A Random Approach to Time-Optimal Control

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This paper concerns the time-optimal control for objects described by a random differential inclusion with discontinous right-hand side, representing the second law of Newtonian mechanics and taking into account a complex model of resistance to motion. Such a task has broad technical applications, especially in robotics. By generalizing the concept of the classic switching curve to the switching region, it is possible to construct in practice a range of convenient suboptimal control structures that provide many advantages, especially in respect to robustness.

1 Introduction

Consider a single-degree-of-freedom mechanical system, playing a vital role in modern engineering, especially as the foundation for the analysis of problems associated with industrial manipulators and robots. Its dynamics can be described by the following differential inclusion:

$$\ddot{y}(t) \in H(\dot{y}(t), y(t), t) + u(t), \tag{1}$$

where u denotes a bounded control function, and the scalar mapping y means the position of the object. (If the function H identically equals zero, the above inclusion is reduced to the planar linear differential equation

$$\ddot{y}(t) = u(t) \tag{2}$$

which expresses the second law of Newtonian mechanics.) The multivalued (set-valued) function H, however, describes resistance to motion. For the majority of cases in engineering practice, this function can be expressed in the form

$$H(\dot{y}(t), y(t), t) = v(\dot{y}(t), y(t), t)F(\dot{y}(t)), \tag{3}$$

where v is a continuous mapping, and F denotes a piecewise continuous function that may be additionally multivalued at the points of discontinuity. In view of the limitations of the available mathematical methods, the typical synthesis of the controlling system must be preceded by a significant simplification of the form of the function v to trivial ones.

In this paper, a probabilistic concept for solving the problem will be proposed. Namely, it is assumed that the function v introduced in Eq. (3) represents the realization of a given stochastic process V with almost all the realizations being continuous and jointly bounded. Such a model regards as probabilistic uncertainty the dependence of resistance on a broad range of factors, not only $\dot{y}(t)$, $y(t)$ and t, but also those which are usually omitted under other conceptions, due to the necessity to simplify the model. Since random control systems take account of the entire set of possible events, they are distinguished in engineering practice by their significant robustness against the imprecisions of the model, the temporal fluctuations of the characteristics, and the perturbations and noise occurring naturally in real objects.

The deterministic approach to time-optimal control for discontinuous systems of type (1) is presented in Hejmo (1994). The classical methods of control synthesis can be found in survey papers by Tourassis (1988) or Ortega and Spong (1989).

2 Main Results

Theorem.
Assume:

(A) $t_0 \in \mathbb{R}$, $T = [t_0, \infty)$, $x_0 \in \mathbb{R}^2$, $v_-, v_+ \in \mathbb{R}$ such that $-1 < v_- \leq v_+ < 1$;

(B) the origin of coordinates constitutes the target set;

(C) (Ω, Σ, P) is a complete probability space, where Ω denotes the set of elementary events ω, Σ — the sigma algebra of its subsets, and P means the probability measure;

(D) $U_a = \{U : \Omega \times T \to [-1, 1]\}$ represents the set of admissible controls;

(E) $f : \mathbb{R} \to [-1, 1]$ denotes a piecewise continuous function fulfilling locally a Lipschitz condition except at the points of discontinuity, and satisfying the condition $z \cdot f(z) \geq 0$ for every $z \in \mathbb{R}$; moreover, let $F : \mathbb{R} \to \mathcal{P}([-1, 1])$ be such that

$$F(z) = \begin{cases} f(z) & \text{if } z \neq z_i \\ F_i & \text{if } z = z_i \end{cases}, \tag{4}$$

where $z_i \in \mathbb{R}$ and $F_i \subset [-1, 1]$ for $i = 1, 2, \ldots, k$, while $\mathcal{P}([-1, 1])$ means the sets of subsets of the interval $[-1, 1]$;

(F) V is a real stochastic process with almost all realizations being continuous, and satisfying the boundary condition $V(\omega, t) \in [v_-, v_+]$ for $t \in T$;

(G) the random differential inclusion

$$\dot{X}_1(\omega, t) = X_2(\omega, t) \tag{5}$$

$$\dot{X}_2(\omega, t) \in U(\omega, t) - V(\omega, t)F(X_2(\omega, t)), \tag{6}$$

with the initial condition

$$\begin{bmatrix} X_1(\omega, t_0) \\ X_2(\omega, t_0) \end{bmatrix} = x_0 \quad \text{for almost all } \omega \in \Omega \tag{7}$$

describes the dynamics of the system submitted to the control U.

