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A STOCHASTIC MODEL OF THE DEVELOPMENT OF ALPINE RHODODENDRON

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Abstract. A stochastic model of the development of the Alpine rhododendrons is discussed here. Two species appearing in the Austrian Alps are considered: Rhododendron hirsutum and Rhododendron ferrugineum. The idea of the presented model is based on the theory of branching processes. The technique of generating functions is also used. The correctness of the model is checked by a computer simulation. In the model, the distribution of the number of living shoots and flowers in bloom in particular years of the life of a shrub, are considered. This may become a basis for a simulated development of an Alpine rhododendron population over many years, and in particular, a forecast of its future fate.

Introduction

High mountain populations of the Alpine rhododendrons: Rhododendron hirsutum and Rh. ferrugineum, have permanent underground organs whose morphological structure makes it possible to estimate the age and years of flowering as well as to observe other features useful in the construction of a model. The normal development, repeated with great regularity, has facilitated the modeling of the vital processes of both reproduction and prognosis. This gives us interesting prospective knowledge of their dynamics over many years and even centuries.

The first studies of a stochastic model of the development of the Alpine rhododendron were carried out under special assumptions that a process characterizing the growth of an individual is Markovian and the number of new shoots has Poisson's distribution [1]. Good results obtained led to the preparation of a more general model in which those restrictive assumptions were dropped. This is the subject of the present paper. The results obtained by a computer simulation confirmed a wide range of application of the model in demographic considerations referring to some perennials with permanent underground organs.

Description of the development of Alpine rhododendrons

Both the species of the considered here Alpine rhododendrons: Rhododendron hirsutum and Rh. ferrugineum, are characterized by a similar scheme of growth [3].

In the first year the germinating seed produces one shoot, at the tip of which a new section grows every year (Fig. A-1,2,3). The side shoots growing on these older sections (Fig. A-4,5,7,10) do not have much influence on the form of the mature individual, as their vitality is limited (Fig. A-7,9,10,13). Every year the plant may wither in whole or in part. Between eleven and sixteen years of age, in the spring, the plant may bloom for the first time (Fig. A-12,13). From this instant the plant begins the fundamental part of its life-cycle, constituting a basis for the construction of a model.

The origin of the shoots below the inflorescence, whose number varies from one to seven (Fig. A-13), is thus connected with the process of florescence. Any of the shoots may wither in any year. The new shoots grow from May to July. It is possible that the shoot may not put out any new shoots after flowering (Fig. A-17,18). Every new shoot may bloom between two and twelve years of age, initiating the next step in this life-cycle with all possible variants. If it does not flower during this time, it dies.

In the process of development of specimens of Rhododendron hirsutum and Rh. ferrugineum there may be deviations from the scheme just specified, resulting from the richness of natural phenomena. Their frequency, however, is so insignificant that in a model aimed for practical application they may be ignored.

