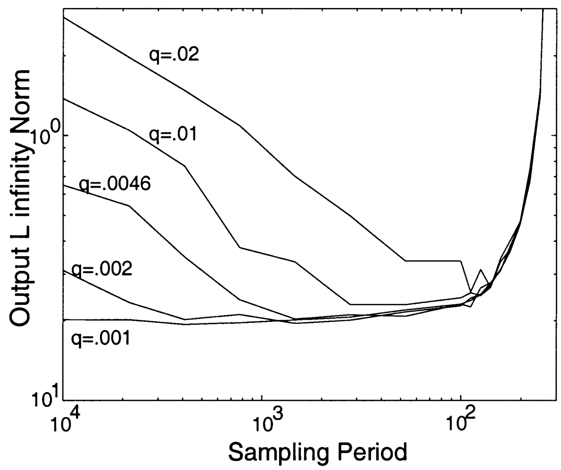

Fig. 7. Lower bounds on L^∞ performance as a function of sampling rate and quantizer precision.

The main feature of the performance data is that performance degrades with decreasing quantizer precision. As the quantizer's precision increases, the sampling rate at which the "onset" of worsening performance occurs also increases. In general however, it seems that for any given quantizer precision, there is a sampling rate fast enough to cause performance to degrade. Furthermore, from this and other data (for various precisions), it seems that as the sampling rate increases, performance degrades unboundedly. Since simulation time increases linearly with the sampling rate, it is not very practical to simulate rates faster than 10^{-4}-10^{-5} using MATLAB on currently available workstations.

The performance data indicates that for any given precision level in the controller's arithmetic, the output norm limits to infinity as the sampling period goes to zero. If this conjecture is true, such systems should be regarded as unstable for fast sampling rates, even though for any fixed rate they are BIBO stable as argued earlier.

Finally, we should point out that as observed in [8], one can significantly improve the finite precision performance of a system by the choice of controller realization. This issue was not addressed in this note. A more judicious choice of controller realization, would have probably improved the performance curves in Fig. 7 (by delaying the onset of worsening performance to higher sampling rates). However, it appears that the basic pattern of degrading performance for sufficiently fast sampling rates would persist. This last point deserves further investigation, and if true, would indicated some fundamental limitation for the use of unstable dynamic sampled-data controllers.

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Rotational Motion Control of a Spacecraft

Rafal Wisniewski and Piotr Kulczycki

***Abstract*—This note describes a systematic procedure for the control synthesis of a rigid spacecraft using the energy shaping method. The geometric concept of a mechanical system in a coordinate-independent form is used to derive a control algorithm for the Euler–Poincaré equations. The main result of this note is a specialization of the method on the unit quaternion group. This note is concluded with the examples of the potential functions and implementation for the three-axis attitude control problem.**

***Index Terms*—Attitude control, differential geometric methods, nonlinear control, stability theory.**

I. INTRODUCTION

Over the last fifty years since the first spacecraft was launched, the subject of attitude control has become mature. A new demand on the aerospace/control engineering has come up. The design phase has to be reduced in time and thereby in cost. A way for achieving this goal is to establish a general design method for an on-board attitude control. Here, energy shaping seems to be a good candidate. The objective of this work is to adopt energy shaping to rotational motion control of a spacecraft.

The energy shaping method in its most common formulation gives a control action, which is the sum of the gradient of potential energy and the dissipation force; [1, Ch. 12]. Such a control law makes the system uniformly asymptotically stable to the desired reference point—the point of minimal potential energy. The key to a precise description of this method is a concept of a mechanical control system. In general, a mechanical system may be approached from the point of view of Riemannian geometry on the tangent bundle or symplectic geometry on the cotangent bundle; see [2] and [3]. An additional generalization of the mechanical system on the cotangent bundle can be accomplished introducing a Poisson structure on the configuration manifold, in particular,

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R. Wisniewski is with the Department of Control Engineering, Aalborg University, DK-9220 Aalborg East, Denmark (e-mail: raf@control.auc.dk).

P. Kulczycki is with the Systems Research Institute, Polish Academy of Sciences, PL-01-447 Warsaw, Poland (e-mail: kulczycki@ibspan.waw.pl).

