Analiza danych

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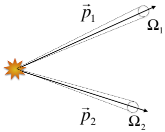
Wykład 13

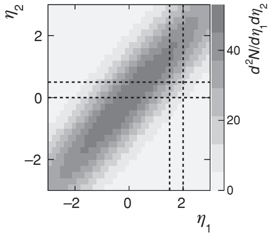
The notion of a correlation function

Lecture based on: W. Kittel, E.A. De Wolf, Soft Multihadron Dynamics, WS 2005 and C.A. Pruneau, Data Analysis Techniques for Physical Sciences, CUP 2017

- Consider a measurement of the numbers of particles N_i produced in volumes Ω_i "centered" around points $\vec{p}_i$, $(i = 1, 2)$ in momentum space:



$$p_{\mathrm{T},i}^{\min} \le p_{\mathrm{T},i} \le p_{\mathrm{T},i}^{\max}$$
$$\eta_i^{\min} \le \eta_i \le \eta_i^{\max}$$
$$\phi_i^{\min} \le \phi_i \le \phi_i^{\max}$$

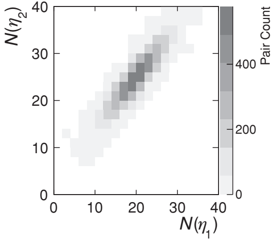


- Given the stochastic nature of particle production, the yields N_i are expected to fluctuate around the means:

$$\langle N_i \rangle = \int_{\Omega_i} \frac{d^3 N_i}{dp_{\mathrm{T}} d\phi d\eta} \, dp_{\mathrm{T}} d\phi d\eta$$

- Fluctuations about the mean are usually characterized by variance or covariance.
- Correlation function is defined as the scaled covariance in the limit in which bin sizes Ω_1 and Ω_2 vanish:

$$C(\vec{p}_1, \vec{p}_2) = \frac{1}{\Omega_1 \Omega_2} \left[\langle N(\vec{p}_1) N(\vec{p}_2) \rangle - \langle N(\vec{p}_1) \rangle \langle N(\vec{p}_2) \rangle \right]$$



- For finite bin sizes, the ratios provide estimators of the single, $\rho_1(\vec{p}_i)$, and the joint two-particle, $\rho_2(\vec{p}_1, \vec{p}_2)$, density functions:

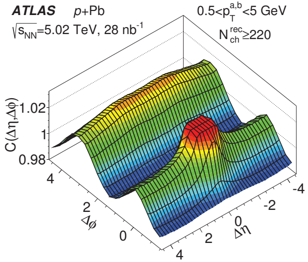
$$\hat{\rho}_1(\vec{p}_i) = \frac{\langle N(\vec{p}_i)\rangle}{\Omega_i} \quad \xrightarrow[\Omega_i \to 0]{} \quad \rho_1(\vec{p}_i) = \frac{d^3 N(\vec{p}_i)}{dp_{\mathrm{T}} d\phi d\eta}$$

$$\hat{\rho}_2(\vec{p}_1, \vec{p}_2) = \frac{\langle N(\vec{p}_1) N(\vec{p}_2)\rangle}{\Omega_1 \Omega_2} \quad \xrightarrow[\Omega_{1,2} \to 0]{} \quad \rho_2(\vec{p}_1, \vec{p}_2) = \frac{d^6 N_{\mathrm{pairs}}(\vec{p}_1, \vec{p}_2)}{dp_{\mathrm{T},1} d\phi_1 d\eta_1 dp_{\mathrm{T},2} d\phi_2 d\eta_2}$$

- Thus, the two-particle correlation function can be expressed in terms of density functions as:

$$C(\vec{p}_1, \vec{p}_2) = \rho_2(\vec{p}_1, \vec{p}_2) - \rho_1(\vec{p}_1)\rho_1(\vec{p}_2)$$

- In its most general form, the two-particle correlation function $C(\vec{p}_1, \vec{p}_2)$ is a function of six coordinates.
- It can be positive, null or negative (as the covariance).
- However, a measurement of correlation function can be reduced to a smalller number of coordinates of interest by integrating (marginalization) over variable that are not of interest.



