{and} \quad C_K = \{c_{k+1}, \ldots, c_n\} \]
- \( b \) - vector of constant terms is given as

\[
b = \begin{bmatrix}
\frac{1}{n-1} m_{1,k+1} \mu(c_{k+1}) + \ldots + \frac{1}{n-1} m_{1,n} \mu(c_n) \\
\frac{1}{n-1} m_{2,k+1} \mu(c_{k+1}) + \ldots + \frac{1}{n-1} m_{2,n} \mu(c_n) \\
\vdots \\
\frac{1}{n-1} m_{k,k+1} \mu(c_{k+1}) + \ldots + \frac{1}{n-1} m_{k,n} \mu(c_n)
\end{bmatrix}
\]
PC method
From algorithm to linear equation

- The matrix $A$ used by the HRE algorithm is:

$$
A = \begin{bmatrix}
1 & -\frac{1}{n-1}m_{1,2} & \cdots & \cdots & -\frac{1}{n-1}m_{1,k} \\
-\frac{1}{n-1}m_{2,1} & 1 & -\frac{1}{n-1}m_{2,3} & \cdots & -\frac{1}{n-1}m_{2,k} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
-\frac{1}{n-1}m_{k,1} & \cdots & \cdots & 1 & -\frac{1}{n-1}m_{k,k-1}
\end{bmatrix}
$$
PC method
Direct solving

- The equation:

\[ A\mu = b \]

can also be solved directly…
PC method
Direct solving – theorems and facts

- **Theorem 1**
  - Equation $A\mu = b$ has exactly one solution if $\det(A) \neq 0$

- **Theorem 2**
  - The Jacobi method is convergent if $A$ is strictly diagonally dominant by rows i.e.
    \[
    1 > \sum_{j=1, j \neq i}^{k} |a_{ij}|
    \]
  - for $i = 1, \ldots, k$
PC method

Direct solving – admissibility of solution

- Observation - result must be positive
  - i.e. $\mu(c_i) > 0$ for $i = 1, \ldots, n$
- That is because of the form of update equation:

$$
\mu_r(c_j) = \frac{1}{|C_j^{r-1}|} \sum_{c_i \in C_j^{r-1}} m_{ji} \mu_{r-1}(c_i)
$$

- Thus $\mu$ obtained by direct solving $A\mu = b$
  is admissible if all $\mu(c_i)$ are strictly positive
- Observation - $A$ is strictly diagonally dominant, then $\mu$
  obtained using direct method is admissible
PC method

Direct solving – solution admissibility

- Is $A$ diagonally dominant?
  - i.e.
    \[ 1 > \sum_{j=1, j \neq i}^{k} |a_{ij}| \text{ where } a_{ij} = -\frac{1}{n-1} m_{ij} \]

- Observation
  - $A$ has a high chance to be diagonally dominant when $a_{ij}$ are not too large.

- Conclusion
  - The more concepts in $C_K$ and the more similar to each other the better (the more likely $A$ is diagonally dominant)
PC method
Intuitiveness of assumptions

- The more concepts in $C_K$ ....
  - Corresponds to the natural desire to have more than the lower number of reference concepts

- The more similar to each other ....
  - Humans (experts) are able to compare (judge) similar things but not different.

- In most cases tested the matrix $A$ was diagonally dominant
PC method, HRE approach

Areas of application

Decision Support Systems
PC method, HRE approach

How to?

- Define the problem
  - Specify problem domain, the set of concepts and the partial function $\mu$
- Define the reference set $C_K$
  - assign the $\mu$ values to the concepts from $C_K$
- Gathering comparative data
- Apply the HRE method
- To remember – this is only heuristics
  - The most important is common sense
PC method, HRE approach
Possible areas of application

- An optimal drug (treatment) selection
  - Reference set: $C_K$
    - Group of drugs with proven efficacy in clinical trials
  - Other concepts $C_U = C \setminus C_K$ and the rest of $M$
    - Comparative opinions of experts (physicians) based on their clinical experience
PC method, HRE approach
Possible areas of application

- Introducing new products into the market
  - Reference set: $C_K$
    - existing products with sales statistics
- Other: $C_U = C \setminus C_K$
  - Comparative opinion survey of the target group
Support for assessment of the value of companies

- Joint stock companies – reference set
  - an actual value is determined by stock exchange
- Companies outside the exchange trading – the rest of concepts
  - Comparative company value estimation
PC method, HRE approach
Possible areas of application

and many more...
PC method, HRE approach
Possible areas of application

- Green AGH Campus
PC method, HRE approach

Additional flavors
PC method, HRE approach
Supporting heuristics

- Minimizing estimation error heuristics
  - Average absolute estimation error
    \[ \hat{e}_\mu = \frac{1}{|C_U|} \sum_{c \in C_U} e_\mu(c) \]
  - Absolute estimation error
    \[ e_\mu(c_j) = \frac{1}{|C_{j-1}|} \sum_{c_i \in C_{j-1}} |\mu(c_j) - \mu(c_i) \cdot m_{ji}| \]
PC method, HRE approach
Supporting heuristics

- Non-reciprocal PC matrix heuristics
  - Lack of reciprocity:
    \[ m_{ij} \neq \frac{1}{m_{ji}} \]
  - Let us transform: \( M \rightarrow \hat{M} \)
    - where:
      \[ \hat{m}_{ij} = \left( m_{ij} \frac{1}{m_{ji}} \right)^{\frac{1}{2}} \]
    - Observation:
      - \( \hat{M} \) is reciprocal
      - If \( M \) is reciprocal then \( M = \hat{M} \)
PC method, HRE approach
Supporting heuristics

- Incomplete PC matrix heuristics:

\[ \mu(c_1) = \frac{1}{2} (m_{13} \mu(c_3) + m_{15} \mu(c_5)) \]

- Impacts the lack of reciprocity heuristics
Bibliography


Questions