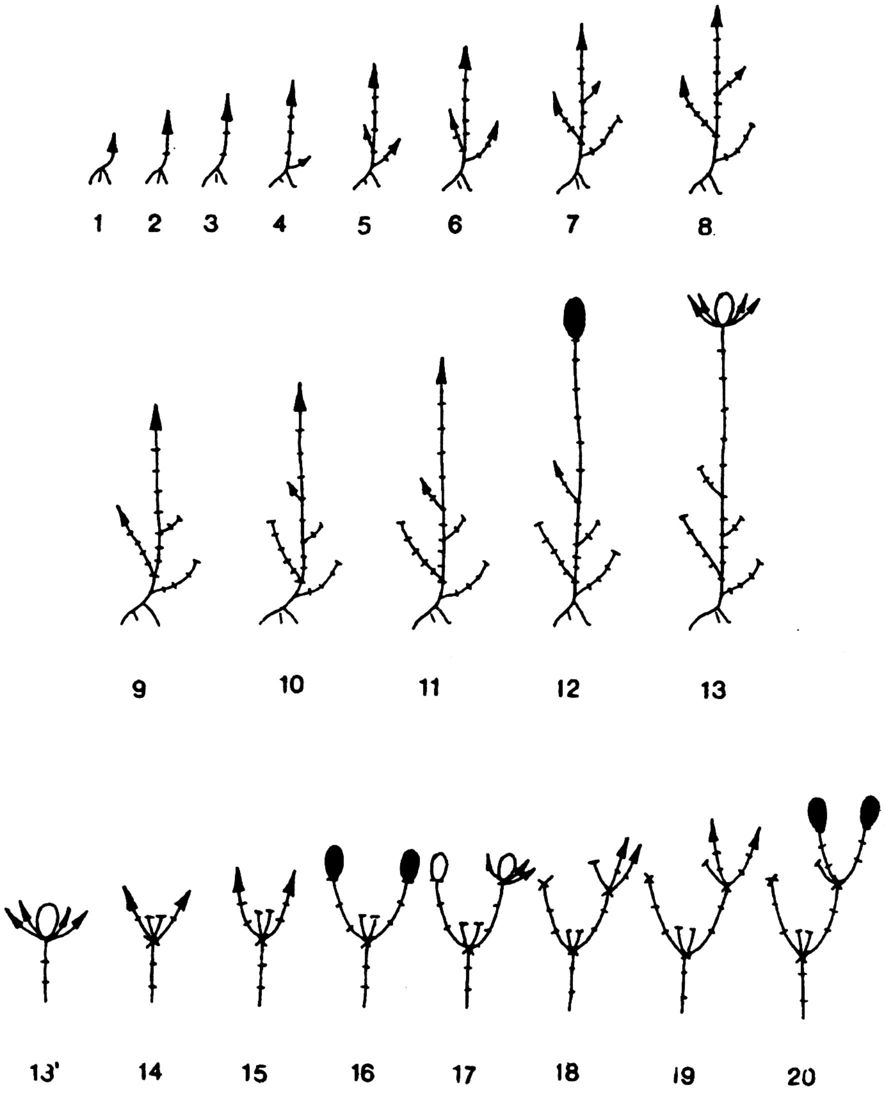

Fig. A. Scheme of shrub development in Rhododendron hirsutum and Rh. ferrugineum (numbers denote age of specimen)

Mathematical model

Let probability distributions of the following quantities characterizing the development of a single specimen of Rhododendron hirsutum or Rh. ferrugineum, be given:

- year of the specimen's first florescence,
- number of shoots appearing after florescence,
- year of florescence of the shoot,
- year of withering of the shoot;

the second of these distributions refers both to the first inflorescence of the specimen and to later flowering of particular shoots. The following assumptions have also been made:

- the fact that the florescence or withering of a given shoot depends solely on how many years ago this shoot flowered the last time,
- the events characterizing the development of various shoots are independent.

In the model proposed, the possibility that the shrubs might die before the first florescence has not been taken into account, since from the point of view of simulation of the life of a single specimen, this case is of no interest.

The subject of the following analysis will be a two-argument family of random variables $\{f_n^k\}(n = 1, 2, \dots$ and $k = 0, 1, \dots, n-1)$, where f_n^k denotes the number of shoots appearing at the end of the n^{th} year of life of a specimen that last flowered k years ago. On the basis of the specified distributions and assumptions, the generating function of the random variables f_n^k, defining also their distributions, will be calculated now. In particular, the expectation of the number of living shoots appearing in the n^{th} year of the life of a shrub, will be specified. The number of living shoots best characterizes the condition of separate individuals.

The functioning of that model will be discussed in the subsequent part of the paper dealing with computer simulation.

Let:

$-a_j (j = 1, 2, \dots)$ denote the probability that the first florescence took place in the j^{th} year of specimen's life,

$-b_j (j = 0, 1, \dots)$ denote the probability that after flowering j new shoots appear,

$-c_j (j = 1, 2, \dots)$ denote the probability that the shoot will flower in j years since the last florescence, on condition that until then it will not die (and will not flower),

$-d_j (j = 0, 1, \dots)$ denote the probability that the shoot will die in j years since the last florescence, on condition that until then it will not flower (and will not die).

Now let:

$$(1) \qquad \widehat{b}_j = \sum_{i=j}^{\infty} \binom{i}{j} b_i (1-d_0)^j \, d_0^{i-j}$$

for $j = 0, 1, \ldots$; $\widehat{b}_j$ is therefore the probability that the value of the random variable, defined as the number of those shoots that appeared after florescence, equals j. The generating function of this random variable is given by the formula:

$$(2) \qquad \begin{aligned} \widehat{B}(x) &= \sum_{j=0}^{\infty} \widehat{b}_j x^j \\ &= \sum_{j=0}^{\infty} \sum_{i=j}^{\infty} \binom{i}{j} b_i (1-d_0)^j d_0^{i-j} x^j \\ &= \sum_{i=0}^{\infty} b_i \sum_{j=0}^{i} \binom{i}{j} (1-d_0)^j x^j d_0^{i-j} \\ &= \sum_{i=0}^{\infty} b_i \left[(1-d_0)x + d_0\right]^i . \end{aligned}$$

Let $\widetilde{f}_n^k$ ($n = 0, 1, \ldots$ and $k = 0, 1, \ldots, n$) denote the number of shoots appearing on a shrub at the end of the n^{th} year, counting from the year of the first florescence, last flowering k years ago. The generating function of this random variable is expressed by the formula:

$$(3) \qquad \widetilde{F}_n^k(x) = \sum_{i=0}^{\infty} P(\widetilde{f}_n^k = i) x^i .$$

Of course:

$$(4) \qquad \begin{aligned} \widetilde{F}_0^0(x) &= \widehat{B}(x) \\ &= \sum_{i=0}^{\infty} \widehat{b}_i x^i . \end{aligned}$$

Now the generating function $\widetilde{F}_n^k$ for $n \geq 1$, will be computed.