Multiparticle Densities and Factorial Moments

- Let $y \equiv \{p_x, p_y, p_z, p_\mathrm{T}, \eta, \phi, ...\}$ denote all particle kinematic variables under interest in a particular study. Then, the joint-probability distribution function for n particles of the same species can be written as $P_n(y_1, y_2, y_3, ..., y_n)$
- The differential desities $\rho_n(y_1, ..., y_n)$ are proportional to the joint probabilities:

$$\rho_n(y_1, ..., y_n) = \langle N(N-1)...(N-n+1)\rangle P_n(y_1, y_2, y_3, ..., y_n)$$

- Integration of densities over the moemntum volume Ω, thus yields the following important relations:

$$\begin{aligned}
\int_\Omega \rho_1(y)dy &= \int_\Omega \frac{d^3N_i}{dp_\mathrm{T}d\phi d\eta} dp_\mathrm{T}d\phi d\eta = \langle N\rangle \\
\iint_\Omega \rho_2(y_1, y_2)dy_1dy_2 &= \langle N(N-1)\rangle \\
&... \\
\int ... \int_\Omega \rho_2(y_1, ..., y_n)dy_1...dy_n &= \langle N(N-1)...(N-n+1)\rangle
\end{aligned}$$

- The averages $\langle N(N-1)...(N-n+1)\rangle$ are called factorial moments of order n.

Cumulants

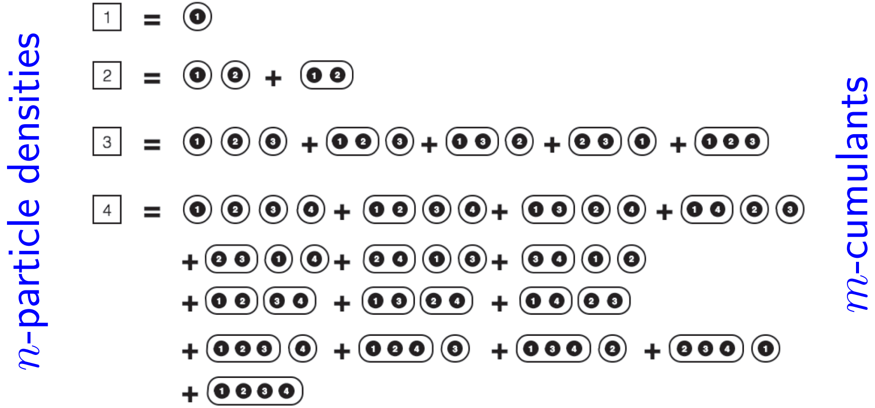
- Inclusive n-particle densities $\rho_n(y_1, ..., y_n)$ are the result of a superposition, in general, of several subprocesses (even from n distinct and uncorrelated subprocesses!).
- Measured n-tuples of particles may then feature a broad variety of correlation sources associated with a plurality of dynamic processes.
- It is a common goal of multiparticle production measurements to identify and study these correlated emissions as distinct subprocesses.
- This can be accomplished by invoking correlation functions known as (factorial) cumulans, expressed either in terms of integral correlators or as differential functions of one or more particle coordinates.
- Digression (statistical independence in terms of particle densities):

  Two variables are said to be statistically independent iff their joint probability density factorizes.

