# Finite Element Analysis for Metal Forming and Material Engineering

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## Annotation

**The finite element method (FEM) is widely used in metal forming and material engineering. This method is an approximate method, that's why it requires the use of theoretical training. The following questions are considered in this course.**

**1. Fundamentals of the FEM.**
**2. Solving a thermal problems.**
**3. Solving problems in the theory of elasticity and plasticity.**
**4. Principles and practical aspects of creating software based on the FEM.**
**5. Review of existing commercial programs.**

**The main part of the lab tutorials are dedicated to the development of FEM codes, dedicated to simple problems in material processing.**

**Initial requirements for students.**
**Fundamentals of programming, the basics of heat transfer, mechanics, numerical methods, the fundamentals of materials engineering and metal forming.**



## Lectures

1.Introduction to FEM. History of FEM. Usage FEM in metal forming and material engineering, industrial examples. Basic conception of FEM. Interpolation in FEM, definition of shape functions.

2.FEM technics. Finite elements of higher order. Isoparametric transformation. Jacobi matrix. Numerical integration.

3. Solving the heat flow problems by FEM. Steady state and non steady state heat flow problems. Equations for stiffness matrix and load vector. Two dimensional FEM code for simulation of heat flow.

4.Solution of elastic problems by FEM. Basics of theory of elasticity. Variation principle. Equations for stiffness matrix and load vector. Example of FEM code for solving plain strain problem by FEM.

5.Solution of rigid plastic problems by FEM. Theory of plasticity on non compressible materials. Variation principle. Equations for stiffness matrix and load vector. Example of FEM code for simulation of plain strain problem in flow formulation. Analogy between flow dynamic and theory of plasticity in flow formulation.

6. Commercial FEM code Qform for simulation of hot metal forming processes. Theoretical basics of Qform FEM program. Structure and interface of Qform. Simulation of forging, shape rolling and extrusion in Qform. Implementation of advanced material models in Qform. Lua scripts in Qform. Implementation of flow stress and fracture models in Qform.

7.Summary of course. Industrial examples of usage FEM in research for metal forming and material engineering. 1h



## Literature

1. O.C.Zienkiewicz, R.L.Taylor The Finite Element Method // Butterworth Heinemann, 3 vol, 5-th Edition, London, 2000
2. K.J. Bathe, Finite Element Procedures in Engineering Analysis, Prentice Hall Inc.
3. Segerlind L. J., Applied Finite Element Analysis // J. Wiley & Sons, New York, 1976, 1984, 1987, 427 pp. ISBN 0-471-80662-5.
4. Kobajashi S., Oh S.I., Altan T., Metal Forming and the Finite Element Method, Oxford University Press, New York, Oxford, 1989.
5. Lenard J.G., Pietrzyk M., Cser L., Mathematical and Physical Simulation of the Properties of Hot Rolled Products, Elsevier, Amsterdam, 1999.
6. http://qform3d.ru
7. www.LCM.agh.edu.pl
8. Finite Element Procedures for Solids and Structures (MIT Open Recourse)
9. A. Milenin Podstawy MES. Zagadnienia termomechaniczne // AGH, Kraków, 2010



Lecture 1. Introduction to FEM. History of FEM. Usage FEM in metal forming and material engineering, industrial examples. Basic conception of FEM. Interpolation in FEM, definition of shape functions.

1. FEM is now widely used structural engineering problem.
2. … In civil, aeronautical, mechanical, mining, metallurgical, biomechanical … engineering.
3. We now see applications in metal forming and material engineering.
4. Various computer programs are available and significant use (Qform, ABAQUS, ADYNA, etc.)



## Historical background

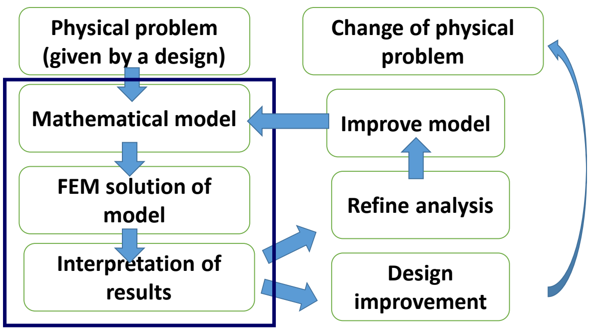
Richard Kurant

R. Kurant VARIATIONAL METHODS FOR THE SOLUTION OF PROBLEMS OF EQUILIBRIUM AND VIBRATIONS, 1942

Walther Ritz

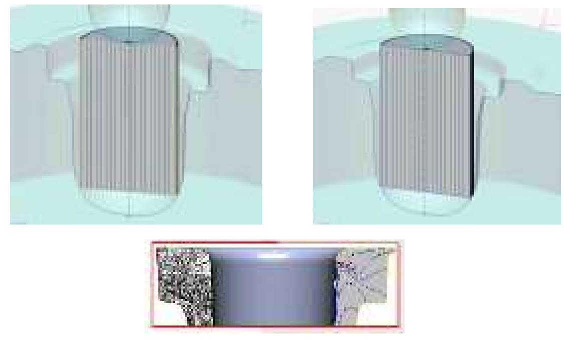
W. Ritz, Ueber eine neue Methode zur Loesung gewisser Variationsprobleme der mathematischen Physik, J. Reine Angew. Math. vol. 135 (1908);

## FEM today. The process of analysis



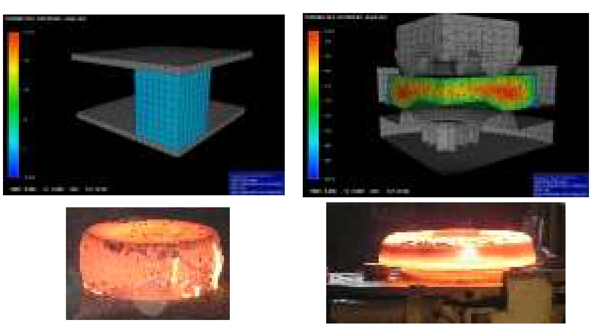



## FEM today. Optimization of metal forming processes (program QForm)





## Manufacture of railway wheels (Forge3d)

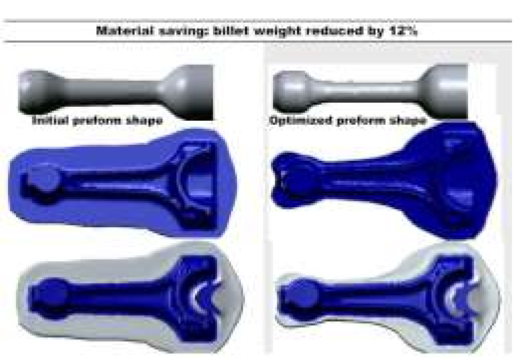




## What we see without FEM simulation…

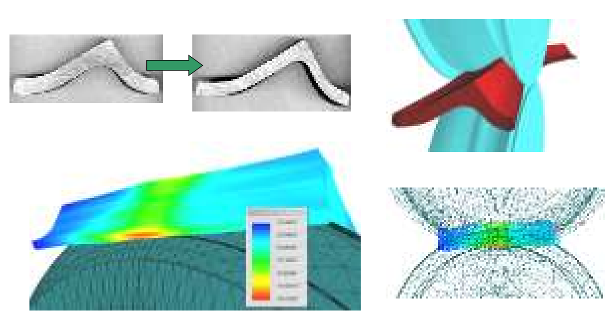


## Saving of materials (QForm)






## Optimization of rolling processes (program Rolling3)



## Structure and problems of FEM simulations

**Formulation of boundary problem** → **Use of FEM techniques** → **Analyze of results**

⇩ **Mistakes in formulation of boundary problem**

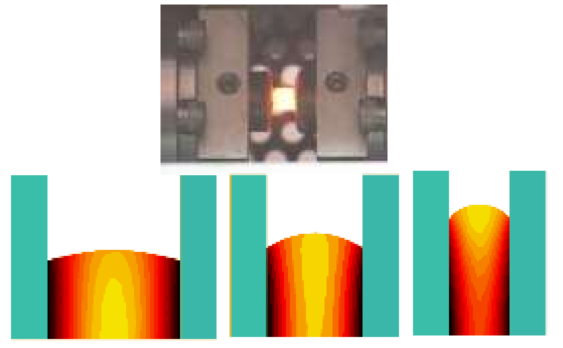
⇩ **Wrong choice of type of FE, solution methods, etc.**

⇩ **Mistakes in interpretation of results**

FEM - this is not the exact method !!!