First let $k \geq 1$. The shoots that appeared in the n^{th} year and flowered exactly k years ago are those that appeared in the $(n-1)^{th}$ year and were then included in the group finally flowering $k-1$ years ago, and since then have neither flowered nor died. As the events of the florescence of individual shoots are unrelated, the conditional distribution of the random variable $\widetilde{f}_n^k$, at the assumed value $\widetilde{f}_{n-1}^{k-1}$, is binomial. Thus:

$$(5) \qquad P\left(\widetilde{f}_n^k = j \mid \widetilde{f}_{n-1}^{k-1} = i\right) = \binom{i}{j}(1-c_k-d_k)^j(c_k+d_k)^{i-j}.$$

Since $\widetilde{f}_n^k \leq \widetilde{f}_{n-1}^{k-1}$, it is true that:

$$(6) \qquad P\left(\widetilde{f}_n^k = j\right) = P\left(\widetilde{f}_n^k = j, \widetilde{f}_{n-1}^{k-1} \geq j\right),$$

and so it follows from the total probability formula:

$$(7) \qquad P\left(\widetilde{f}_n^k = j\right) = \sum_{i=j}^{\infty}\binom{i}{j}(1-c_k-d_k)^j(c_k+d_k)^{i-j}P\left(\widetilde{f}_{n-1}^{k-1} = i\right).$$

Substituting the probability obtained in formula (3) and using Newton's binomial equation one can obtain:

$$(8) \qquad \begin{aligned} \widetilde{F}_n^k(x) &= \sum_{j=0}^{\infty}x^j\sum_{i=j}^{\infty}\binom{i}{j}(1-c_k-d_k)^j(c_k+d_k)^{i-j}P\left(\widetilde{f}_{n-1}^{k-1} = i\right) \\ &= \sum_{i=0}^{\infty}P\left(\widetilde{f}_{n-1}^{k-1} = i\right)\sum_{j=0}^{i}\binom{i}{j}(1-c_k-d_k)^j x^j(c_k+d_k)^{i-j} \\ &= \sum_{i=0}^{\infty}P\left(\widetilde{f}_{n-1}^{k-1} = i\right)\left[(1-c_k-d_k)x+c_k+d_k\right]^i \\ &= \widetilde{F}_{n-1}^{k-1}\left[(1-c_k-d_k)x+c_k+d_k\right]. \end{aligned}$$

Hence for $n = 1, 2, \ldots$ and $k = 1, 2, \ldots, n$ the following recurrent pattern is true:

$$(9) \qquad \widetilde{F}_n^k(x) = \widetilde{F}_{n-1}^{k-1}\left[(1-c_k-d_k)x+c_k+d_k\right].$$

Finally, the demarcated generating function $\widetilde{F}_n^0$ remains to be calculated for $n = 1, 2, \ldots$. The random variable $\widetilde{f}_n^0$ characterizes the number of shoots that originated from shoots flowered in the n^{th} year, and so it is the sum of n independent random variables $\{g_n^m\}$ $(m = 1, 2, \ldots, n)$, where g_n^m denotes the number of new shoots out of those shoots which had last flowered m years ago, that did not die in the n^{th} year. Since the generating function of the sum of independent random variables is equal to the product of their generating functions, one obtains:

$$\widetilde{F}_n^0(x) = \prod_{m=1}^{n} G_n^m(x), \tag{10}$$

where:

$$G_n^m(x) = \sum_{i=0}^{\infty} P(g_n^m = i)\, x^i \tag{11}$$

is the generating function of the random variable g_n^m. In turn, g_n^m is the sum of p independent random variables with a distribution whose generating function is $\widehat{B}$, so that p denotes the number of shoots flowering in the n^{th} year. Hence for the fixed value r the provisory generating function may be calculated:

$$\begin{aligned} G_n^m(x \mid p = r) &= \sum_{i=0}^{\infty} P(g_n^m = i \mid p = r)\, x^i \\ &= [\widehat{B}(x)]^r. \end{aligned} \tag{12}$$

Because the shoots that originated in the n^{th} year but did not flower for m years, came from those that appeared towards the end of the year $n-1$ and did not flower for $m-1$ years, so:

$$P\left(p = r \mid \widetilde{f}_{n-1}^{m-1} = q\right) = \binom{q}{r} c_m^r (1 - c_m)^{q-r}, \tag{13}$$

which yields:

$$\begin{aligned} G_n^m\left(x \mid \widetilde{f}_{n-1}^{m-1} = q\right) &\sum_{i=0}^{q} \binom{q}{i} c_m^i (1 - c_m)^{q-i} [\widehat{B}(x)]^i \\ &= \left[c_m \widehat{B}(x) + 1 - c_m\right]^q, \end{aligned} \tag{14}$$

and therefore, using the total probability formula one obtains:

$$(15)\qquad \begin{aligned} G_n^m(x) &= \sum_{q=0}^{\infty} P\left(\widetilde{f}_{n-1}^{m-1} = q\right)\left(c_m\widehat{B}(x) + 1 - c_m\right)^q \\ &= \widetilde{F}_{n-1}^{m-1}\left(c_m\widehat{B}(x) + 1 - c_m\right). \end{aligned}$$

Finally:

$$(16)\qquad \widetilde{F}_n^0(x) = \prod_{m=1}^{n} \widetilde{F}_n^m\left(c_m\widehat{B}(x) + 1 - c_m\right)$$

for $n = 1, 2, \ldots$.