  The statistical independence for two particles means $\rho_2(y_1, y_2) = \rho_1(y_1)\rho_1(y_2)$

  Similarly for n particles we have $\rho_n(y_1, ..., y_n) = \rho_1(y_1)...\rho_1(y_n)$

- Cumulants of order m, C_m, are defined as m-particle densities representing emission (production) of m correlated particles originating from a common process.
- An n-particle density can then be expressed as a sum of several terms yielding n particles, but each with its own "cluster" decomposition into products of m-cumulants:

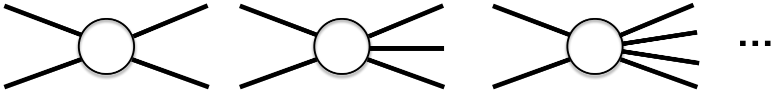


Particle denstities and cumulants

- In general n-particle densities can be expressed in terms of cumulants using the formula (shorthand notation $y_i \to i$):

$$\begin{aligned}\rho_n(1,...,n) = {} & C_n(1,...,n) + \sum_{\text{perm}} C_1(1)C_{n-1}(2,...,n) \\ & + \sum_{\text{perm}} C_1(1)C_1(2)C_{n-2}(3,...,n) \\ & + \sum_{\text{perm}} C_2(1,2)C_{n-2}(3,...,n) + ... + \prod_{i=1}^{n} C_1(i)\end{aligned}$$

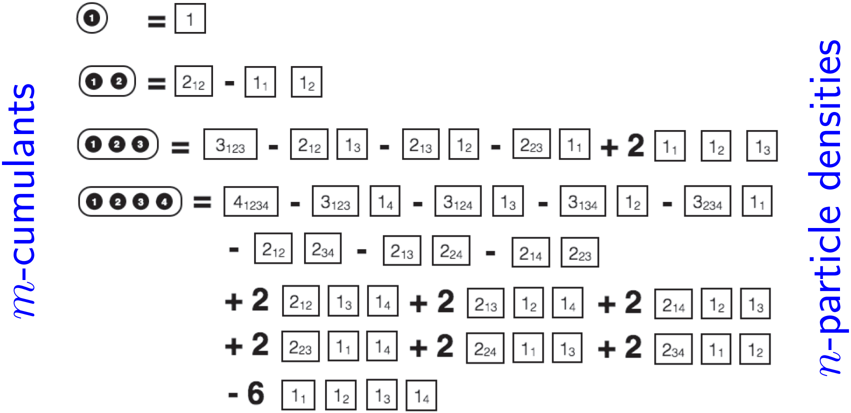
- m-cumulants represent fractions of the particle production cross-section associated with processes yielding m correlated particles (which cannot be further factorized).
- m-cumulants are directly calculated based on theoretical models:



- Experimentally measured quantities are n-particle densities, not cumulants.

- Cumulants can be obtained from measured densities using “reverse engineering”:

$$C_1(1) = \rho_1(1)$$
$$C_2(1,2) = \rho_2(1,2) - \rho_1(1)\rho_1(2)$$
$$C_3(1,2,3) = \rho_3(1,2,3) - \sum_{(3)} \rho_1(1)\rho_2(2,3) + 2\rho_1(1)\rho_1(2)\rho_1(3)$$



Cumulants scaling with source multiplicity

- Cumulants $C_n(y_1, ..., y_n)$ feature a simple scaling property for collision systems consisting of a superposition of m_s independent (but otherwise identical) sources.
- Example: Heavy ion collisions (A+A) can be regarded (to first approximation) as a superposition of m_s nucleon-nucleon (pp) interactions, each of which produces clusters consisting of n correlated particles.

  Assume that production of such clusters in pp may be described by cumulant C_n^{pp}.