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the underlying manifold needs not to be even dimensional. If the system allows symmetry, it is an advantage to use the reduced Lie–Poisson dynamics [4].

Stabilization by the energy shaping of a mechanical system was first proposed in [2]. At present, it stands for numerous methods reliant on the aforementioned descriptions of the mechanical system. References [1] and [2] presented the energy shaping for manifolds with Poisson structure. The concept of the energy shaping on a Riemannian manifold was addressed in [3] and [5]. The first work relied on the Lagrangian approach and provided earliest coordinate free formulation of the method, the latter gave a description of the energy shaping for a general mechanical system expressed in terms of a Riemannian connection.

This note brings motion control of a rigid body into focus. The contribution is an application of the potential energy shaping for the systems described by the Euler–Poincaré equations with forcing. The work places the stress on the configuration manifold given by the group of unit quaternions. The choice of the unit quaternion is made as it is the most commonly used attitude representation in the literature on aerospace. The findings of this work are implemented for three-axis attitude control of a rigid spacecraft.

This note is organized as follows. Section II provides a general description of an Euler–Poincaré control system. The energy shaping method for the configuration space of the unit quaternions is formulated in Section III. Section IV gives an application of the results for the three-axis control of a rigid spacecraft.

II. Euler–Poincaré Control System

A description of a general mechanical system with forcing is addressed in this section. The final part deals with modeling of a particular system, a rigid body. A rigid body belongs to the class of simple mechanical systems, which Hamiltonian is the sum of potential and kinetic energy. It has been exhaustively analyzed in the literature of classical mechanics. This gives freedom to treat it from a Hamiltonian or a Lagrangian point of view, as motion on: Riemannian, symplectic, or Poisson manifold. Following the baselines of [6] and [5], a simple mechanical system on a configuration manifold Q is defined by a Riemannian metric g playing the role of the kinetic energy, a twice differentiable function ϕ on Q identified with the potential energy, and m one-forms, $F^1, \ldots, F^m$, on Q defining the inputs to the system. There exists a unique linear connection $\nabla : \tau(Q) \times \tau(Q) \rightarrow \tau(Q)$ that is compatible with g and symmetric, called the Levi–Civita connection [7]. The forced Euler–Lagrange equation ([4] and [8]) can then be expressed by means of the covariant derivative D/Dt related to ∇

$$g^{\flat} \frac{D\dot{q}}{Dt} = -d\phi(q) + F(q, \dot{q}) \tag{1}$$

where $g^{\flat} : TQ \rightarrow T^*Q$ indicates the following map associated to the Remannian metric g:

$$g(V, W)_q = (g^{\flat} V)(q) \cdot W, \forall W \in T_q Q \tag{2}$$

$D\dot{q}/Dt$ in (1) denotes the covariant derivative of $\dot{q}$ along $\dot{q}$, $d\phi$ is the differential of the potential function $\phi(q)$ and $F(t, q, \dot{q}) = \sum F^j(q) u_j(t)$ stands for the input force. Notice that if there are no external forces and the potential energy equals zero then $D\dot{q}/Dt = 0$ gives the equation of the geodesics with respect to ∇.

If the configuration manifold is a Lie group, (1) can be transformed using Euler–Poincaré reduction to two sets of equations: kinematics and dynamics; see [4, Ch. 13.6]. The following theorem formulates the Euler–Poincaré equation with a force field, a fiber preserving map, $F : TG \rightarrow T^*G$.

Theorem 1: Let G be a Lie group with Lie algebra $\mathfrak{g}$, $L : TG \rightarrow \mathbb{R}$ be a left invariant Lagrangian, $l : \mathfrak{g} \rightarrow \mathbb{R}$ be its restriction to the Lie algebra and $F : TG \rightarrow T^*G$ a force field. For a curve $g(t) \in G$, let $\xi(t) = T_{g(t)} L_{g(t)^{-1}} \dot{g}(t)$.