Formulas (4), (9) and (16) used recurrently provide the generating function of number of the shoots when the year is counted relatively, i.e. from the first florescence. It will now be replaced by the absolute age.

For consistency of a notation, for $n = 0, 1, \ldots$ and $k = n+1, n+2, \ldots$ let the random variables $\widetilde{f}_n^k$ be defined as equal to zero with probability 1.

Now let $n = 1, 2, \ldots$ and let l $(l = 1, 2, \ldots, n)$ denote the year of the first florescence. Then for $n = 1, 2, \ldots$ and $k = 0, 1, \ldots, n-1$ the equality $f_n^k = \widetilde{f}_{n-l}^k$ is true, and so:

$$(17)\qquad \begin{aligned} P\left(f_n^k = j\right) &= \sum_{i=1}^{n} P\left(\widetilde{f}_{n-i}^k = j\right) P\left(l = i\right) \\ &= \sum_{i=1}^{n} a_i P\left(\widetilde{f}_{n-i}^k = j\right), \end{aligned}$$

which implies:

$$(18)\qquad F_n^k(x) = \sum_{i=1}^{n} a_i \widetilde{F}_{n-i}^k(x),$$

for $n = 1, 2, \ldots$ and $k = 0, 1, \ldots, n-1$.

Using formulas (4), (9), (16) and (18) one can recurrently calculate the generating functions of all the random variables f_n^k, which is equivalent to finding their distributions. In particular, it follows from equation (18) that:

$$
(19) \qquad \begin{aligned} E\left(f_n^k\right) &= \left(F_n^k\right)'(1) \\ &= \sum_{i=1}^{n} a_i \left(\widetilde{F}_{n-i}^k\right)'(1) \\ &= \sum_{i=1}^{n} a_i E\left(\widetilde{f}_{n-i}^k\right) \end{aligned}
$$

for $n = 1, 2, \ldots$ and $k = 0, 1, \ldots, n-1$. From formula (4) one obtains:

$$
(20) \qquad \begin{aligned} E\left(\widetilde{f}_0^0\right) &= \left(\widetilde{F}_0^0\right)'(1) \\ &= \sum_{i=0}^{\infty} i\widehat{b}_i. \end{aligned}
$$

In turn, for $n = 1, 2, \ldots$ equation (16) yields:

$$
(21) \qquad \begin{aligned} E\left(\widetilde{f}_n^0\right) &= \left(\widetilde{F}_n^0\right)'(1) \\ &= \sum_{m=1}^{n} \frac{d}{dx} \widetilde{F}_{n-1}^{m-1} \left[c_m \widehat{B}(x) + (1 - c_m)\right] \Big|_{x=1} \\ &= \sum_{m=1}^{n} \left(\widetilde{F}_{n-1}^{m-1}\right)'(1) c_m \widehat{B}'(1) \\ &= (1 - d_0)\Big(\sum_{i=1}^{\infty} i b_i\Big)\Big[\sum_{m=1}^{n} c_m E\left(\widetilde{f}_{n-1}^{m-1}\right)\Big] \end{aligned}
$$

and formula (9) implies:

$$
(22) \qquad \begin{aligned} E\left(\widetilde{f}_n^k\right) &= \left(\widetilde{F}_n^k\right)'(1) \\ &= \frac{d}{dx} \widetilde{F}_{n-1}^{k-1} [(1 - c_k - d_k)x + c_k + d_k]\Big|_{x=1} \\ &= \left(\widetilde{F}_{n-1}^{k-1}\right)'(1) \frac{d}{dx} \left[(1 - c_k - d_k)x + c_k + d_k\right]\Big|_{x=1} \\ &= E\left(\widetilde{f}_{n-1}^{k-1}\right)(1 - c_k - d_k) \end{aligned}
$$

for $k = 1, 2, \ldots, n-1$. Since the expected value of the number of living shoots in the n^{th} year of the life of an individual is:

$$\sum_{k=0}^{n-1} E\left(f_n^k\right), \tag{23}$$

it may be recurrently calculated on the basis of formulas (20)–(23). Their interpretation is evidently in agreement with intuition.

The random variables characterizing the number of flowers blooming and as a consequence the number of seeds, may be considered in a similar way.

Computer simulation. Conclusions

To check the accuracy of the presented model, a computer simulation has been prepared. It simulates the development of a single individual, recording its picture symbolically. Fig. B presents an exemplary result. The symbol "-" denotes the annual growth of the shoot, "+" or "#" its death, "*" or "#" symbolize the blooming flower, "∧" and "|" are connected with the shoots appearing in the meantime (thus the symbol "#" denotes the withering of the shoot in the same year in which it flowered). The changes taking place in a specified year are contained in one column of figures. Fig. C shows an exemplary interpretation of the architecture of the shrub described symbolically in Fig. B.

