Length Contraction Paradox

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(Received January 29, 1961)

A certain man walks very fast—so fast that the relativistic length contraction makes him very thin. In the street he has to pass over a grid. A man standing at the grid fully expects the fast thin man to fall into the grid. Yet to the fast man the grid is much narrower even than to the stationary man, and he certainly does not expect to fall in. Which is correct? The answer hinges on the relativity of rigidity.

SOME two or three years ago I proposed to colleagues at Cornell a simple paradox on relativistic length contraction which I had already proposed several years earlier to students at London University. It seemed the kind of paradox that must occur to anyone concerned with the subject, but I failed to find it mentioned in the literature. At a recent professional meeting it still aroused some interest, and I therefore offer it now.

A 10-in. long "rigid" rod moves longitudinally over a flat table. In its path is a hole 10 in. wide. Suppose the rod moves so fast that its Lorentz contraction factor is 10. To an observer B moving with the rod the hole is only 1 in. wide, and the rod, being "rigid," might be expected to pass unhindered over the hole. To an observer A at rest relative to the table, however, it is the rod that is only 1 in. long; in passing over the 10-in. hole it is bound to fall somewhat under gravity, and it will consequently strike the far edge of the hole and so be stopped. Which description is correct?

The resolution of the paradox has already been hinted at by setting the word *rigid* in quotation marks. There is no doubt that A's description of events is correct. The rod simply cannot remain rigid in B's inertial frame (see Fig. 1). This illustrates well the difficulties encountered in the search for a satisfactory definition of rigidity in relativity.

Before proving our assertion, let us make the experiment more concrete. The hole shall be filled with a trap door which will be removed (downward, and with sufficient acceleration to allow the rod to fall freely) by the observer A at the instant when to him the hind end of the rod passes into the hole. This precaution eliminates the tendency of the rod to topple over the edge. All points of the rod will then fall equally fast, and the rod will remain horizontal, in the frame of A. The gravitational field can be replaced by a magnetic field acting on an iron rod, or even by a uniform vertical sand blast from above, if it be held that special relativity is inapplicable to gravitation. It must be stressed, however, that special relativity is perfectly applicable to accelerated bodies: what it cannot do is cope with nonflat space times.

Now let it be understood that the rod is originally a rectangular parallelepiped and that the observer B uses an internal frame fixed to the hind end of the rod. Call this frame S', call

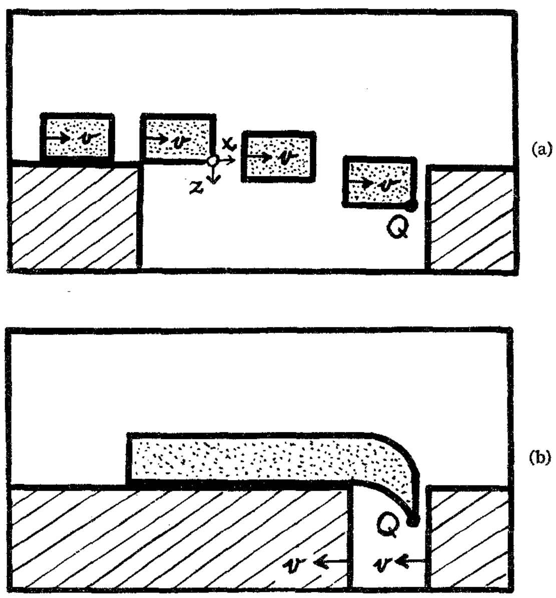

FIG. 1. (a) Sequence of four observations made by A at equal intervals of time t. (b) Observation made by B at one particular instant t'. (For convenience these diagrams are drawn for the case $\gamma=4$, not $\gamma=10$ as in the text.)

A's frame S, and let their relative velocity be v. Take as common origin event a front-bottom corner Q of the rod at the instant when the trap door separates from Q, measure z, z' down from the top of the table, and x, x' along the initial path of Q. Then the standard Lorentz transformation equations

$$z=z', \quad t=\gamma(t'+vx'/c^2), \quad \gamma=(1-v^2/c^2)^{-\frac{1}{2}} \quad (1)$$

apply to S and S'. The equations of the bottom edge of the rod in S are

$$z=0 \quad \text{when} \quad t<0, \quad z=\tfrac{1}{2}at^2 \quad \text{when} \quad t\geqslant 0, \quad (2)$$

where a is the acceleration produced by the field or sandblast. (A uniform field in relativity will only approximately produce uniform acceleration, but the small error is quite irrelevant here.) By use of (1), we can immediately transform Eqs. (2) into

$$z'=0 \quad \text{when} \quad x'<-c^2t'/v,$$
$$z'=\tfrac{1}{2}a\gamma^2(t'+vx'/c^2)^2 \quad \text{when} \quad x'\geqslant c^2t'/v. \quad (3)$$

The interpretation of Eqs. (3) is as follows. In S', imagine a parabola with vertex at Q, axis vertically down, and latus rectum $2c^4/a\gamma^2v^2$. The vertex of this parabola moves along the rod with velocity c^2/v starting at $t'=0$; and the rod, as it passes over that vertex, "flows" down the parabola. Its horizontal extent clearly remains constant until it hits the far edge of the hole. It can easily be shown that the near edge of the hole at time t' is at $x'=-L/\gamma^2-vt'$, where L is the rest length of the rod. Consequently, this edge, moving with velocity v along the rod, leads the vertex of the parabola and is overtaken by the latter exactly at $x'=-L$, i.e., at the hind end of the rod. A sizable compression of the rod must eventually occur in S' because, as can be seen from the description in S, the hind end of the rod passes well into the hole.

ORINS Summer Symposium

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Leading representatives of universities, industry, and government will discuss the various types of subcritical assemblies, the techniques of their use, their applications in university research and education programs, and other facets of obtaining and operating subcritical assemblies on university campuses.

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