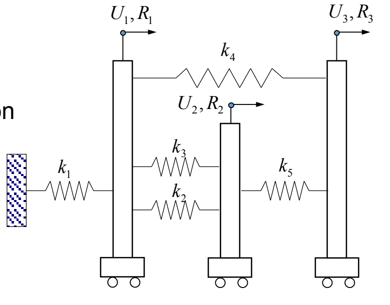
## FEM simulation of material tests on GLEEBLE machine





## Algorithm of FEM for discrete system. Analysis of system of rigid carts interconnected by springs.

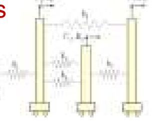
1. Equilibrium requirement.
2. Interconnection requirement.
3. Compatibility requirement.

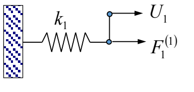



*K.J. Bathe, Finite Element Procedures in Engineering Analysis, Prentice Hall Inc.
http://www.youtube.com/watch?v=oNqSzzycRhw&feature=relmfu

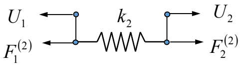


## 1. Equilibrium requirement of each springs (Finite Elements)



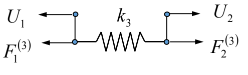



$$k_1 U_1 = F_1^{(1)}$$



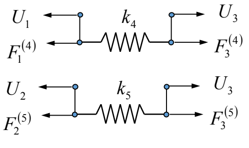

$$k_2(U_1 - U_2) = F_1^{(2)}$$
$$k_2(-U_1 + U_2) = F_2^{(2)}$$

$$k_2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} F_1^{(2)} \\ F_2^{(2)} \end{bmatrix}$$



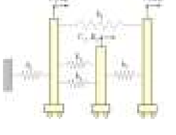

$$k_3\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} F_1^{(3)} \\ F_2^{(3)} \end{bmatrix}$$






$$k_4\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} U_1 \\ U_3 \end{bmatrix} = \begin{bmatrix} F_1^{(4)} \\ F_3^{(4)} \end{bmatrix}$$

$$k_5\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} F_2^{(5)} \\ F_3^{(5)} \end{bmatrix}$$



### 2. Element interconnection requirement.

$$\left.\begin{aligned} F_1^{(1)} + F_1^{(2)} + F_1^{(3)} + F_1^{(4)} &= R_1 \\ F_2^{(2)} + F_2^{(3)} + F_2^{(5)} &= R_2 \\ F_3^{(4)} + F_3^{(5)} &= R_3 \end{aligned}\right\}$$

### 3. Compatibility requirement.

$$KU = R$$
$$U^T = [U_1 \quad U_2 \quad U_3]$$
$$R^T = [R_1 \quad R_2 \quad R_3]$$



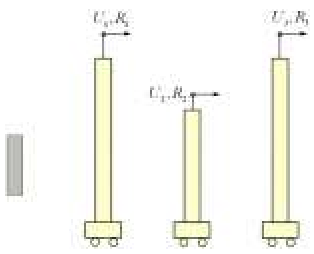
## Assemblage process. Direct stiffness method

$$K = \sum_{i=1}^{5} k^{(i)}$$

$$k = \begin{bmatrix} k_1 + k_2 + k_3 + k_4 & -k_2 - k_3 & -k_4 \\ -k_2 - k_3 & k_2 + k_3 + k_5 & -k_5 \\ -k_4 & -k_5 & k_4 + k_5 \end{bmatrix}$$

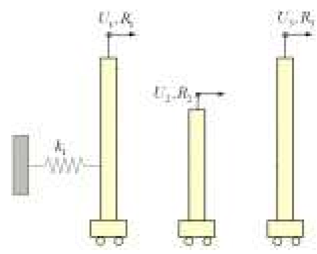
$$k^{(1)} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad k^{(2)} = \begin{bmatrix} k_2 & -k_2 & 0 \\ -k_2 & k_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad k^{(3)} = \begin{bmatrix} k_3 & -k_3 & 0 \\ -k_3 & k_3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$k^{(4)} = \begin{bmatrix} k_4 & 0 & -k_4 \\ 0 & 0 & 0 \\ -k_4 & 0 & k_4 \end{bmatrix} \quad k^{(5)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_5 & -k_5 \\ 0 & -k_5 & k_5 \end{bmatrix}$$



$$k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

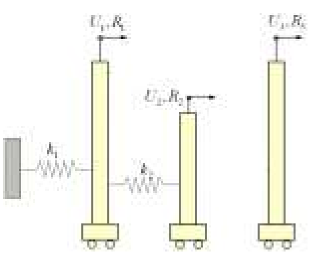




$$k^{(1)} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$k = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

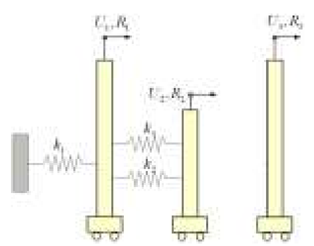




$$k^{(2)} = \begin{bmatrix} k_2 & -k_2 & 0 \\ -k_2 & k_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$k = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

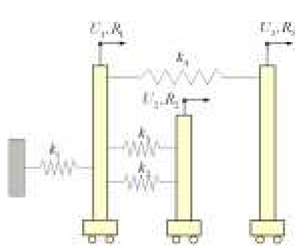




$$k^{(3)} = \begin{bmatrix} k_3 & -k_3 & 0 \\ -k_3 & k_3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$k = \begin{bmatrix} k_1 + k_2 + k_3 & -k_2 - k_3 & 0 \\ -k_2 - k_3 & k_2 + k_3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

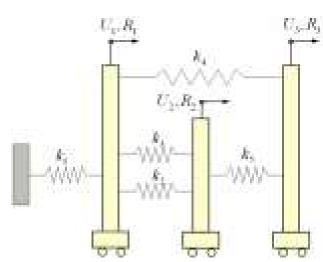




$$k^{(4)} = \begin{bmatrix} k_4 & 0 & -k_4 \\ 0 & 0 & 0 \\ -k_4 & 0 & k_4 \end{bmatrix}$$

$$k = \begin{bmatrix} k_1 + k_2 + k_3 + k_4 & -k_2 - k_3 & -k_4 \\ -k_2 - k_3 & k_2 + k_3 & 0 \\ -k_4 & 0 & k_4 \end{bmatrix}$$





$$k^{(5)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_5 & -k_5 \\ 0 & -k_5 & k_5 \end{bmatrix}$$

$$k = \begin{bmatrix} k_1 + k_2 + k_3 + k_4 & -k_2 - k_3 & -k_4 \\ -k_2 - k_3 & k_2 + k_3 + k_5 & -k_5 \\ -k_4 & -k_5 & k_4 + k_5 \end{bmatrix}$$

## Alternative approach - Variational formulation of the problem:

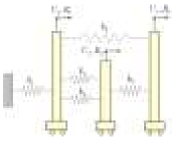
$$J = V - W$$

$$V = \frac{1}{2} U^T K U \qquad W = U^T R \qquad J = \frac{1}{2} U^T K U - U^T R$$

Extremum formulation: $\dfrac{\partial J}{\partial U_i} = 0$

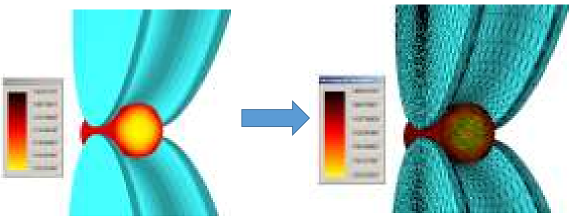
$$\frac{\partial J}{\partial U_i} = KU - R = 0$$

$$KU = R \qquad K = \sum_{i=1}^{5} k^{(i)}$$





## Main conception of FEM for continues systems



Continues object -> Discrete model



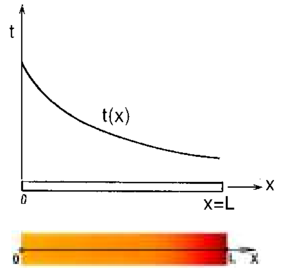
## Algorithm FEM (for heat transfer problem)

1. In the continuum, we are taking a limited number of points (nodes).
2. The values of temperature in each node is defined as a parameters, which we designate.
3. Zone designation of temperature (volume) is composed of a limited number of zones, which are celled finite elements.
4. The temperature is approximated for each FE by using the polynomial, which is designated using nodal temperatures.
5. Value of temperature on nodes must be selected in such a way as to ensure the best approximation to the actual field temperatures. Such selection is performed by means of minimizing a functional, which corresponds to both the equation conduction of heat.



## Discretization

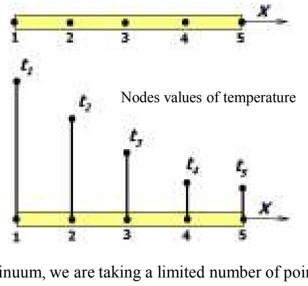
One-dimensional example:




1d temperature distribution in bar



## Discretization



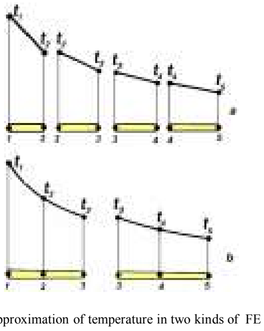

1. In the continuum, we are taking a limited number of points (nodes).
2. The values of temperature on each node is defined as a parameters, which we designate.