The data introduced into the program have been taken in a form different from those appearing directly in the model, but more convenient for identification, and so the following random variables have been taken as a data:

- the year of the first florescence of a specimen,
- the number of shoots appearing after a florescence,
- the year of florescence of the shoot on condition that it flowers sometimes (and does not die),
- the year when the shoot dies, on condition that it sometimes dies (and does not flower),

as well as the number r – the total probability of the death of the shoot ($1 - r$ is thus the total probability of its florescence).

On the basis of the observational material, it may be assumed that the probability measures of those random variables are concentrated on the sets $\{1, 2, \ldots, 20\}$, $\{0, 1, \ldots, 20\}$, $\{1, 2, \ldots, 20\}$ and $\{0, 1, \ldots, 20\}$ respectively. On account of this, the probability that these random variables take the value j is denoted by a_j – $j = 1, 2, \ldots, 20$, b_j – $j = 0, 1, \ldots, 20$, c_j^* – $j = 1, 2, \ldots, 20$ and d_j^* – $j = 0, 1, \ldots, 20$, respectively. In particular, d_0^* is a conditional probability of the death of the shoot in the same year in which earlier (in spring) the flower bloomed.

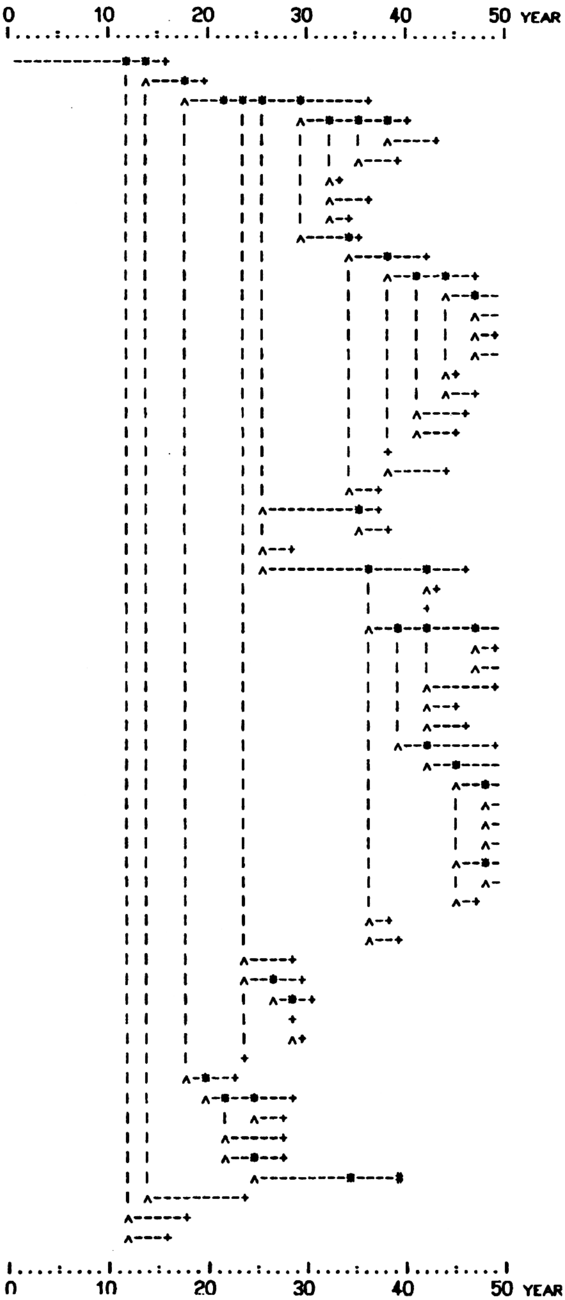

Fig. B. Symbolic picture of shrub obtained by computer simulation

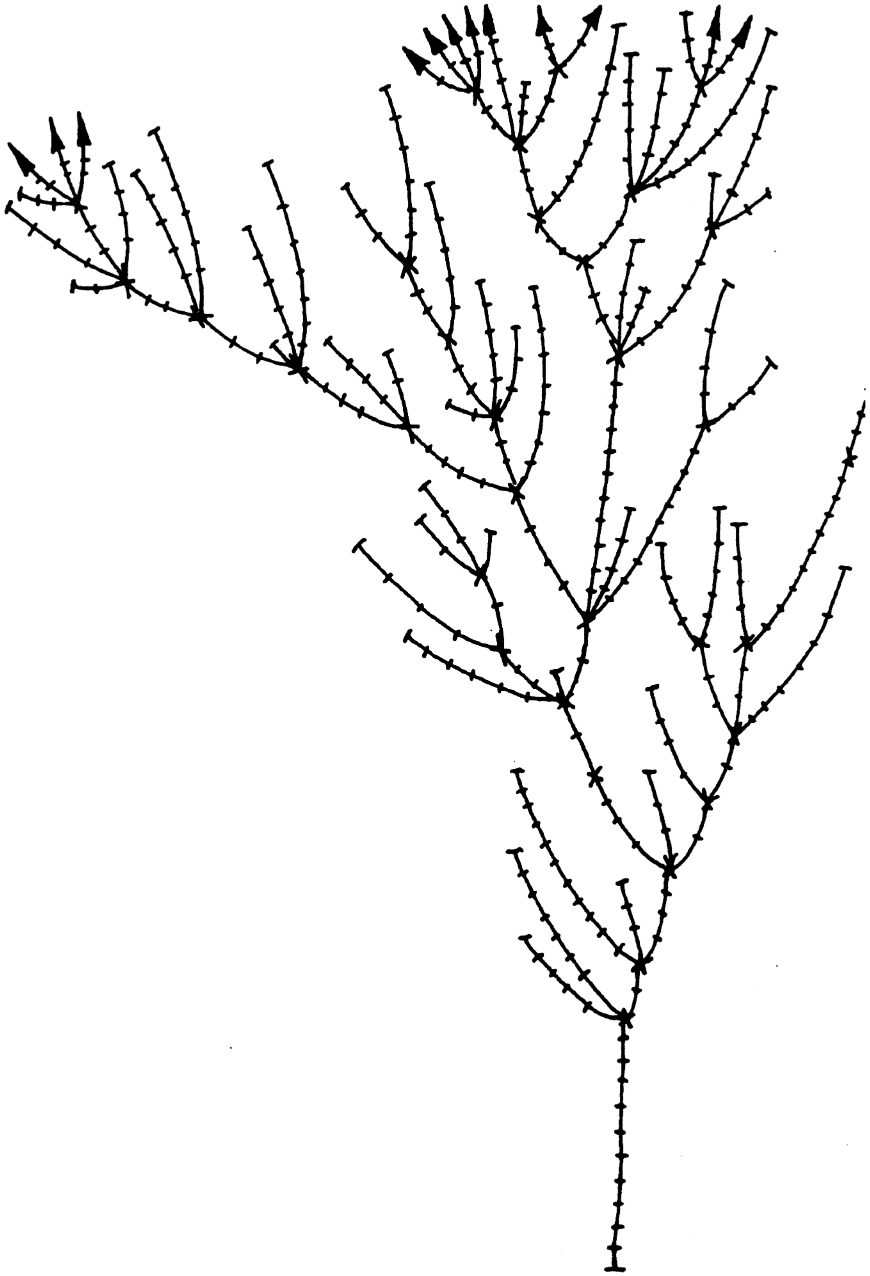

Fig. C. Example of interpretation of shrub architecture recorded symbolically in Fig. B

The identification of the distributions of the random variables can be made with no difficulty. For instance, given a considerable number of observations of dead shoots it is easy to mark how many of these died in particular years and hence calculate the values of the parameters d_j^*. For both the species studied, these distributions were so much alike that they were recognized as the same. The results are shown in Fig. D.

j	$\mathbf{a}_j$	$\mathbf{b}_j$	$\mathbf{c}_j^*$	$\mathbf{c}_j$ $r_h=0.615$	$\mathbf{c}_j$ $r_f=0.635$	$\mathbf{d}_j^*$	$\mathbf{d}_j$ $r_h=0.615$	$\mathbf{d}_j$ $r_f=0.635$