Then, the integral Lagrange–d'Alembert principle

$$\delta \int_a^b L(g(t), \dot{g}(t)) dt = \int_a^b F(g(t), \dot{g}(t)) \cdot \delta g \, dt \tag{3}$$

holds for all proper variations $\delta g(t)$, is equivalent to the following Euler–Poincaré equation with forcing:

$$\frac{d}{dt} dl_{\xi}(t) = ad_{\xi}^* dl_{\xi}(t) + T_e^* L_g \, F(g(t), \dot{g}(t)) \tag{4}$$

$$\dot{g}(t) = T_e L_{g(t)} \, \xi(t). \tag{5}$$

Proofs of Theorem 1: Since L is left invariant

$$L(g(t), \dot{g}(t)) = L(e, T_{g(t)} L_{g(t)^{-1}} \dot{g}(t))$$

the left-hand side of (3) becomes

$$\delta \int_a^b L(g(t), \dot{g}(t)) dt = \delta \int_a^b l(\xi(t)) dt = \int_a^b \frac{\delta l}{\delta \xi} (\delta \xi) dt. \tag{6}$$

According to [9, Th. 6.1] the variation of

$$\xi(t, \epsilon) = T L_{g(t,\epsilon)^{-1}} (\partial g(t, \epsilon))/(\partial t)$$

takes the form

$$\delta \xi = \frac{\partial \xi(t, \epsilon)}{\partial \epsilon} = \frac{\partial \eta(t, \epsilon)}{\partial t} + ad_{\xi(t,\epsilon)} \eta(t, \epsilon) \tag{7}$$

where $\eta(t, \epsilon) = T L_{g(t,\epsilon)^{-1}} (\partial g(t, \epsilon))/(\partial \epsilon)$. Substituting (7) into (6) gives

$$\int_a^b \frac{\delta l}{\delta \xi} (\delta \xi) dt = \int_a^b \frac{\delta l}{\delta \xi} \left(\frac{\partial \eta}{\partial t} + ad_{\xi} \eta \right) dt$$

$$= \int_a^b \left(-\frac{d}{dt} \frac{\partial l}{\partial \xi} + ad_{\xi}^* \frac{\partial l}{\partial \xi} \right) (\eta) dt. \tag{8}$$

The right-hand side of (3) can be rewritten as

$$\int_a^b F(\delta g) dt = \int_a^b F(T_e L_g \eta) dt = \int_a^b T_e^* L_g F(\eta) dt. \tag{9}$$

Comparing (8) and (9) with (3), the Euler–Poincaré equation (4) follows. ■

In the remaining part of this section, the equation of the rigid body are formulated with use of Theorem 1. The attitude is parameterized by the unit quaternion. The coordinates in $\mathbb{R}^4$ are denoted by q^i, and the bases for the tangent space at q, $T_q \mathbb{R}^4$, consist of the partial derivative operators $\partial/\partial q^1$. The unit quaternions $S^3 = \{q \in \mathbb{R}^4 : \|q\| = 1\}$ form a Lie group with the multiplication defined by the following formula:

$$q \cdot p = Q(q) p \tag{10}$$

where

$$Q(q) = \begin{bmatrix} q^0 & -q^1 & -q^2 & -q^3 \\ q^1 & q^0 & -q^3 & q^2 \\ q^2 & q^3 & q^0 & -q^1 \\ q^3 & -q^2 & q^1 & q^0 \end{bmatrix}. \tag{11}$$

The physical interpretation of the unit quaternion arises from the Euler's theorem. It states that the general displacement of a rigid body with one point fixed is uniquely given by a unit vector ϵ, defining an axis of rotation, and an angle of rotation ϕ. A unit quaternion can be interpreted as a combination of the components of the unit vector and the angle of rotation

$$q = [\cos \tfrac{\phi}{2} \quad \epsilon^1 \sin \tfrac{\phi}{2} \quad \epsilon^2 \sin \tfrac{\phi}{2} \quad \epsilon^3 \sin \tfrac{\phi}{2}]^{\mathrm{T}}. \tag{12}$$