  At a given impact parameter b (centrality), A+A collisions should involve an average of $\langle m_s \rangle$ pp interactions.

  m_s fluctuates from event to event, but the n-cumulant for A+A collisions, at fixed m_s may be written as:

  $$C_n^{\mathsf{AA}}(y_1, ..., y_n \,|\, m_s) = m_s C_n^{pp}(y_1, ..., y_n)$$

  Averaging over all A+A collisions (and assuming a superposition of independent and unmodified pp collisions, and such that produced particles do not interact with one another) yields:

  $$C_n^{\mathsf{AA}}(y_1, ..., y_n) = \langle m_s \rangle C_n^{pp}(y_1, ..., y_n)$$

- The total multiplicity of produced particles in A+A also features the same scaling with m_s: $\rho_1^{\mathsf{AA}}(y) = m_s \rho_1^{pp}(y) \quad \Rightarrow \quad \langle n \rangle_{\mathsf{AA}} = m_s \langle n \rangle_{pp}$

Cumulants scaling with source multiplicity

- For the pairs of particles, one can form pairs from each of m_s individual pp collisions, but one can also mix particles from independent sources:

$$\rho_2^{\mathsf{AA}}(y_1, y_2) = m_s \rho_2^{pp}(y_1, y_2) + m_s(m_s - 1)\rho_1^{pp}(y_1)\rho_1^{pp}(y_2)$$

- The same result can be obtained using the cumulant decomposition of

$$\begin{aligned}
\rho_2^{\mathsf{AA}}(y_1, y_2) &= C_1^{\mathsf{AA}}(y_1)C_1^{\mathsf{AA}}(y_2) + C_2^{\mathsf{AA}}(y_1, y_2) \\
&= m_s^2 C_1^{pp}(y_1)C_1^{pp}(y_2) + m_s C_2^{pp}(y_1, y_2) \\
&= m_s^2 \rho_1^{pp}(y_1)\rho_1^{pp}(y_2) + m_s\left[\rho_2^{pp}(y_1, y_2) - \rho_1^{pp}(y_1)\rho_1^{pp}(y_2)\right] \\
&= m_s(m_s - 1)\rho_1^{pp}(y_1)\rho_1^{pp}(y_2) + m_s \rho_2^{pp}(y_1, y_2)
\end{aligned}$$

- At fixed value of m_s, integration over y_1 and y_2 yields:

$$\langle n(n-1)\rangle_{\mathsf{AA}} = m_s \langle n(n-1)\rangle_{pp} + m_s(m_s - 1)\langle n\rangle_{pp}^2$$

For large m_s, the scaling is dominated by uncorrelted combinatorial pairs from particles produced in different pp interactions and approximatelly scales by m_s^2.

Cumulants scaling with source multiplicity

- Similarly, in the case of triplets, one can show that:

$$\begin{aligned}
\rho_3^{\mathsf{AA}}(1,2,3) &= C_1^{\mathsf{AA}}(1)C_1^{\mathsf{AA}}(2)C_1^{\mathsf{AA}}(3) + C_1^{\mathsf{AA}}(1)C_2^{\mathsf{AA}}(2,3) + \\
&\quad + C_1^{\mathsf{AA}}(2)C_2^{\mathsf{AA}}(1,3) + C_1^{\mathsf{AA}}(3)C_2^{\mathsf{AA}}(1,2) + C_3^{\mathsf{AA}}(1,2,3) \\
&= m_s^3 C_1^{pp}(1)C_1^{pp}(2)C_1^{pp}(3) + m_s^2 \sum_{\text{perm}} C_1^{pp}(1)C_2^{pp}(2,3) + m_s C_3^{pp}(1,2,3) \\
&= (m_s^3 - m_s^2 + 2m_s)\rho_1^{pp}(1)\rho_1^{pp}(2)\rho_1^{pp}(3) \\
&\quad + (m_s^2 - m_s)\sum_{\text{perm}} \rho_1^{pp}(1)\rho_2^{pp}(2,3) + m_s \rho_3^{pp}(1,2,3)
\end{aligned}$$

- At fixed m_s, after integration over coordinates y_1, y_2 and y_3 one gets:

$$\begin{aligned}
\langle n(n-1)(n-2)\rangle_{\mathsf{AA}} &= (m_s^3 - m_s^2 + 2m_s)\langle n\rangle_{pp}^3 \\
&\quad + 3(m_s^2 - m_s)\langle n(n-1)\rangle_{pp}\langle n\rangle_{pp} + m_s\langle n(n-1)(n-2)\rangle_{pp}
\end{aligned}$$

  The average number of triplets in A+A collisions is dominated by combinatorics and essentially scales as $m_s^3\langle n\rangle_{pp}^3$.