0		0.02				0.06	0.037	0.038
1	0	0.07	0	0	0	0.08	0.051	0.053
2	0	0.325	0.215	0.091	0.086	0.013	0.096	0.099
3	0	0.325	0.585	0.3	0.285	0.285	0.333	0.337
4	0	0.17	0.105	0.115	0.108	0.15	0.297	0.3
5	0	0.065	0.04	0.071	0.066	0.115	0.349	0.352
6	0	0.02	0.015	0.044	0.041	0.07	0.341	0.345
7	0	0.005	0.015	0.07	0.065	0.06	0.478	0.482
8	0	0	0.005	0.048	0.045	0.015	0.24	0.244
9	0	0	0.005	0.066	0.062	0.015	0.338	0.344
10	0	0	0.005	0.107	0.1	0.01	0.381	0.388
11	0.25	0	0.005	0.192	0.182	0.005	0.381	0.388
12	0.295	0	0.005	0.385	0.365	0.005	1	1
13	0.295	0	0	0	0	0	0	0
14	0.04	0	0	0	0	0	0	0
15	0.015	0	0	0	0	0	0	0
16	0.015	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0

Fig. D. Parameters of distributions of random variables considered in simulation

The parameter r turned out to be the most difficult to determine. It may be realized intuitively that it is a markedly conditional parameter, i.e. the model under consideration is very sensitive to possible errors in identification. It was not possible to apply a direct method, since in the younger parts of the plant there appear shoots which may still either flower or wither, while in the older parts some of the withered shoots rot and become imperceptible. For various reasons indirect methods did not give results. Finally, the parameter r, different for the two species, was determined on the basis of the analysis of the simulation results for its individual values and, in particular, on observations of the number of living shoots in consecutive years. Preliminary estimations have been obtained on the base of the fact that for:

$$r_0 = 1 - \frac{1}{E(b)} = 1 - \frac{1}{\sum_{i=0}^{20} ib_i} \cong 1 - \frac{1}{2.85} \cong 0.65 \tag{24}$$

the expected value of the number of living shoots is constant in consecutive years. Therefore the parameter r should have a slightly lower value. Finally $r_h = 0.615$ was accepted for the species of Rhododendron hirsutum and $r_f = 0.635$ for Rh. ferrugineum.