Denoting the body angular velocity by ω the kinematics of a rigid body motion take the celebrated formula [10]:

$$\dot{q} = \frac{1}{2}Q(q)i(\omega) \tag{13}$$

where $i : \mathbb{R}^3 \hookrightarrow \mathbb{R}^4$ is the inclusion $[\omega^1\ \omega^2\ \omega^3]^{\mathrm{T}} \mapsto [0\ \omega^1\ \omega^2\ \omega^3]^{\mathrm{T}}$. Notice that (13) corresponds to (5) for $\xi = (1/2)i(\omega)$ and

$$T_e L_q \beta = Q(q)\beta, \text{ where } \beta \in T_e S^3. \tag{14}$$

The Lagrangian of the rigid body consists of the kinetic energy only

$$l(\omega) = T(\omega) = \frac{1}{2}\omega^{\mathrm{T}} J \omega \tag{15}$$

where ω is the body angular velocity, and J indicates the inertia tensor. To derive the Euler–Poincaré equation two facts are used. The Lie algebra of S^3 is $\mathbb{R}^3$ relative to the Lie bracket given by twice cross product of vectors

$$ad^*_\xi \alpha = 2i(\pi(\xi) \times \pi(\alpha)) \tag{16}$$

where $\xi \in T_e S^3$, $\alpha \in T^*_e S^3$, and $\pi : \mathbb{R}^4 \to \mathbb{R}^3$ is the projection $[q^0\ q^1\ q^2\ q^3]^{\mathrm{T}} \mapsto [q^1\ q^2\ q^3]^{\mathrm{T}}$. Furthermore, the following equality holds:

$$T^*_e L_q \alpha = Q^{\mathrm{T}}(q)\alpha, \ \forall \alpha \in T^*_e S^3. \tag{17}$$

It is worthy of note that in (16) the tangent space $T_e S^3$ and the cotangent space $T^*_e S^3$ were identified with $\mathbb{R}^3$.

Finally, the specialization of the Euler–Poincaré equation (4) for a rigid body and identifying $T_e S^3$ as well as $T^*_e S^3$ with $\mathbb{R}^3$ gives

$$J\dot{\omega} = \omega \times J\omega + M(q, \dot{q}) \tag{18}$$

where

$$M(q, \dot{q}) = \frac{1}{2}\pi Q^{\mathrm{T}}(q) F(q, \dot{q}). \tag{19}$$

Notice that $M(q, \dot{q})$ in (18) is merely a torque. It has been decided to keep the original force field $F(q, \dot{q})$ in the formulation of the rigid body motion to make an implementation of the standard energy shaping method straightforward.

III. Control Synthesis

The energy shaping has been formulated for a general mechanical system (1) in [3] and [6]. The control consists of a differential of a potential function $\phi : G \to \mathbb{R}$ and a dissipative force field $F_d : TG \to T^*G$ as indicated in the following equation:

$$F(g, \dot{g}) = -d\phi(g) + F_d(g, \dot{g}). \tag{20}$$

The dissipative force field F_d satisfies $\langle F_d(v), v\rangle < 0$ for all nonzero $v \in TG$. If g_0 is a local minimum of ϕ, then according to Theorem 1 in [3], $(g_0, 0)$ is asymptotically stable equilibrium state of the closed-loop system.

A. Energy Shaping on Three-Sphere

The control law (20) applies to the systems described by the Euler–Poincaré equations. The control input is

$$M(g, \dot{g}) = M_c(g) + M_d(g, \dot{g}) \tag{21}$$

where $M_c(g) = -T^*_e L_g d\phi(g)$ denotes the conservative force and $M_d(g, \dot{g}) = T^*_e L_g\ F_d(g, \dot{g})$ represents the dissipation. To compute the explicit form for the conservative force for the unit quaternions, the following equation for the differential of ϕ applies:

$$T^*_e L_q d\phi\left(\frac{\partial}{\partial q_i}\right) = d\phi\left(T_e L_q \frac{\partial}{\partial q_i}\right) = T_e L_q \frac{\partial \phi}{\partial q_i}. \tag{22}$$

Making use of (17) and (22), the conservative force equals

$$M_c = -\frac{1}{2}[\mathrm{d}^1\phi\ \mathrm{d}^2\phi\ \mathrm{d}^3\phi]^{\mathrm{T}} \tag{23}$$

where

$$[\mathrm{d}^0\phi(q)\ \ \mathrm{d}^1\phi(q)\ \ \mathrm{d}^2\phi(q)\ \ \mathrm{d}^3\phi(q)] = \frac{\partial \phi(q)}{\partial q} Q(q). \tag{24}$$

