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Invited paper

Topological phase transitions: An Outlook

1. Topological phases and their transitions

The usual paradigm to describe phase transitions is that due to Landau [1]. This is based on the concept of an order parameter that is finite in a broken symmetry phase. A functional that plays the role of a free energy is expanded in powers of the order parameter, expected to be small close to the phase transition. The terms in this expansion are dictated by symmetry. For example, if the magnetisation is the order parameter and the system has rotational invariance only even powers of the magnetisation are allowed in the functional in the absence of an external magnetic field. Landau's theory has been and continues to be extremely useful. It can be generalised to describe quantum phase transitions, where the functional is the action of the system. In the quantum case, since static and dynamics are coupled due to the Heisenberg uncertainty relation, the expansion of the action takes into account both, the frequency and momentum dependence of the order parameter. Then, the dynamics of the system in the relevant low frequency regime is incorporated in the description of the quantum critical behaviour. This introduces a new *dynamical critical exponent z* that is related to usual static critical exponents by the quantum hyperscaling relation [2]

$$2 - \alpha = \nu(d + z). \tag{1}$$

In this equation, ν is the correlation length exponent, d the dimensionality of the system and the exponent α characterises the behaviour of the singular part of the free energy density close to the transition, i.e., $f_s \propto |g|^{2-\alpha}$. The quantity g measures the distance to the quantum critical point (QCP), at g=0, in terms of some control variable. This is the quantum equivalent of $(T - T_c)$ in thermal phase transitions.

The study and characterisation of topological phases make it clear that transitions between these phases do not conform to the Landau paradigm. Probably, the best way to show this is to look at a specific example. Consider Kitaev's model for a one-dimensional p-wave superconductor described by the Hamiltonian [3]

$$\mathcal{H} = -t\sum_{ij} c_i^\dagger c_j - \sum_{ij} (\Delta_{ij} c_i^\dagger c_j^\dagger + \Delta_{ij}^* c_i c_j) - \mu \sum_i c_i^\dagger c_i. \tag{2}$$

This model has three parameters, the hopping t of the spinless fermions in the chain, the antisymmetric interaction Δ_{ij} ($=-\Delta_{ji}$), and the chemical potential μ. At T=0 this system is always a p-wave superconductor. However, for $\mu/2t = 1$ it presents a topological quantum phase transition between two superconducting phases with radically different physical (topological) properties [4–6]. The phase for $\mu/2t < 1$ is topologically non-trivial and the main feature of this phase is the presence of fermionic modes, called Majoranas, at the ends of the chain [4–6]. These are quasi-particles with zero energy excitation and the property of being their own anti-particles. The phase for $\mu/2t > 1$ is still a superconductor but topologically trivial, with no Majoranas. We can now appreciate the difficulty of using Landau approach to this problem. On both sides of the transition we have superconducting phases and there is no symmetry change at $\mu/2t = 1$ where the transition occurs. Of course the superconducting order parameter cannot be used to distinguish between these phases. Although they are associated with different values of a *topological invariant* [4], this changes discontinuously at the transition and is not useful as an order parameter.

From the point of view of the theory of critical phenomena, the situation is not hopeless, since we can identify in general a diverging length and a diverging time at topological quantum phase transitions. This allows to use the concepts of scaling to approach these phenomena. In Kitaev's model the diverging length is the penetration depth of the Majorana modes inside the chain. It diverges as $\xi \propto |g|^{-\nu}$, where the correlation length exponent takes the value $\nu = 1$ [2]. Since the transition is driven by varying the chemical potential, $g = (\mu/2t) - (\mu/2t)_c$, with $(\mu/2t)_c = 1$ the quantum critical point at which the topological transition occurs. From the spectrum of excitations, which is Dirac-like at the transition one can identify the dynamic exponent z=1 for the Kitaev model [2].

The term topological transition also appears in the condensed matter literature as referring to transitions where bands in a solid merge or disappear below the Fermi surface as an external control parameter like pressure, magnetic field or doping is varied [7,8]. These transitions, also known as Lifshitz transitions, share with the previous one the absence of a clear order parameter or of symmetry breaking. They can be associated in general with metal-insulator but also with a metal-to-metal transitions. The Landau approach is again of no use here. Also in this case, one can in general identify diverging length and time with their respective critical exponents, ν and z. Due to the quadratic form of the electronic spectrum near the band edges, it turns out that z=2. Since the gap exponent $\nu z = 1$, this implies for the correlation exponent the value $\nu = 1/2$ [8].

Besides the scaling approach, the renormalisation group (RG) theory provides the necessary tools to tackle the problem of topological transitions. In the Kitaev model, for example, the flow of the RG equation for the interaction in the trivial superconducting phase is towards a strong coupling fixed point, while in the non-trivial topological phase the flow is to a chaotic attractor. The RG also allows to obtain the critical exponents at the transition [2].

Finally, a relevant question is whether we can expect to find any thermodynamic anomaly at a topological phase transition. Notice that these

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transitions are strictly zero temperature phenomenon or quantum phase transitions. In spite of that, they can leave their signatures in physical quantities at finite temperatures as is now well known for quantum phase transitions in general [2]. A key role here is played by the quantum hyperscaling relation, Eq. (1), which relates the exponent of the free energy to other critical exponents. Consider the non-analytic part of the free energy of the Kitaev model close to the topological transition, $f_s \propto |g|^{2-\alpha}$. The compressibility $\kappa = \partial^2 f_s/\partial\mu^2 \sim |g|^{-\alpha}$ and using Eq. (1), with d=1, $\nu = 1$ and z=1, we obtain $\alpha = 0$. Then we can expect a jump or at most a logarithmic singularity in the compressibility of the Kitaev model at the topological transition.

In the case of a magnetic field-driven Lifshitz transition, the quantity $g \propto (h - h_c)$ where h_c is the critical magnetic field at which the Lifshitz transition occurs. If we write the free energy as before, $f_s \propto |g|^{2-\alpha}$ and take the second derivative with respect to the magnetic field, we obtain, $\chi \propto \partial^2 f_s/\partial h^2 \sim |g|^{-\alpha}$. Using the values of ν and z obtained above and the quantum hyperscaling relation, Eq. (1), for a 3d system we find $\alpha = -1/2$, implying a negligible anomaly of the magnetic susceptibility at the transition. Notice that at finite temperatures, temperature appears in the free energy in the scaling form [2], $T/|g|^{\nu z}$ and this will be important to determine the form of crossover lines close to topological transitions in finite temperature phase diagrams [2].

The field of topological insulators and superconductors is experiencing an impressive growth, as new materials are discovered and detailed experiments probe the exotic excitations associated with non-trivial topological phases [5]. These excitations with their *topological protection* [4] have promising applications in the field of quantum computers [9]. Theoretical progress in understanding topological phases is being made at the same pace. Lifshitz transitions [8,10,11], on the other hand, are now ubiquitous when probing electronic systems with high magnetic fields, or under pressure. The study of these different topological phase transitions extends our investigation of quantum critical phenomena beyond the Landau paradigm. It can enrich our knowledge of the topological phases and test the limits of the concept of universality, as we know it today.

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