- By extension, we conclude that the average number of n-ntuplets in A+A collisions scales as $m_s^n\langle n\rangle_{pp}^n$.

Normalized cumulants and normalized factorial moments

- Normalized inclusive densities and normalized cumulants are defined as:

$$r_n(y_1, ..., y_n) = \frac{\rho_n(y_1, ..., y_n)}{\rho_1(y_1)...\rho_1(y_n)} \qquad R_n(y_1, ..., y_n) = \frac{C_n(y_1, ..., y_n)}{\rho_1(y_1)...\rho_1(y_n)}$$

- It is also common to use reduced (normalized) factorial moments:

$$f_n = \frac{\langle N(N-1)...(N-n+1)\rangle}{\langle N\rangle^n}$$

- For systems consisting of m identical subprocesses the normalized n-cumulant scales inversely as m^{n-1} times the n-cumulant of the subsystem ($R_n^{(m)}$ are diluted by power m^{n-1} relative to the subsystems' $R_n^{(1)}$):

$$R_n^{(m)}(y_1, ..., y_n) = \frac{C_n^{(m)}(y_1, ..., y_n)}{\rho_1^{(m)}(y_1)...\rho_1^{(m)}(y_n)} = \frac{1}{m^{n-1}} R_n^{(1)}(y_1, ..., y_n)$$

- A simple relationship exists between the normalized desities and cumulants:

$$r_2(1,2) = 1 + R_2(1,2)$$

$$r_3(1,2,3) = 1 + \sum_{(3)} R_2(1,2) + R_3(1,2,3)$$

$$r_4(1,2,3,4) = 1 + \sum_{(6)} R_2(1,2) + \sum_{(3)} R_2(1,2)R_2(3,4) + \sum_{(3)} R_3(1,2,3) + R_4(1,2,3,4)$$

Particle probability densities

- Particle probability densities have been defined as (slide 3):

$$P_n(y_1,...,y_n) \equiv \frac{\rho_n(y_1,...,y_n)}{\langle N(N-1)...(N-n+1)\rangle}$$

- If the production of particles 1 to n is statistically independent, then the ratio:

$$q_n(y_1,...,y_n) \equiv \frac{P_n(y_1,...,y_n)}{P_1(y_1)...P_1(y_n)} = 1$$

- From the above one can see that:

$$r_n(y_1,...,y_n) = \underbrace{\frac{\langle N(N-1)...(N-n+1)\rangle}{\langle N\rangle^n}}_{\text{multiplicity fluctuations if } \neq 1} \quad \underbrace{q_n(y_1,...,y_n)}_{\text{genuine correlations if } \neq 1}$$

- $q_n \neq 1$ is required to yield nonvanishig normalized cumulants, e.g.:

$$R_2(y_1,y_2) = \frac{\langle N(N-1)\rangle}{\langle N\rangle^2} q_2(y_1,y_2) - 1$$

The strength of two-particle correlations is thus determined both by the function $q_2(y_1,y_2)$ and the amplitude of multiplicity fluctuations, $\langle N(N-1)\rangle/\langle N\rangle^2 \neq 1$.