The interest in this note is confined to a particular choice of the dissipative force field

$$F_d = -D\dot{q} \tag{25}$$

where D indicates a positive–definite matrix. Combining (21), (23), and (25), the control law follows:

$$M = -\frac{1}{2}[\mathrm{d}_1\phi\ \mathrm{d}_2\phi\ \mathrm{d}_3\phi]^{\mathrm{T}} - \frac{1}{2}\pi Q^{\mathrm{T}}(q) D\dot{q}. \tag{26}$$

B. Potential Functions

Finding an appropriate potential function constitutes a major task in the construction of a control algorithm with use of the energy shaping. An immediate choice points at a Morse function, which all critical points are non degenerated. Since the three-sphere is a compact manifold, one shall seek for potential functions having only one minimum and one maximum.

It seems reasonably easy to design a positive definite function on $\mathbb{R}^n$. Quadratic forms are frequent examples. It appears, however, much more difficult to find a desired positive definite function on the three-sphere. As a matter of fact especially one has gained a great attention in the literature of aerospace and robotics: $\phi(q) = 1 - q_0$; see [11].

The procedure outlined below provides another example of a potential function. One may design a potential function $\phi_R(q)$ in the Euclidean space with the minimum at the required point q_e and then restrict it to the three-sphere $\phi(q) = \phi_R(q)\,|_{S^3}$. One choice could be a quadratic form

$$\phi_R(q) = \frac{1}{2}(Q(q_e)q - e)^{\mathrm{T}} P(Q(q_e)q - e) \tag{27}$$

where $P > 0$ denotes a positive–definite matrix and $e = [1\ 0\ 0\ 0]^{\mathrm{T}}$ indicates the identity. The necessary condition for existence of critical points, $\mathrm{d}\phi(q) = 0$, is equivalent saying that there exists a real k such that

$$\left.\frac{\partial \phi(q)}{\partial q}\right|_{q=q_e} = k q_e^{\mathrm{T}} \tag{28}$$

since then

$$\left.\frac{\partial \phi(q)}{\partial q}\right|_{q=q_e} Q(q_e) = [\,k \quad 0 \quad 0 \quad 0\,] \tag{29}$$

which follows from the orthogonality of the matrix $Q(q) : Q^{\mathrm{T}}(q)$ $Q(q) = Q(q)Q^{\mathrm{T}}(q) = E_{4\times4}$ and because q defines the first column of $Q(q)$ in (11). Using the definition of the differential in (22) together with (17), it is seen that $\mathrm{d}\phi(q) = 0$.

(28) permits two solutions: the first one for $k = 0$ and the second for $k \neq 0$. Therefore $\phi(q)$ has two critical points. Equations (27) and (28) show that the function $\phi(q)$ reaches minimum for $k = 0$ and $q = q_e$. The potential function $\phi(q)$ is continuous and S^3 is compact, hence, both a minimum and a maximum of $\phi(q)$ exist on the 3-sphere. It was already shown that the minimum is determined by $k = 0$. The maximum can be computed by solving (28) for $k \neq 0$.

For a particular choice of $P = E_{4\times4}$ and $q_e = e$ the potential function $\phi(q) = \phi_R(q)\,|_{S^3}$, where $\phi_R(q)$ was defined in (27), is equivalent to $\phi(q) = 1 - q_0$, which has the global minimum at the identity e and the maximum at $-e$.

IV. Spacecraft Attitude Control

The findings developed in the preceding sections are applied to the three-axis attitude control in the inertial frame. The objective is to stabilize the spacecraft to the desired attitude given by q_e. The restriction of $\phi_R(q)$ defined by (27) to the 3-sphere gives the potential energy $\phi(q) = \phi_R(q)\,|_{S^3}$. The control law (26) takes the following form:

$$\begin{aligned} M = & -\frac{1}{2}\pi Q^{\mathrm{T}}(q)DQ(q)i(\omega) \\ & - [\mathrm{d}^1\phi\ \mathrm{d}^2\phi\ \mathrm{d}^3\phi]^{\mathrm{T}} \end{aligned} \tag{30}$$

where

$$[\mathrm{d}^0\phi\ \mathrm{d}^1\phi\ \mathrm{d}^2\phi\ \mathrm{d}^3\phi] = \frac{1}{2}(Q(q_e)q - e)^{\mathrm{T}} PQ(q_e)Q(q). \tag{31}$$