## Finite Elements

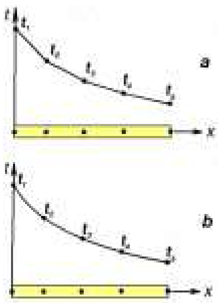
3. Zone designation of temperature is composed of a limited number of zones, which are finite elements.

4. The temperature is approximated for each FE by using the polynomial, which is designated using nodal temperatures.



Approximation of temperature in two kinds of FE

## Global interpolation of temperature





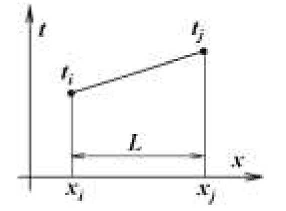

## 1-D finite element

$$t = a_1 + a_2 X$$

$$t = t_i \ \ for \ \ X = X_i$$
$$t = t_j \ \ for \ \ X = X_j$$

$$\left.\begin{aligned} t_i &= a_1 + a_2 X_i \\ t_j &= a_1 + a_2 X_j \end{aligned}\right\}$$

$$a_1 = \frac{t_i X_j - t_j X_i}{L} \qquad a_2 = \frac{t_j - t_i}{L}$$






## Shape functions

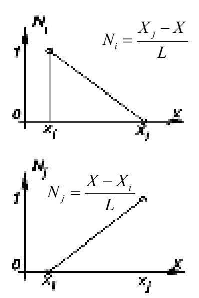
$$t = \left(\frac{t_i X_j - t_j X_i}{L}\right) + \left(\frac{t_j - t_i}{L}\right) X$$

$$t = \left(\frac{X_j - X}{L}\right) t_i + \left(\frac{X - X_i}{L}\right) t_j$$

$$N_i = \frac{X_j - X}{L} \qquad N_j = \frac{X - X_i}{L}$$

$$t = N_i t_i + N_j t_j = [N]\{t\}$$

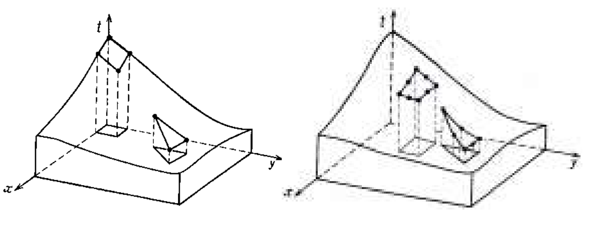
$$\{t\} = \begin{Bmatrix} t_i \\ t_j \end{Bmatrix} \qquad [N] = \lfloor N_i ; N_j \rfloor$$




$$\sum_{\beta=1}^{n} N_\beta = 1$$

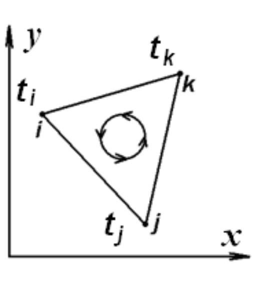


## 2D Finite Elements





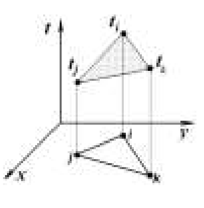
## 2D simplex finite element






## Shape functions of 2d simplex element

$$\left.\begin{aligned} &for \ X = X_i, Y = Y_i \ \ t = t_i, \Rightarrow t_i = a_1 + a_2 X_i + a_3 Y_i \\ &for \ X = X_j, Y = Y_j \ \ t = t_j, \Rightarrow t_j = a_1 + a_2 X_j + a_3 Y_j \\ &for \ X = X_k, Y = Y_k \ \ t = t_k, \Rightarrow t_k = a_1 + a_2 X_k + a_3 Y_k \end{aligned}\right\}$$



$$a_1 = \frac{1}{2A}\left[(X_j Y_k - X_k Y_j) t_i + (X_k Y_i - X_i Y_k) t_j + (X_i Y_j - X_j Y_i) t_k\right]$$

$$a_2 = \frac{1}{2A}\left[(Y_j - Y_k) t_i + (Y_k - Y_i) t_j + (Y_i - Y_j) t_k\right]$$

$$a_3 = \frac{1}{2A}\left[(X_k - X_j) t_i + (X_i - X_k) t_j + (X_j - X_i) t_k\right]$$

$$2A = \begin{vmatrix} 1 & X_i & Y_i \\ 1 & X_j & Y_j \\ 1 & X_k & Y_k \end{vmatrix} = X_j Y_k + X_i Y_j + X_k Y_i - X_j Y_i - X_k Y_j - X_i Y_k$$

A – area of FE

## Shape functions of 2d simplex element

$$\varphi = \varphi_i N_i + \varphi_j N_j + \varphi_k N_k = \{N\}^T \cdot \{\varphi\}$$

$$t = t_i N_i + t_j N_j + t_k N_k = \{N\}^T \cdot \{t\}$$

$$N_i = \frac{1}{2A}(a_i + b_i X + c_i Y)$$

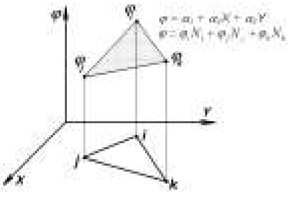
$$a_i = X_j Y_k - X_k Y_j \quad b_i = Y_j - Y_k \quad c_i = X_k - X_j$$

$$N_j = \frac{1}{2A}(a_j + b_j X + c_j Y)$$

$$a_j = X_k Y_i - X_i Y_k \quad b_j = Y_k - Y_i \quad c_j = X_i - X_k$$

$$N_k = \frac{1}{2A}(a_k + b_k X + c_k Y)$$

$$a_k = X_i Y_j - X_j Y_i \quad b_k = Y_i - Y_j \quad c_k = X_j - X_i$$



Finite Element Analysis for Metal Forming and Material Engineering, A.Milenin

## Practical example

**Problem:**

$\sigma_i$ = 40 MPa
$X_i$ = 0 mm
$Y_i$ = 0 mm

$\sigma_j$ = 34 MPa
$X_j$ = 4 mm
$Y_j$ = 0.5 mm

$\sigma_k$ = 46 MPa
$X_k$ = 2 mm
$Y_k$ = 5 mm

$X_B$=2 mm
$Y_B$=1.5 mm

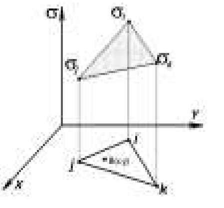
$\sigma_B$ = ?

$$\sigma = \sigma_i N_i + \sigma_j N_j + \sigma_k N_k$$

$$a_i = X_j Y_k - X_k Y_j = 4 \cdot 5 - 2 \cdot 0.5 = 19$$
$$b_i = Y_j - Y_k = 0.5 - 5 = -4.5$$
$$c_i = X_k - X_j = 2 - 4 = -2$$

$$a_j = X_k Y_i - X_i Y_k = 2 \cdot 0 - 0 \cdot 5 = 0$$
$$b_j = Y_k - Y_i = 5 - 0 = 5$$
$$c_j = X_i - X_k = 0 - 2 = -2$$

$$a_k = X_i Y_j - X_j Y_i = 0 \cdot 0.5 - 4 \cdot 0 = 0$$
$$b_k = Y_i - Y_j = 0 - 0.5 = -0.5$$
$$c_k = X_j - X_i = 4 - 0 = 4$$



Finite Element Analysis for Metal Forming and Material Engineering, A.Milenin

## Practical example

$$2A = \begin{vmatrix} 1 & X_i & Y_i \\ 1 & X_j & Y_j \\ 1 & X_k & Y_k \end{vmatrix} = X_j Y_k + X_i Y_j + X_k Y_i - X_j Y_i - X_k Y_j - X_i Y_k = 19$$

$$N_i = \frac{1}{2A}(a_i + b_i X + c_i Y) = \frac{7}{19}$$

$$N_j = \frac{1}{2A}(a_j + b_j X + c_j Y) = \frac{7}{19}$$

$$N_k = \frac{1}{2A}(a_k + b_k X + c_k Y) = \frac{5}{19}$$

$$\sigma_B = \sigma_i N_i^{(B)} + \sigma_j N_j^{(B)} + \sigma_k N_k^{(B)} = \frac{7}{19}40 + \frac{7}{19}34 + \frac{5}{19}46 = 39.4$$

$$N_i + N_j + N_k = \frac{7}{19} + \frac{7}{19} + \frac{5}{19} = 1$$

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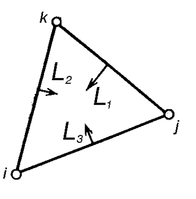
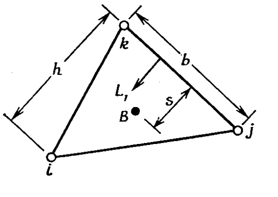
## L – coordinates (natural coordinates)