The distributions characterizing the first florescence of a specimen and the number of shoots originating after the florescence do not need elaboration, and may be applied directly. In accordance with the biological cycle, it has been assumed that in a given year the first florescence may possibly occur, and then the withering is possible. Then (Fig. D.):

$$c_j = \begin{cases} \frac{(1-r)c_j^*}{c^{\sim}} & \text{if } c^{\sim} \neq 0 \\ 0 & \text{if } c^{\sim} = 0 \end{cases} \tag{25}$$

$$d_j = \begin{cases} \frac{rd_j^*}{d^{\sim}} & \text{if } d^{\sim} \neq 0 \\ 0 & \text{if } d^{\sim} = 0, \end{cases} \tag{26}$$

where:

$$c^{\sim} = 1 - (1-r)\sum_{i=1}^{j-1} c_i^* - r\sum_{i=0}^{j-1} d_i^* \tag{27}$$

$$d^{\sim} = 1 - (1-r)\sum_{i=1}^{j} c_i^* - r\sum_{i=0}^{j-1} d_i^*. \tag{28}$$

First, the computations of the program give the value of the random variable defining the year of the first florescence on the basis of the parameters a_j, as well as the value of the random variable deciding the number of shoots appearing on the basis of the parameter b_j. Next, the changes taking place in consecutive years are defined; in each year all the living shoots are considered in turn. Thus, after determining the number of years which have passed since the last inflorescence, the fact whether a shoot flowered or not in the year under observation is established on the basis of the parameters c_j. If it did, the number of shoots appearing on it is calculated on the basis of the parameters b_j. Next, on the basis of the parameter d_0, it should be established for each of them whether it withers in the year of appearance or not. If, however, the shoot in question does not flower, it should be established whether it withers or not, on the basis of the parameters d_j.

In accordance with the results, the symbolic picture of the shrub is supplemented by the symbols: "*", "∧", "-", "+" or "#". Its printed letter is preceded by the symbol "|" in order to facilitate the interpretation. The results are recorded in the form of a symbolic picture (Fig. B) and tables. Fig. E shows such a table for a shrub in Fig. B and C, covering a period of 50 years. The variety of the changes occurring is worth noticing. As in nature, there are years of prosperity and years of crisis. What is more, they have a tendency to appear alternately. Fig. F gives a few results of the simulating program for the species of Rhododendron hirsutum, as does Fig. G for Rh. ferrugineum. Their diversity is evident, as are the cases of shrubs dying at various ages, resembling the modeled nature again.

In conclusion it should be noted that the computer simulation displayed a satisfactory consistence with the reality of the results obtained by using the presented model. This consistence however, is difficult to perceive with the help of numerical criteria.

One problem which emerged during the simulated verification of the correctness of the model, is overmuch frequent death of the shrub between 15 and 40 years of age. This may be explained by the existence of fluctuations, natural in probabilistic models. They proceed around the slowly increasing expected value of the number of living shoots, at this stage still near zero. The survival of the shrub indicates that there was not a single year in which the number of living shoots assumed the value zero, which is the product of many events. It should be remarked that symmetry is not preserved here, because there is a lack of similar limitation "from above".

Year	Number of living shoots	Number of dead shoots	Numbers of flowers	Increase in number of living shoots	Increase in number of dead shoots	Increase in number of flowers
1	1	0	0	1	0	0
2	1	0	0	0	0	0
3	1	0	0	0	0	0
4	1	0	0	0	0	0
5	1	0	0	0	0	0
6	1	0	0	0	0	0
7	1	0	0	0	0	0
8	1	0	0	0	0	0
9	1	0	0	0	0	0
10	1	0	0	0	0	0
11	1	0	0	0	0	0
12	3	0	1	2	0	1
13	3	0	1	0	0	0
14	5	0	2	2	0	1
15	5	0	2	0	0	0
16	3	2	2	-2	2	0
17	3	2	2	0	0	0
18	4	3	3	1	1	1
19	4	3	3	0	0	0
20	4	4	4	0	1	1
21	4	4	4	0	0	0
22	6	4	6	2	0	2
23	5	5	6	-1	1	0
24	6	7	7	1	2	1
25	8	7	9	2	0	2
26	11	7	10	3	0	1
27	12	7	11	1	0	1
28	9	10	11	-3	3	0
29	7	14	12	-2	4	1
30	7	16	13	0	2	1
31	6	17	13	-1	1	0
32	6	17	13	0	0	0
33	9	17	14	3	0	1
34	8	18	14	-1	1	0
35	9	19	16	1	1	2
36	10	20	18	1	1	2
37	11	22	19	1	2	1
38	9	24	19	-2	2	0
39	10	27	21	1	3	2
40	8	30	23	-2	3	2
41	7	31	23	-1	1	0
42	9	31	24	2	0	1
43	13	33	27	4	2	3
44	11	35	27	-2	2	0
45	13	36	28	2	1	1
46	13	39	29	0	3	1
47	10	42	29	-3	3	0
48	12	45	31	2	3	2
49	16	45	33	4	0	2
50	12	49	33	-4	4	0