This seemingly a complex control law has an ordinary PD control. To see this the following example is considered. Let the reference be the unit quaternion, the gains $D = 4k_d E_{4\times 4}$ and $P = 2k_p E_{4\times 4}$. Then, the differential $\mathrm{d}\phi(q)$ equals

$$\begin{aligned} [\mathrm{d}^0\phi\ \mathrm{d}^1\phi\ \mathrm{d}^2\phi\ \mathrm{d}^3\phi] &= k_p(qQ^{\mathrm{T}}(q) - eQ^{\mathrm{T}}(q)) \\ &= k_p[\,1 - q^0 \quad q^1 \quad q^2 \quad q^3\,] \end{aligned} \tag{32}$$

and the control law reduces to the PD form

$$M = -k_p[q^1\ q^2\ q^3]^{\mathrm{T}} - k_d\omega. \tag{33}$$

This shows that the energy shaping approach presented in this note agrees with the previous results on the three-axis attitude control summarized in [12].

V. Conclusion

This note further enhanced the energy shaping method for the Euler–Poincaré systems. The resultant algorithm was applied for the configuration manifold of the unit quaternions. A general scheme for control design of a rigid body was proposed and implemented for three-axis attitude control of a spacecraft.

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Exact Computation of Traces and $\mathcal{H}^2$ Norms for a Class of Infinite-Dimensional Problems

Bassam Bamieh and Mohammed Dahleh

***Abstract*—We derive a formula for the trace of a class of differential operators defined by forced two point boundary value problems. The formula involves finite-dimensional computations with matrices whose dimension is no larger than the order of the differential operator. Thus, we achieve an exact reduction of an infinite-dimensional problem to a finite-dimensional one. We relate this trace calculation to computation of the $\mathcal{H}^2$ norm for certain infinite-dimensional systems. An example from fluid dynamics is included to illustrate the method.**

I. Introduction

Certain differential operators on functions of one variable are typically given in terms of a two point boundary value problem (TPBVP) with a forcing function (an input). For certain problems in control theory and systems analysis it is desirable to compute the trace of operators related to the given differential operator. Such computations are typically done numerically after appropriate discretization of the underlying differential operator. This results in a finite-dimensional approximations to the underlying infinite-dimensional problem.

In this note, we show that in certain computations involving only the trace, it is possible to circumvent the approximation procedures. Special cases when this can be done turn out to be useful in computing the $\mathcal{H}^2$ of a certain class of infinite-dimensional systems, and in problems involving transition to turbulence in fluid dynamics [1]. An example of the latter is included in the last section.

II. Computing the Trace from State-Space Realizations

Differential operators in one variable with mixed boundary conditions can always be transformed into the so-called "first-order" representation, which is a state space realization. An example of this transformation is given in the last section of this note. Thus, we assume that we are given an operator $H : f \longmapsto g$ in the form of a two point boundary value state-space realization (TPBVSR), i.e., $g = Hf$ is represented by

$$\begin{aligned} \frac{d}{dy}\begin{bmatrix} x_1(y) \\ x_2(y) \end{bmatrix} &= A\begin{bmatrix} x_1(y) \\ x_2(y) \end{bmatrix} + Bf(y) \\ g(y) &= Cx(y), \qquad y \in [-1, 1] \\ \begin{bmatrix} x_2(+1) \\ x_2(-1) \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned} \tag{1}$$

where $x^T(y) := [x_1^T(y)\ x_2^T(y)]$ is an equal partitioning of the state variables. The independent variable y is typically a spatial position which takes values in a finite interval, which in our notation has been standardized to $[-1, 1]$. We will assume that the given TPBVSR is well posed, that is, it has a unique solution for every input function f.

We note that the TPBVSR in (1) is not in the most general form possible. In particular, the boundary conditions could be given as a general

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B. Bamieh is with the Department of Mechanical Engineering, University of California, Santa Barbara, CA 93106 USA (e-mail: bamieh@engineering.ucsb.edu).

M. Dahleh, deceased, was with the Department of Mechanical Engineering, University of California, Santa Barbara, CA 93106 USA.

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