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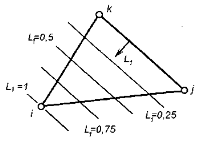
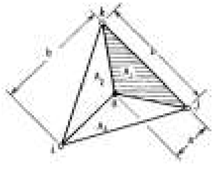
## L – coordinates (natural coordinates)



$$A = \frac{bh}{2} \quad A_1 = \frac{bs}{2} \quad \frac{A_1}{A} = \frac{s}{h} = L_1 \quad L_1 = \frac{A_1}{A}$$

$$A_1 + A_2 + A_3 = A$$

$$L_1 + L_2 + L_3 = 1$$

$$N_i = L_1 \quad L_1 = 1$$
$$N_j = L_2 \quad L_1 = 0$$
$$N_k = L_3 \quad L_1 = 0$$

$$x = L_1 X_i + L_2 X_j + L_3 X_k$$
$$y = L_1 Y_i + L_2 Y_j + L_3 Y_k$$
$$1 = L_1 + L_2 + L_3$$

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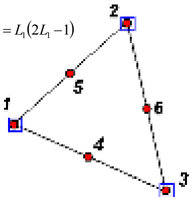
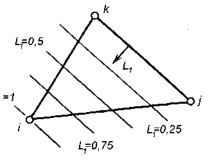
## 6 – nodes FE

$$N_1 = L_1(2L_1 - 1)$$




$$N_2 = L_2(2L_2 - 1)$$
$$N_3 = L_3(2L_3 - 1)$$
$$N_4 = 4L_1L_3$$
$$N_5 = 4L_1L_2$$
$$N_6 = 4L_2L_3$$

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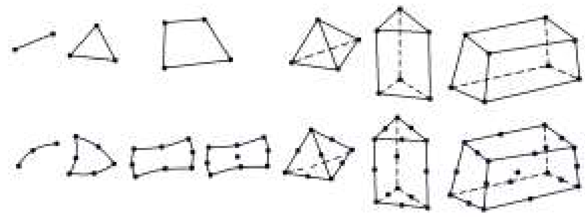
Lecture 2.

FEM technics.
- Finite elements of higher order.
- Isoparametric transformation.
- Jacobi matrix.
- Numerical integration.



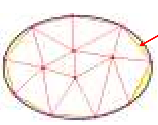
## Types of FE





## Motivation of usage of higher order FE

1. Error in approximation



$\Delta S_j$ $\quad \Delta S = \sum_{N_j} \Delta S_j$

2. Error during differentiation

$$\sigma_x = \sigma_0 + \frac{2\bar{\sigma}}{3\bar{\xi}}\xi_x; \quad \xi_x = \frac{\partial v_x}{\partial x}$$

3. Locking


DOF = 2*10+7=27

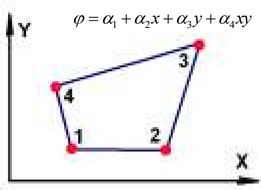
Ncond = ne = 30

DOF<Ncond



## 4-nodes FE

$$\varphi = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$




$$\varphi_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1 + \alpha_4 x_1 y_1$$
$$\varphi_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2 + \alpha_4 x_2 y_2$$
$$\varphi_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3 + \alpha_4 x_3 y_3$$
$$\varphi_4 = \alpha_1 + \alpha_2 x_4 + \alpha_3 y_4 + \alpha_4 x_4 y_4$$

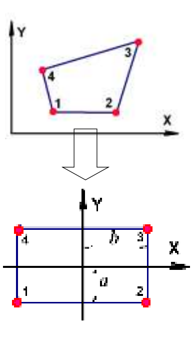
$$\frac{\partial \varphi}{\partial x} \neq const \qquad \frac{\partial \varphi}{\partial y} \neq const$$

Shape functions ?

$$\varphi = N_1\varphi_1 + N_2\varphi_2 + N_3\varphi_3 + N_4\varphi_4$$



## Rectangular FE




$$\varphi = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$

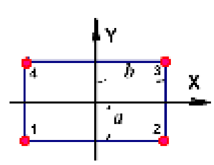
$$\alpha_1 = \frac{\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4}{4}$$
$$\alpha_2 = \frac{-\varphi_1 + \varphi_2 + \varphi_3 - \varphi_4}{4b}$$
$$\alpha_3 = \frac{-\varphi_1 - \varphi_2 + \varphi_3 + \varphi_4}{4a}$$
$$\alpha_4 = \frac{\varphi_1 - \varphi_2 + \varphi_3 - \varphi_4}{4ab}$$



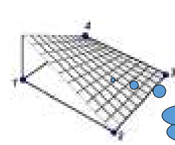
## Rectangular FE

Shape functions:

$$\varphi = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$
$$\varphi = N_1\varphi_1 + N_2\varphi_2 + N_3\varphi_3 + N_4\varphi_4$$




$$N_1 = \frac{1}{4ab}(b-x)(a-y)$$
$$N_2 = \frac{1}{4ab}(b+x)(a-y)$$
$$N_3 = \frac{1}{4ab}(b+x)(a+y)$$
$$N_4 = \frac{1}{4ab}(b-x)(a+y)$$



$$N_1 = \frac{1}{4ab}(b-x)(a-y)$$

## Shape functions in local coordinate system

$$N_1=\frac{1}{4ab}(b-x)(a-y)=\frac{1}{4}\left(1-\frac{x}{b}\right)\left(1-\frac{y}{a}\right)=\frac{1}{4}(1-\xi)(1-\eta)$$

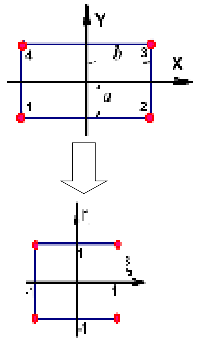
$$N_2=\frac{1}{4ab}(b+x)(a-y)=\frac{1}{4}\left(1+\frac{x}{b}\right)\left(1-\frac{y}{a}\right)=\frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3=\frac{1}{4ab}(b+x)(a+y)=\frac{1}{4}\left(1+\frac{x}{b}\right)\left(1+\frac{y}{a}\right)=\frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4=\frac{1}{4ab}(b-x)(a+y)=\frac{1}{4}\left(1-\frac{x}{b}\right)\left(1+\frac{y}{a}\right)=\frac{1}{4}(1-\xi)(1+\eta)$$

$$\xi=\frac{x}{b}\quad \eta=\frac{y}{a}$$

$$x=b\xi\quad y=a\eta$$

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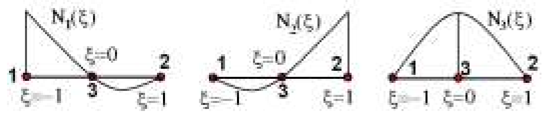
## Types of FE




$$N_1(\xi)=-\frac{1}{2}\xi(1-\xi)$$

$$N_3(\xi)=1-\xi^2$$

$$N_2(\xi)=\frac{1}{2}\xi(1+\xi)$$

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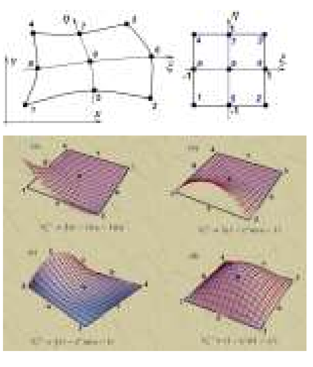
## Types of FE



$$N_1=\frac{1}{4}(1-\xi)(1-\eta)\xi\eta$$

$$N_2=-\frac{1}{4}(1+\xi)(1-\eta)\xi\eta$$

$$N_3=\frac{1}{4}(1+\xi)(1+\eta)\xi\eta$$

$$N_4=-\frac{1}{4}(1-\xi)(1+\eta)\xi\eta$$

$$N_5=-\frac{1}{2}(1-\xi^2)(1-\eta)\eta$$

$$N_6=\frac{1}{2}(1+\xi)(1-\eta^2)\xi$$

$$N_7=\frac{1}{2}(1-\xi^2)(1+\eta)\eta$$

$$N_8=-\frac{1}{2}(1-\xi)(1-\eta^2)\xi$$

$$N_9=(1-\xi^2)(1-\eta^2)$$

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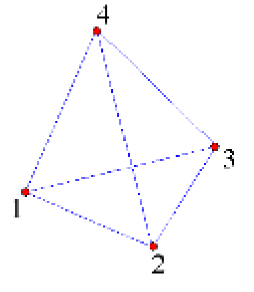
## 4-nodes 3d simplex FE