Then, there exists an almost certain time-optimal control (i.e., a real stochastic process such that almost all its realizations are time-optimal controls for proper deterministic systems obtained at the fixed random factor $\omega \in \Omega$), whose realizations take on the values $+1$ or -1, and have at most one point of discontinuity. This control generates a unique almost certain C-solution (i.e., a two-dimensional stochastic process such that almost all its realizations are solutions in the Caratheodory sense for proper deterministic systems obtained at the fixed factor $\omega \in \Omega$), which is also a unique almost certain F-solution (as above for solutions in the Filippov sense) and a unique almost certain K-solution (as above for solutions in the Krasovski sense). ■

Definitions of Caratheodory, Filippov, and Krasovski solutions (in the deterministic case) can be found in Kulczycki (1996b). System (5)–(6) is the random counterpart of differential equation (1) with the substitution of dependence (3); in this sense, the stochastic processes X_1 and X_2 can be identified with the functions y and $\dot{y}$, respectively. The proof of the theorem presented above, including the existence of the stochastic processes X_1, X_2, U and a characterization of the control U, is presented in Kulczycki (1996a, d). The sets of all states which can be brought to the origin by the control $U \equiv +1$, if $V \equiv v_-$ or $V \equiv v_+$, respectively, have been defined here as K_{+-}, K_{++}; analogously, K_{--} and K_{-+} for $U \equiv -1$, if $V \equiv v_-$ or $V \equiv v_+$. These are, accordingly, the sets of states brought to the target by the control $+1$ or -1, given the minimum and maximum values of resistance to motion (for the relative localization of those sets see also Fig. 1). Because the change of sign in the particular realizations of the almost certain time-optimal control (switching of the control) can occur only

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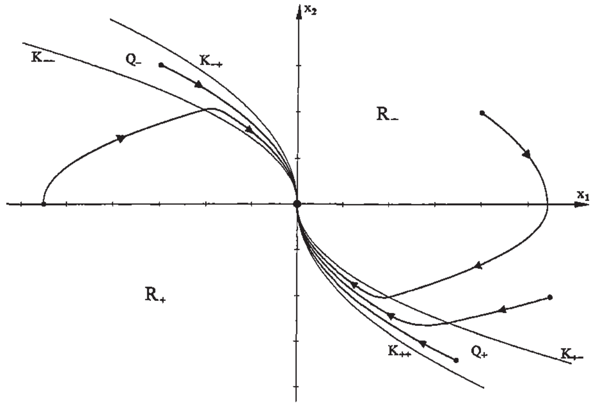

Fig. 1 Trajectories generated by control (8)

when the system state belongs to the set (closed region) restricted by K_{+-}, K_{++} and K_{--}, K_{-+}, this set is called the switching region. It constitutes the generalization of a switching curve familiar from the classic case of time-optimal transfer of a mass (Athans and Falb, 1966; Section 7.2). (The region is reduced to this curve when $v_- = v_+ = 0$.)

Therefore, the function H introduced in formula (1), which represents the model of resistance to motion, has been decomposed into two factors: $F(\dot{y}(t))$ and $V(\omega, t)$. The former, a deterministic one, makes it possible to incorporate the properties of discontinuity and multivalency of friction phenomena. The latter one, thanks to its probabilistic nature, includes among other things approximations and identification errors (of the first factor also), the dependence of resistance on position, time, and temperature, as well as perturbations and noise naturally occurring in real systems. The switching curve implied by the first — deterministic — factor has been "blurred" by the second — random — to the switching region.

The almost certain time-optimal control obviously ensures the realization of the minimum expected value of the time to reach the target set; however, besides specific cases, its direct practical realization encounters difficulties because of the dependence on the random factor, in fact unknown *a priori.* However, thanks to the results of the above Theorem, the presented material constitutes a useful basis for the creation of technical constructions of suboptimal control structures, in which such a dependence is removed.

The first conception of these structures can be based on the classical idea of the switching curve, though its parameters are optimally calculated by using statistical decision theory. A detailed description is found in Kulczycki (1996c).