Factorial and cumulant moment-generating functions

- It is known, that for the moment generating function we have:

$$M_{\mathrm{X}}(t) = \mathcal{E}\left[e^{t\mathrm{X}}\right] = \int_{-\infty}^{\infty} e^{tx} f(x) dx \quad \Rightarrow \quad m_k = \frac{d^k}{dt^k}\mathcal{E}\left[e^{t\mathrm{X}}\right]\bigg|_{t=0} = \frac{d^k}{dt^k} M_{\mathrm{X}}(t)\bigg|_{t=0}$$

- Inclusive densities of order n may be written as:

$$\rho_n(y_1, ..., y_n) = \sum_m P_m \rho_n^{(m)}(y_1, ..., y_n) \quad \Leftarrow \quad P_m \equiv \frac{\sigma_m}{\sum_m \sigma_m} = \frac{\sigma_m}{\sigma_{\mathsf{inel}}}$$

  where σ_m is the cross section for a the process yielding m particles, and $\rho_n^{(m)}(...)$ are n-particle densities for processes that produce exactly m particles ($m \geq n$).

- Integration of inclusive n-particle density yields:

$$\begin{aligned}\tilde{F}_n &\equiv \int_\Omega \rho_n(y_1, ..., y_n) dy_1 ... dy_n = \sum_m P_m \int_\Omega \rho_n^{(m)}(y_1, ..., y_n) dy_1 ... dy_n \\ &= \sum_m P_m\, m(m-1)...(m-n+1) = \langle m(m-1)...(m-n+1)\rangle \equiv \langle m^{[n]} \rangle\end{aligned}$$

- Assuming there is a value $n = N$ beyond which all probabilities vanish and since terms in $P_{n<N}$ cannot contribute to $\tilde{F}_N$ one can write: $P_N = \tilde{F}_N / N!$

- Proceeding recusively, one finds: $P_n = \dfrac{1}{n!} \displaystyle\sum_{k=0}^{N-n} (-1)^k \frac{\tilde{F}_{k+n}}{k!}$, for $n = 0, 1, ..., N$

Factorial cumulants

- The factorial moment generating function should have the form:

$$G(z) = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \tilde{F}_n = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \int_\Omega \rho_n(y_1, ..., y_n) dy_1 ... dy_n, \quad \tilde{F}_n = \left. \frac{d^n G(z)}{dz^n} \right|_{z=0}$$

- Factorial cumulants are defined as: $\quad f_n = \int_\Omega dy_1 ... \int_\Omega dy_n \, C_n(y_1, ..., y_n)$

- Factorial moments may be expressed in terms of factorial cumulants:

$$\begin{aligned}
\tilde{F}_1 &= f_1 \\
\tilde{F}_2 &= f_2 + f_1^2 \\
\tilde{F}_3 &= f_3 + 3 f_2 f_1 + f_1^3 \\
\tilde{F}_4 &= f_4 + 4 f_3 f_1 + 3 f_2^2 + 6 f_2 f_1^2 + f_1^4 \\
\tilde{F}_5 &= f_5 + 5 f_4 f_1 + 10 f_3 f_2 + 10 f_3 f_1^2 + 15 f_2^2 f_1 + 10 f_2 f_1^3 + f_1^5 \\
&... \\
\tilde{F}_n &= n! \sum_{\{l_i\}_n} \prod_{j=1}^{n} \left(\frac{f_j}{j!} \right)^{l_j} \frac{1}{l_j!}
\end{aligned}$$

where summation is done over permutations satisfying $\sum_{i=1}^{n} i l_i = n$.

Factorial cumulants

- Factorial cumulant generating functions are defined as:

$$\ln G(z) = \langle n \rangle z + \sum_{k=2}^{\infty} \frac{z^k}{k!} f_k \quad \Rightarrow \quad f_n = \left. \frac{d^n \ln G(z)}{dz^n} \right|_{z=0}$$

- Example: Generating function for a Poisson distribution $P_n = \frac{\langle n \rangle^n}{n!} e^{-\langle n \rangle}$:

$$\Rightarrow \quad G(z) = \sum_{n=0}^{\infty} P_n (1+z)^n = e^{-\langle n \rangle} \sum_{n=0}^{\infty} \frac{\langle n \rangle^n}{n!} (1+z)^n = \exp\left(\langle n \rangle z\right)$$

$$\Rightarrow \quad \tilde{F}_m = \left. \frac{d^n G(z)}{dz^n} \right|_{z=0} = \langle n \rangle^m$$

$\Rightarrow \quad f_1 = \langle n \rangle$ and $f_m \equiv 0$, for $m > 1$ - expected, since Poisson statistics implies production of uncorrelated particles and cumulants of order $m \geq 2$ must vanish.