$$L_1+L_2+L_3+L_4=1$$

$$N_1=L_1$$
$$N_2=L_2$$
$$N_3=L_3$$
$$N_4=L_4$$

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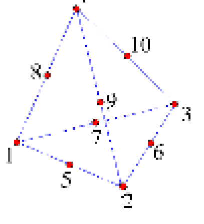
## 10-nodes 3d FE

$$N_1=(2L_1-1)L_1$$
$$N_2=(2L_2-1)L_2$$
$$N_3=(2L_3-1)L_3$$
$$N_4=(2L_4-1)L_4$$

$$N_5=4L_1L_2\quad N_8=4L_1L_4$$
$$N_6=4L_2L_3\quad N_9=4L_2L_4$$
$$N_7=4L_1L_3\quad N_{10}=4L_3L_4$$




$$L_1+L_2+L_3+L_4=1$$

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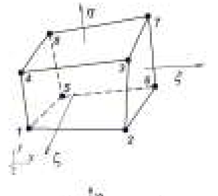
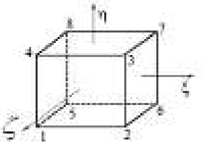
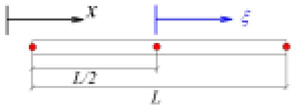
## 8-nodes 3d FE





$$N_1=\frac{1}{8}(1-\xi)(1-\eta)(1+\zeta)$$
$$N_2=\frac{1}{8}(1+\xi)(1-\eta)(1+\zeta)$$
$$N_3=\frac{1}{8}(1+\xi)(1+\eta)(1+\zeta)$$
$$N_4=\frac{1}{8}(1-\xi)(1+\eta)(1+\zeta)$$
$$N_5=\frac{1}{8}(1-\xi)(1-\eta)(1-\zeta)$$
$$N_6=\frac{1}{8}(1+\xi)(1-\eta)(1-\zeta)$$
$$N_7=\frac{1}{8}(1+\xi)(1+\eta)(1-\zeta)$$
$$N_8=\frac{1}{8}(1-\xi)(1+\eta)(1-\zeta)$$

## Transformation of coordinates. Jacobian matrix



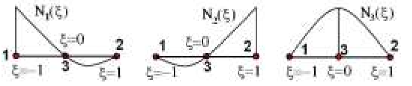

$$-1 \le \xi \le 1$$

$$X = X(\xi) \qquad \xi = \xi(x)$$

$$J = \left|\frac{dX(\xi)}{d\xi}\right| \qquad J = \begin{vmatrix} \frac{\partial X(\xi,\eta)}{\partial \xi} & \frac{\partial Y(\xi,\eta)}{\partial \xi} \\ \frac{\partial X(\xi,\eta)}{\partial \eta} & \frac{\partial Y(\xi,\eta)}{\partial \eta} \end{vmatrix} \qquad J = \begin{vmatrix} \frac{\partial X(\xi,\eta,\zeta)}{\partial \xi} & \frac{\partial Y(\xi,\eta,\zeta)}{\partial \xi} & \frac{\partial Z(\xi,\eta,\zeta)}{\partial \xi} \\ \frac{\partial X(\xi,\eta,\zeta)}{\partial \eta} & \frac{\partial Y(\xi,\eta,\zeta)}{\partial \eta} & \frac{\partial Z(\xi,\eta,\zeta)}{\partial \eta} \\ \frac{\partial X(\xi,\eta,\zeta)}{\partial \zeta} & \frac{\partial Y(\xi,\eta,\zeta)}{\partial \zeta} & \frac{\partial Z(\xi,\eta,\zeta)}{\partial \zeta} \end{vmatrix}$$



## Transformation of coordinates.




$$N_1(\xi) = -\frac{1}{2}\xi(1-\xi) \qquad N_2(\xi) = \frac{1}{2}\xi(1+\xi) \qquad N_3(\xi) = 1-\xi^2$$

$$x = x(\xi) = N_1X_1 + N_2X_2 + N_3X_3 \qquad \frac{\partial N_i}{\partial x} = ?$$

$$\frac{dN_i}{d\xi} = \frac{dN_i}{dx}\frac{dx(\xi)}{d\xi}$$

$$\frac{dN_i}{dx} = \frac{1}{\frac{dx(\xi)}{d\xi}}\frac{dN_i(\xi)}{d\xi}$$

$$\frac{dx}{d\xi} = [J] = \frac{dN_1}{d\xi}X_1 + \frac{dN_2}{d\xi}X_2 + \frac{dN_3}{d\xi}X_3$$

$$\boxed{\frac{dN_i}{dx} = [J]^{-1}\frac{dN_i}{d\xi}}$$



## Calculation of global derivatives

$$\begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix} = J \begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} \qquad \begin{aligned} \frac{\partial N_i}{\partial \xi} &= \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} &= \frac{\partial N_i}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial N_i}{\partial y}\frac{\partial y}{\partial \eta} \end{aligned} \qquad \boxed{\begin{Bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{Bmatrix} = J^{-1} \begin{Bmatrix} \frac{\partial N_i}{\partial \xi} \\ \frac{\partial N_i}{\partial \eta} \end{Bmatrix}}$$

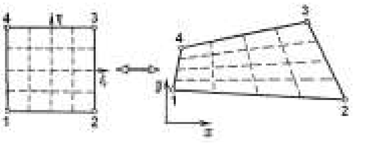
$$J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \qquad J^{-1} = \frac{1}{\det J}\begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi} \\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi} \end{bmatrix}$$

$$S = \int_S \mathrm{dxdy} = \int_{-1}^{1}\int_{-1}^{1} \det J d\xi d\eta$$

$$V = \int_V dXdYdZ = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \det[J] d\xi d\eta d\zeta$$



## Isoparametrical transformation




$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

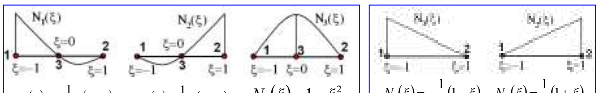
$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

$$x = \sum_{i=1}^{N_{nodes}} N_i x_i = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4$$

$$y = \sum_{i=1}^{N_{nodes}} N_i y_i = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4$$



## Subparametrical, isoparametrical and superparametrical elements




$$N_1(\xi) = -\frac{1}{2}\xi(1-\xi) \qquad N_2(\xi) = \frac{1}{2}\xi(1+\xi) \qquad N_3(\xi) = 1-\xi^2$$

$$N_1(\xi) = -\frac{1}{2}(1-\xi) \qquad N_2(\xi) = \frac{1}{2}(1+\xi)$$

| | |
|---|---|
| $\varphi = N_1\varphi_1 + N_2\varphi_2 + N_3\varphi_3$<br>$x = f(\xi) = N_1X_1 + N_2X_2$ | **Subparametrical** |
| $\varphi = N_1\varphi_1 + N_2\varphi_2 + N_3\varphi_3$<br>$x = f(\xi) = N_1X_1 + N_2X_2 + N_3X_3$ | **Isoparametrical** |
| $\varphi = N_1\varphi_1 + N_2\varphi_2$<br>$x = f(\xi) = N_1X_1 + N_2X_2 + N_3X_3$ | **Superparametrical** |



## Isoparametrical transformation

$$N_1 = -\frac{1}{4}(1-\xi)(1-\eta)(\xi+\eta+1)$$

$$N_2 = \frac{1}{2}(1-\xi^2)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1-\eta)(\xi-\eta-1)$$

$$N_4 = \frac{1}{2}(1-\eta^2)(1+\xi)$$

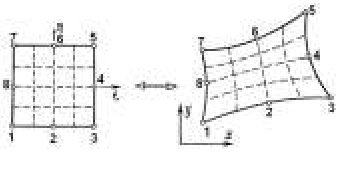
$$N_5 = \frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-1)$$

$$N_6 = \frac{1}{2}(1-\xi^2)(1+\eta)$$

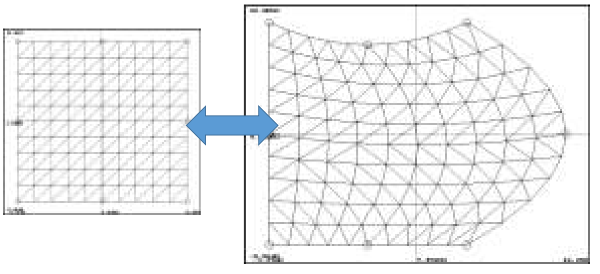
$$N_7 = -\frac{1}{4}(1-\xi)(1+\eta)(\xi-\eta+1)$$

$$N_8 = \frac{1}{2}(1-\eta^2)(1-\xi)$$

$$x = \sum_{i=1}^{N_{nodes}} N_i x_i \qquad y = \sum_{i=1}^{N_{nodes}} N_i y_i$$



## Using for FE grid meshing





## Numerical integration in FEM

$$\int_{-1}^{1} f(\xi)d\xi \approx W_0 f(\xi_0) + W_1 f(\xi_1) + \ldots + W_n f(\xi_n)$$