For the case when the used actuator allows all control values from the interval $[-1, 1]$, the above structure can be modified to eliminate frequent switchings between $+1$ and -1 occurring on the sliding trajectories. (Such switchings often have a negative impact on the endurance of the device and the convenience of its use.) Namely, let the sets K_{+-}, K_{++} and K_{--}, K_{-+} remain unchanged; moreover, the sets Q_+, Q_- and R_-, R_+ are given as shown on Fig. 1. The control is now defined by the formula

$$U(\omega, t) = \begin{cases} -1 & \text{if } [X_1(\omega, t), X_2(\omega, t)]^T \in R_- \\ -g(-x_1, -x_2) & \text{if } [X_1(\omega, t), X_2(\omega, t)]^T \in Q_- \\ 0 & \text{if } [X_1(\omega, t), X_2(\omega, t)]^T \in \{[0, 0]^T\}, \\ g(x_1, x_2) & \text{if } [X_1(\omega, t), X_2(\omega, t)]^T \in Q_+ \\ +1 & \text{if } [X_1(\omega, t), X_2(\omega, t)]^T \in R_+ \end{cases} \tag{8}$$

where the function $g : \mathbb{R}^2 \to \mathbb{R}$ takes on the fixed value $g_* \in [-1, 1 - v_+ + v_-]$

$$g(x_1, x_2) = g_* \tag{9}$$

on the sets K_{+-} and K_{--}, after which it shows a continuous and monotonic increase to the value 1 on the sets K_{++} and K_{-+}. This idea can be regarded as the result of averaging in respect to the random factor of the results from the Theorem presented above. Condition (9) is sufficient to guarantee the existence of a C-solution, which in practice entails the avoidance of the sliding trajectories that can appear in the opposite case along the curves K_{+-} and K_{--}. When that condition is stated more precisely as

$$g(x_1, x_2) = -1, \tag{10}$$

one obtains the continuity of control (8). This control can then be interpreted as a smoothed version of the "bang-bang" conception, presented in Kulczycki (1996c).

3 Conclusions

In this paper, a probabilistic concept has been presented for the synthesis of a time-optimal control for objects described by random discontinuous differential inclusion (1). Two suboptimal closed-loop structures have been proposed as applicational conclusions from the presented Theorem. The first of these is intended for cases where the conditioning of an actuator makes it possible to use only the extreme values of the admissible controls. The second, not encumbered by any such limitations, allows for the avoidance of sliding trajectories, which in practice are often inconvenient. The above structures are endowed with all the advantages of random control systems, especially as to robustness, a matter of exceptional importance in the practice of contemporary engineering.

References

Athans, M., and Falb, P. L., 1966, *Optimal Control,* McGraw-Hill, New York.

Hejmo, W., 1994, "Time-optimal Feedback System Controlling a Discontinuous Dynamic Object Reaching a Moving Target," *Automatica,* Vol. 30, pp. 1937–1941.

Kulczycki, P., 1996a, "Almost Certain Time-Optimal Positional Control," *IMA Journal of Mathematical Control & Information,* Vol. 13, pp. 63–77.

Kulczycki, P., 1996b, "Some Remarks on Solutions of Discontinuous Differential Equations Applied in Automatic Control," *Industrial Mathematics,* Vol. 46, pp. 119–128.

Kulczycki, P., 1996c, "Time-Optimal Positional Feedback Controller for Random Systems," *Proceedings of the National Science Council ROC(A),* Vol. 20, pp. 79–89.

Kulczycki, P., 1996d, "Time-Optimal Stabilization of a Discontinuous and Non-Autonomous Dynamic Object," *Control and Cybernetics,* Vol. 25, pp. 707–720.

Ortega, R., and Spong, M. W., 1989, "Adaptive Motion Control of Rigid Robots: A Tutorial," *Automatica,* Vol. 25, pp. 877–888.

Tourassis, V. D., 1988, "Principles and Design of Model-Based Robot Controllers," *International Journal of Control,* Vol. 47, pp. 1267–1275.

A Simplified Robust Circle Criterion Using the Sensitivity-Based Quantitative Feedback Theory Formulation

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The circle criterion provides a sufficient condition for global asymptotic stability for a specific class of nonlinear systems, those

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