Fig. E. Table characterizing development of shrub shown in Fig. *B* and C

Number	Year	Number of living shoots	Number of dead shoots	Numbers of flowers
1	80	83	218	145
2	80	50	248	163
3	18 +	0 +	5 +	2 +
4	36 +	0 +	16 +	10 +
5	80	39	185	119
6	80	81	273	176
7	80	73	229	155
8	80	103	333	232
9	80	108	447	285
10	80	65	273	179
11	59 +	0 +	86 +	40 +
12	27 +	0 +	13 +	4 +
13	80	7	70	44
14	80	52	337	196
15	32 +	0 +	11 +	7 +
16	80	147	438	323
17	80	35	124	97
18	80	99	366	245
19	80	199	590	405
20	80	113	625	392
21	27 +	0 +	8 +	2 +
22	80	173	363	277
23	80	60	386	251
24	80	128	336	240
25	80	44	274	163
26	80	79	369	237
27	32 +	0 +	21 +	8 +
28	80	69	488	306
29	42 +	0 +	18 +	9 +
30	80	133	425	283
31	80	159	599	403
32	80	22	361	211
33	80	93	217	153
34	80	71	335	220
35	80	55	348	225
36	73 +	0 +	77 +	46 +
37	16 +	0 +	4 +	1 +
38	26 +	0 +	12 +	5 +
39	80	25	174	116
40	80	75	297	201
41	80	118	391	267
42	23 +	0 +	3 +	2 +
43	80	12	115	78
44	80	46	335	196
45	80	84	349	228
46	60 +	0 +	81 +	46 +
47	19 +	0 +	2 +	1 +
48	80	86	420	274
49	80	96	338	235
50	49 +	0 +	28 +	18 +

Fig. F. Results of simulation obtained for Rhododendron hirsutum (symbol "+" distinguishes dead shrubs)

Number	Year	Number of living shoots	Number of dead shoots	Numbers of flowers
1	80	77	241	170
2	43 +	0 +	36 +	21 +
3	80	11	170	90
4	80	41	134	88
5	80	30	166	100
6	80	9	110	60
7	19 +	0 +	3 +	3 +
8	80	32	480	285
9	32 +	0 +	13 +	6 +
10	29 +	0 +	14 +	6 +
11	80	21	212	123
12	80	33	385	241
13	80	101	443	285
14	80	39	339	200
15	56 +	0 +	56 +	34 +
16	80	15	251	146
17	36 +	0 +	16 +	9 +
18	80	8	102	55
19	26 +	0 +	5 +	3 +
20	45 +	0 +	26 +	16 +
21	27 +	0 +	8 +	4 +
22	80	51	245	154
23	80	27	232	144
24	30 +	0 +	11 +	5 +
25	80	17	84	53
26	80	22	165	102
27	80	53	313	192
28	80	25	257	145
29	35 +	0 +	27 +	13 +
30	40 +	0 +	27 +	12 +
31	76 +	0 +	68 +	41 +
32	27 +	0 +	17 +	8 +
33	80	60	230	148
34	80	28	276	168
35	80	44	296	179
36	23 +	0 +	8 +	4 +
37	80	47	287	183
38	80	5	154	90
39	80	36	299	188
40	80	83	472	287
41	34 +	0 +	14 +	9 +
42	80	155	631	407
43	21 +	0 +	5 +	3 +
44	20 +	0 +	5 +	2 +
45	15 +	0 +	3 +	1 +
46	80	59	250	173
47	80	65	235	149
48	66 +	0 +	55 +	31 +
49	80	15	101	66
50	80	44	515	311

Fig. G. Results of simulation obtained for Rhododendron ferrugineum (symbol "+" distinguishes dead shrubs)

Solution of that problem can be the dependence of the random variable distribution of withering or florescence of shoots on the number of living shoots, e.g. by the value of the parameter r. Such behavior unfortunately increases the complexity of the model, which may make render its application difficult, on increasing the number of parameters requiring identification. In addition to this, the exploitation of this theory and the formulas becomes more complicated. In spite of these reservations, a creation of such a model will be the subject of further investigations of the modeling of biological processes by probabilistic models, although it should be emphasized that by means of the model which is simpler and so easier to use, quite satisfactory results have been achieved.

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