$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)d\xi d\eta \approx W_0 f(\xi_0,\eta_0) + W_1 f(\xi_1,\eta_1) + \ldots + W_n f(\xi_n,\eta_n)$$

For Gauss integration $p+1 = 2(n+1)$

W – weights of integration points

n+1 – number of integration points



## Gauss integration

$$p+1 = 2(n+1)$$

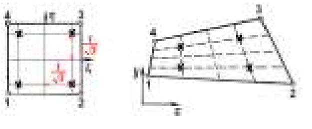
| Number of points for integration, $n$+1 | Rang of exactly integrated polinomial function, $p$ |
|---|---|
| 1 | 1 |
| 2 | 3 |
| 3 | 5 |
| 4 | 7 |

n=1

$$\xi_1 = -\xi_0 = \frac{1}{\sqrt{3}} \approx 0{,}577350269$$

$$W_0 = W_1 = 1{,}0$$

$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)d\xi d\eta = \sum_{i=1}^{n}\sum_{j=1}^{n} W_i W_j f(\xi_i,\eta_j)$$





## Gauss integration

n=2

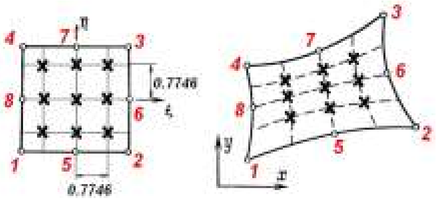
$$\xi_2 = -\xi_0 = \sqrt{\frac{3}{5}} \approx 0{,}774596669$$

$$\xi_1 = 0{,}0$$

$$W_0 = W_2 = \frac{5}{9} \approx 0{,}5555555556$$

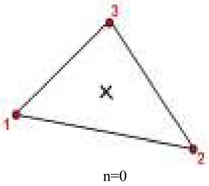
$$W_1 = \frac{8}{9} \approx 0{,}8888888889.$$

$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)d\xi d\eta = \sum_{i=1}^{n}\sum_{j=1}^{n} W_i W_j f(\xi_i,\eta_j)$$

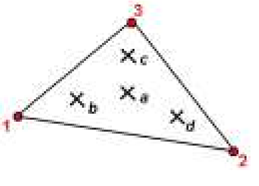





## Example of numerical integration for triangular elements



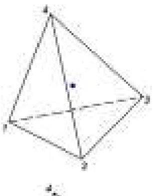
n=0
$L_1$ =1/3;
$L_2$=1/3;
$L_3$=1/3;
W=1/2.



a) $L_1$ =1/3; $L_2$=1/3; $L_3$=1/3; W= -27/96
b) $L_1$ =11/15; $L_2$=2/15; $L_3$=2/15; W= 25/96
c) $L_1$ =2/15; $L_2$=2/15; $L_3$=11/15; W= 25/96
d) $L_1$ =2/15; $L_2$=11/15; $L_3$=2/15; W=25/96



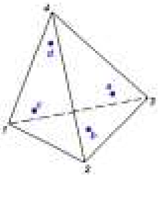
## Example of numerical integration for tetrahedral elements



For n=0 :

**$L_1$ =1/4; $L_2$=1/4; $L_3$=1/4; $L_4$=1/4; W=1;**

For n=1 :



**α = 0.58541020; β = 0.13819660;**

**a) $L_1$ = α; $L_2$=β; $L_3$=β; $L_4$=β; W=1/4;**
**b) $L_1$ = β; $L_2$= α; $L_3$=β; $L_4$=β; W=1/4;**
**c) $L_1$ = β; $L_2$=β; $L_3$= α; $L_4$=β; W=1/4 ;**
**d) $L_1$ = β; $L_2$=β; $L_3$=β; $L_4$= α; W=1/4 ;**

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## Example of code

```
DO P=1, N_p
CALL Jacob_2d(J_, J_inv, P, N_p, NBN, N1, N2, X, Y, DetJ);
        DO i=1, NBN
                Ndx(i)= N1(i,p)*J_inv(1,1) + N2(i,p)*J_inv(1,2)
                Ndy(i)= N1(i,p)*J_inv(2,1) + N2(i,p)*J_inv(2,2)
        END DO;
        detJ = detJ* W(p);
        DO N=1, NBN
                DO I=1, NBN
                Hin = mK*(Ndx(n)*Ndx(i) + Ndy(n)*Ndy(i))*DetJ;
                est(n,i) = est(n,i) + Hin;
                END DO
         END DO
END DO

! ******************************************************************
! Milenin Andre, AGH, 2008, milenin@agh.edu.pl
! ******************************************************************
SUBROUTINE Jacob_2d(J_, J_inv, P, N_p, NBN, N1, N2, X, Y, DetJ);
IMPLICIT NONE;
REAL*8, DIMENSION(2,2) ::              J_, J_inv;
INTEGER*4                              P, N_p, NBN;
REAL*8, DIMENSION(NBN, N_p) ::         N1, N2;
REAL*8, DIMENSION(NBN)    ::           X,Y;
REAL*8                                 DetJ;
integer*4                              i;
J_=0.0;
J_inv=0.0;
DO I=1, NBN
        J_(1,1) = J_(1,1)+N1(i,p)*X(i);
        J_(1,2) = J_(1,2)+N1(i,p)*Y(i);
        J_(2,1) = J_(2,1)+N2(i,p)*X(i);
        J_(2,2) = J_(2,2)+N2(i,p)*Y(i);
END DO;
detJ = J_(1,1)*J_(2,2)-J_(1,2)*J_(2,1);
I=2;
CALL Inv_MAT(i, J_, J_inv);
END SUBROUTINE Jacob_2d;
```

$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{Bmatrix} = J^{-1} \begin{Bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{Bmatrix}$$

$$J = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}$$

$$J^{-1} = \frac{1}{\det J} \begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial y}{\partial \xi} \\ -\dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \xi} \end{bmatrix}$$

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## Lecture 3.

- Solving the heat flow problems by FEM.
- Steady state and non steady state heat flow problems.
- Equations for stiffness matrix and load vector.
- Two dimensional FEM code for simulation of heat flow.

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## Temperature distribution

$$\frac{\partial}{\partial x}\left(k_x(t)\frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y(t)\frac{\partial t}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z(t)\frac{\partial t}{\partial z}\right) + Q = c_{eff}\,\rho\frac{\partial t}{\partial \tau}$$

$$q = \alpha_{konw}(t - t_\infty) \qquad \alpha_{konw}\ (\mathrm{W/m^2\,K});$$

$$q = \sigma_{rad}(t^4 - t_\infty^4) \qquad \sigma_{rad}\ (\mathrm{W/m^2K^4}).$$

$$J = \int_V \left[\frac{k(t)}{2}\left(\left(\frac{\partial t}{\partial x}\right)^2 + \left(\frac{\partial t}{\partial y}\right)^2 + \left(\frac{\partial t}{\partial z}\right)^2\right) - Qt\right] dV + \int_S \frac{\alpha}{2}(t - t_\infty)^2 dS + \int_S qt\,dS$$

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## Dyscretisation

$$t = \sum_{i=1}^{n} N_i t_i = \{N\}^T\{t\} = [N]\{t\}$$

$$\frac{\partial t}{\partial x} = \frac{\partial}{\partial x}\left(\{N\}^T\{t\}\right) = \left\{\frac{\partial\{N\}}{\partial x}\right\}^T\{t\}$$

we substitute this results in the original functional instead t and dt/dx, dt/dy. dt/dz:

$$J = \int_V \left[\frac{k(t)}{2}\left(\left(\left\{\frac{\partial\{N\}}{\partial x}\right\}^T\{t\}\right)^2 + \left(\left\{\frac{\partial\{N\}}{\partial y}\right\}^T\{t\}\right)^2 + \left(\left\{\frac{\partial\{N\}}{\partial z}\right\}^T\{t\}\right)^2\right) - Q\{N\}^T\{t\}\right] dV +$$

$$+ \int_S \frac{\alpha}{2}\left(\{N\}^T\{t\} - t_\infty\right)^2 dS + \int_S q\{N\}^T\{t\}dS.$$