Two-particle azimuthal correlations

- Energy-momentum conservation (e.g. resonances' decays,jets).
- Restricting the variables y_1 and y_2 to represent the azimuthal production angles ϕ_1 and ϕ_2, we have:

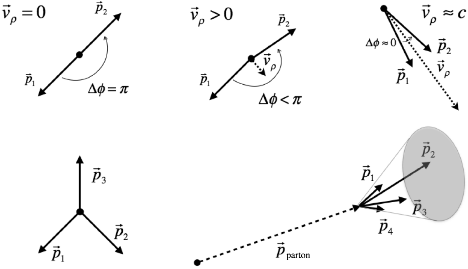
$$C_2(\phi_1,\phi_2) = \rho_2(\phi_1,\phi_2) - \rho_1(\phi_1)\rho_1(\phi_2)$$

$$R_2(\phi_1,\phi_2) = \frac{\rho_2(\phi_1,\phi_2)}{\rho_1(\phi_1)\rho_1(\phi_2)} - 1$$

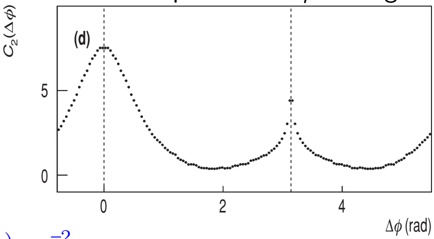
where the densities $\rho_1(\phi_i)$ and $\rho_2(\phi_1,\phi_2)$ are measured for specific ranges of $p_{\mathrm{T}}^{\mathrm{min}} \le p_{\mathrm{T}} \le p_{\mathrm{T}}^{\mathrm{max}}$ and $\eta_{\mathrm{min}} \le \eta \le \eta_{\mathrm{max}}$.

- In the absence of polarization or other discriminating direction, one expects that $\rho_1(\phi_1) = \rho_1(\phi_2) \equiv \bar{\rho}_1$, and C_2 should depend on $\Delta\phi = \phi_1 - \phi_2$:
- $\phi_1, \phi_2 \;\rightarrow\; \Delta\phi,\; \bar{\phi} = (\phi_1+\phi_2)/2$

$$C_2(\Delta\phi) = \rho_2(\Delta\phi) - \frac{1}{2\pi}\int_0^{2\pi} \bar{\rho}_1^2 d\bar{\phi} = \rho_2(\Delta\phi) - \bar{\rho}_1^2$$



Example of $C_2(\Delta\phi)$ in $\rho^0 \to \pi^+\pi^-$ with broad spectrum of ρ^0 energies.



Correlations from anisotropic flow

- Two-particle correlations may be very much influenced by collective effects as in collisions of heavy nuclei.

- $$\rho_1(\phi_i|\psi) = \bar{\rho}\left\{1 + 2\sum_{n=1}^{\infty} v_n \cos\left(n(\phi_i - \psi)\right)\right\} \;\Rightarrow\; \rho_1(\phi_i) = \int_0^{2\pi} d\psi\, \rho_1(\phi_i|\psi)P(\psi) = \bar{\rho}$$

- $$\rho_2(\phi_1,\phi_2) = \frac{1}{2\pi}\int_0^{2\pi} d\psi\, \rho_2(\phi_1,\psi_2|\psi)P(\psi) = \bar{\rho}^2\left\{1 + 2\sum_{n=1}^{\infty} v_n^2 \cos\left(n(\phi_1 - \phi_2)\right)\right\}$$