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## Minimization of functional

we differentiate the functional by vector {t} and equate the result to zero

$$\frac{\partial J}{\partial\{t\}} = \int_V \left[k(t)\left(\left\{\frac{\partial\{N\}}{\partial x}\right\}\left\{\frac{\partial\{N\}}{\partial x}\right\}^T + \left\{\frac{\partial\{N\}}{\partial y}\right\}\left\{\frac{\partial\{N\}}{\partial y}\right\}^T + \left\{\frac{\partial\{N\}}{\partial z}\right\}\left\{\frac{\partial\{N\}}{\partial z}\right\}^T\right)\{t\} - Q\{N\}\right] dV +$$

$$\int_S \alpha\left(\{N\}^T\{t\} - t_\infty\right)\{N\}dS + \int_S q\{N\}dS = 0$$

$$[H]\{t\} + \{P\} = 0$$

## Stiffness matrix

$$[H] = \int_V k(t)\left(\left\{\frac{\partial\{N\}}{\partial x}\right\}\left\{\frac{\partial\{N\}}{\partial x}\right\}^T + \left\{\frac{\partial\{N\}}{\partial y}\right\}\left\{\frac{\partial\{N\}}{\partial y}\right\}^T + \left\{\frac{\partial\{N\}}{\partial z}\right\}\left\{\frac{\partial\{N\}}{\partial z}\right\}^T\right)dV + \int_S \alpha\{N\}\{N\}^T dS,$$

$$\{P\} = -\int_S \alpha\{N\}t_\infty dS - \int_V Q\{N\}dV + \int_S q\{N\}dS$$

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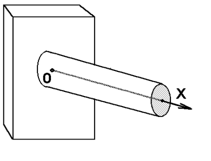
## 1d Example:

$$[H] = \int_V k\left\{\frac{\partial\{N\}}{\partial x}\right\}\left\{\frac{\partial\{N\}}{\partial x}\right\}^T dV + \int_S \alpha\{N\}\{N\}^T dS$$

$$\{P\} = -\int_S \alpha\{N\}t_\infty dS + \int_S q\{N\}dS$$




$$\{N\} = \begin{Bmatrix} N_i \\ N_j \end{Bmatrix} = \begin{Bmatrix} \dfrac{x_j - x}{L} \\ \dfrac{x - x_i}{L} \end{Bmatrix}$$

$$\{N\}^T = \{N_i \quad N_j\} = \left\{\frac{x_j - x}{L}; \quad \frac{x - x_i}{L}\right\}$$

$$\left\{\frac{\partial\{N\}}{\partial x}\right\} = \begin{Bmatrix} -\dfrac{1}{L} \\ \dfrac{1}{L} \end{Bmatrix}$$

$$\left\{\frac{\partial\{N\}}{\partial x}\right\}^T = \left\{-\frac{1}{L} \quad \frac{1}{L}\right\}$$

$$[H] = k\begin{Bmatrix} -\dfrac{1}{L} \\ \dfrac{1}{L} \end{Bmatrix}\left\{-\frac{1}{L} \quad \frac{1}{L}\right\}SL + \alpha\begin{Bmatrix} N_i \\ N_j \end{Bmatrix}\{N_i \quad N_j\}S$$

$$[H] = \int_V k\begin{Bmatrix} -\dfrac{1}{L} \\ \dfrac{1}{L} \end{Bmatrix}\left\{-\frac{1}{L} \quad \frac{1}{L}\right\}dV + \int_S \alpha\begin{Bmatrix} \dfrac{x_j - x}{L} \\ \dfrac{x - x_i}{L} \end{Bmatrix}\left\{\frac{x_j - x}{L}; \quad \frac{x - x_i}{L}\right\}dS$$

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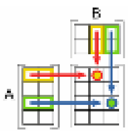
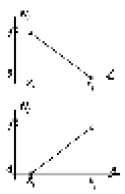
$$[H]=k\begin{Bmatrix}-\frac{1}{L}\\ \frac{1}{L}\end{Bmatrix}\begin{Bmatrix}-\frac{1}{L} & \frac{1}{L}\end{Bmatrix}SL+\alpha\begin{Bmatrix}N_i\\ N_j\end{Bmatrix}\begin{Bmatrix}N_i & N_j\end{Bmatrix}S$$

$$[H]^{(1)}=k\begin{bmatrix}\frac{1}{L^{(1)2}} & -\frac{1}{L^{(1)2}}\\ -\frac{1}{L^{(1)2}} & \frac{1}{L^{(1)2}}\end{bmatrix}SL^{(1)}=\begin{bmatrix}\frac{Sk}{L^{(1)}} & -\frac{Sk}{L^{(1)}}\\ -\frac{Sk}{L^{(1)}} & \frac{Sk}{L^{(1)}}\end{bmatrix}$$

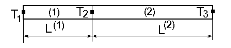
$$[H]^{(2)}=k\begin{bmatrix}\frac{1}{L^{(2)2}} & -\frac{1}{L^{(2)2}}\\ -\frac{1}{L^{(2)2}} & \frac{1}{L^{(2)2}}\end{bmatrix}SL^{(2)}+\alpha\begin{Bmatrix}N_iN_i & N_iN_j\\ N_iN_j & N_jN_j\end{Bmatrix}S$$

$$[H]^{(2)}=\begin{bmatrix}\frac{Sk}{L^{(2)}} & -\frac{Sk}{L^{(2)}}\\ -\frac{Sk}{L^{(2)}} & \frac{Sk}{L^{(2)}}\end{bmatrix}+\begin{Bmatrix}0 & 0\\ 0 & \alpha S\end{Bmatrix}=\begin{bmatrix}\frac{Sk}{L^{(2)}} & -\frac{Sk}{L^{(2)}}\\ -\frac{Sk}{L^{(2)}} & \frac{Sk}{L^{(2)}}+\alpha S\end{bmatrix}$$








$$\{P\}=-\int_S\alpha\{N\}t_\infty dS+\int_S q\{N\}dS$$

$$\{P\}=-\int_S\alpha\begin{Bmatrix}N_i\\ N_j\end{Bmatrix}t_\infty dS+\int_S q\begin{Bmatrix}N_i\\ N_j\end{Bmatrix}dS$$

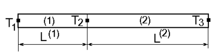
$$\{N\}=\begin{Bmatrix}N_i\\ N_j\end{Bmatrix}=\begin{Bmatrix}\frac{x_j-x}{L}\\ \frac{x-x_i}{L}\end{Bmatrix}\qquad \{N\}^T=\{N_i \quad N_j\}=\left\{\frac{x_j-x}{L};\quad \frac{x-x_i}{L}\right\}$$

$$\{P\}^{(1)}=q\begin{Bmatrix}N_i\\ N_j\end{Bmatrix}S=q\begin{Bmatrix}1\\ 0\end{Bmatrix}S=\begin{Bmatrix}qS\\ 0\end{Bmatrix}$$

$$\{P\}^{(2)}=-\alpha t_\infty S\begin{Bmatrix}N_i\\ N_j\end{Bmatrix}=-\alpha t_\infty S\begin{Bmatrix}0\\ 1\end{Bmatrix}=\begin{Bmatrix}0\\ -\alpha t_\infty S\end{Bmatrix}$$



## Global stiffness matrix




$$[H]=\sum_{e=1}^{n_e}[H]^{(e)}$$

Positions of stiffness matrix for FE number 2

| | 1 | 2 |
|---|---|---|
| 1 | 2,2 | 2,3 |
| 2 | 3,2 | 3,3 |



$$[H]=[H]^{(1)}+[H]^{(2)}=\begin{bmatrix}\frac{Sk}{L^{(1)}} & -\frac{Sk}{L^{(1)}} & 0\\ -\frac{Sk}{L^{(1)}} & \frac{Sk}{L^{(1)}} & 0\\ 0 & 0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0 & 0\\ 0 & \frac{Sk}{L^{(2)}} & -\frac{Sk}{L^{(2)}}\\ 0 & -\frac{Sk}{L^{(2)}} & \frac{Sk}{L^{(2)}}+\alpha S\end{bmatrix}=$$

$$=\begin{bmatrix}\frac{Sk}{L^{(1)}} & -\frac{Sk}{L^{(1)}} & 0\\ -\frac{Sk}{L^{(1)}} & Sk\left(\frac{1}{L^{(1)}}+\frac{1}{L^{(2)}}\right) & -\frac{Sk}{L^{(2)}}\\ 0 & -\frac{Sk}{L^{(2)}} & \frac{Sk}{L^{(2)}}+\alpha S\end{bmatrix}$$

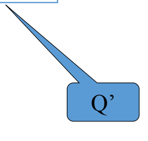
$$\{P\}=\{P\}^{(1)}+\{P\}^{(2)}=\begin{Bmatrix}qS\\ 0\\ 0\end{Bmatrix}+\begin{Bmatrix}0\\ 0\\ -\alpha t_\infty S\end{Bmatrix}=\begin{Bmatrix}qS\\ 0\\ -\alpha t_\infty S\end{Bmatrix}$$



## Non- stationary problem

$$\frac{\partial}{\partial x}\left(k_x(t)\frac{\partial t}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y(t)\frac{\partial t}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_z(t)\frac{\partial t}{\partial z}\right)+\left(Q-c_{eff}\rho\frac{\partial t}{\partial \tau}\right)=0$$

$$t=\{N\}^T\{t\}$$




$$Q'=Q-c_{eff}\rho\frac{\partial t}{\partial \tau}=Q-c_{eff}\rho\frac{\partial}{\partial \tau}\{t\}\{N\}^T$$

We substitute the result into initial equation





## Stiffness matrix :

$$[H]=\int_V k(t)\left(\left\{\frac{\partial\{N\}}{\partial x}\right\}\left\{\frac{\partial\{N\}}{\partial x}\right\}^T+\left\{\frac{\partial\{N\}}{\partial y}\right\}\left\{\frac{\partial\{N\}}{\partial y}\right\}^T+\left\{\frac{\partial\{N\}}{\partial z}\right\}\left\{\frac{\partial\{N\}}{\partial z}\right\}^T\right)dV+\int_S\alpha\{N\}\{N\}^T dS,$$

$$\{P\}=-\int_S\alpha\{N\}t_\infty dS-\int_V Q\{N\}dV+\int_S q\{N\}dS$$

$$Q'=Q-c_{eff}\rho\frac{\partial t}{\partial \tau}=Q-c_{eff}\rho\frac{\partial}{\partial \tau}\{t\}\{N\}^T$$

$$[H]\{t\}+\{P\}=0 \quad \Rightarrow \quad [H]\{t\}+[C]\frac{\partial}{\partial \tau}\{t\}+\{P\}=0$$

$$[C]=\int_V\{N\}\rho c_{eff}\{N\}^T dV$$

## Derivative in time

$$\{t\} = \{N_0, N_1\}\begin{Bmatrix}\{t_0\}\\ \{t_1\}\end{Bmatrix}$$

$$N_0 = \frac{\Delta\tau - \tau}{\Delta\tau}$$

$$N_1 = \frac{\tau}{\Delta\tau}$$

$$\Rightarrow \quad \frac{\partial\{t\}}{\partial\tau} = \left\{\frac{\partial N_0}{\partial\tau}, \frac{\partial N_1}{\partial\tau}\right\}\begin{Bmatrix}\{t_0\}\\ \{t_1\}\end{Bmatrix} = \frac{1}{\Delta\tau}\{-1,1\}\begin{Bmatrix}\{t_0\}\\ \{t_1\}\end{Bmatrix}$$



$$[H]\{t\} + [C]\frac{\partial}{\partial\tau}\{t\} + \{P\} = 0 \qquad (1)$$

$$\frac{\partial\{t\}}{\partial\tau} = \left\{\frac{\partial N_0}{\partial\tau}, \frac{\partial N_1}{\partial\tau}\right\}\begin{Bmatrix}\{t_0\}\\ \{t_1\}\end{Bmatrix} = \frac{1}{\Delta\tau}\{-1,1\}\begin{Bmatrix}\{t_0\}\\ \{t_1\}\end{Bmatrix} = \frac{\{t_1\}-\{t_0\}}{\Delta\tau}. \qquad (2)$$

## Integration in time

1. Explicit method: $\{t\}=\{t_0\}$ ,

$$[H]\{t_0\} + [C]\frac{\{t_1\}-\{t_0\}}{\Delta\tau} + \{P\} = 0$$

$$\{t_1\} = \{t_0\} - \frac{\Delta\tau}{[C]}\left([H]\{t_0\} + \{P\}\right)$$



$$[H]\{t\} + [C]\frac{\partial}{\partial\tau}\{t\} + \{P\} = 0 \qquad (1)$$

$$\frac{\partial\{t\}}{\partial\tau} = \left\{\frac{\partial N_0}{\partial\tau}, \frac{\partial N_1}{\partial\tau}\right\}\begin{Bmatrix}\{t_0\}\\ \{t_1\}\end{Bmatrix} = \frac{1}{\Delta\tau}\{-1,1\}\begin{Bmatrix}\{t_0\}\\ \{t_1\}\end{Bmatrix} = \frac{\{t_1\}-\{t_0\}}{\Delta\tau}. \qquad (2)$$

2. Implicit method $\{t\}=\{t_1\}$ :

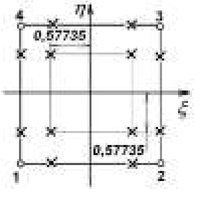
$$[H]\{t_1\} + [C]\frac{\{t_1\}-\{t_0\}}{\Delta\tau} + \{P\} = 0$$

$$\left([H] + \frac{[C]}{\Delta\tau}\right)\{t_1\} - \left(\frac{[C]}{\Delta\tau}\right)\{t_0\} + \{P\} = 0$$



**EXAMPLE OF FEM CODE**

FORMULATION OF THERMAL PROBLEM




$$\left([H] + \frac{[C]}{\Delta\tau}\right)\{t_1\} - \left(\frac{[C]}{\Delta\tau}\right)\{t_0\} + \{P\} = 0,$$

where

$$[H] = \int_V k\left(\left\{\frac{\partial\{N\}}{\partial x}\right\}\left\{\frac{\partial\{N\}}{\partial x}\right\}^T + \left\{\frac{\partial\{N\}}{\partial y}\right\}\left\{\frac{\partial\{N\}}{\partial y}\right\}^T\right)dV + \int_S \alpha\{N\}\{N\}^T dS,$$

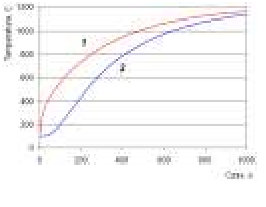
$$\{P\} = -\int_S \alpha\{N\}t_\infty dS,$$

$$[C] = \int_V c\rho\{N\}\{N\}^T dV.$$



DATA FOR SIMULATION

| | | |
|---|---|---|
| 100 | mTbegin | Initial temperature, C |
| 1000 | mTime | Time of process, s |
| 1 | mdTime | Time step, s |
| 1200 | mT_otoczenia | Temperature of invironment C |
| 300 | mAlfa | W/Cm2 |
| 100 | mH0 | h, mm |
| 100 | mB0 | b, mm |
| 25 | mNhH | nh |
| 25 | mNhH | nb |
| 700 | mC | c, J/Ckg |
| 25 | mK | k |
| 7800 | mR | ro kg/m3 |





## Lection 4.

1.Solution of elastic problems by FEM. Basics of theory of elasticity.
Variation principle.
Equations for stiffness matrix and load vector.
Example of FEM code for solving plain strain problem by FEM.

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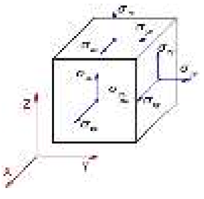
## Theoretical basis of the linear theory of elasticity

**Stress tensor. Couchy stresses. Effective stress (intensity).**



$$\sigma_{ij} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \sigma_0 \delta_{ij} + s_{ij} = A_\sigma + D_\sigma$$

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

$$\sigma_0 = \frac{1}{3}\left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right) = \frac{1}{3}\sigma_{ij}\delta_{ij}$$

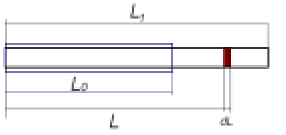
$$\bar{\sigma} = \sqrt{\frac{3}{2} s_{ij} s_{ij}} = \frac{1}{\sqrt{2}}\sqrt{\left(\sigma_{xx} - \sigma_{yy}\right)^2 + \left(\sigma_{yy} - \sigma_{zz}\right)^2 + \left(\sigma_{zz} - \sigma_{xx}\right)^2 + 6\left(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2\right)}$$

$$\bar{\tau} = T = \sqrt{\frac{1}{2} s_{ij} s_{ij}} = \frac{\bar{\sigma}}{\sqrt{3}}$$



***Strains. Engineer approach. Logarithmical strains.***

$$\varepsilon_x = \frac{L_1 - L_0}{L_0} = \frac{\Delta L}{L_0}$$



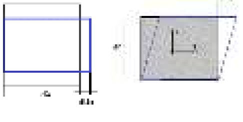

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix}$$

$$e_x = \int_{L_0}^{L_1} \frac{dL}{L} = \ln\left(\frac{L_1}{L_0}\right) \qquad e_L = \ln\left(\frac{L_1}{L_0}\right) = \ln\left(\frac{L_0 + \Delta L}{L_0}\right) = \ln\left(1 + \varepsilon_L\right)$$



## Cauchy equations (Augustin Louis Cauchy)

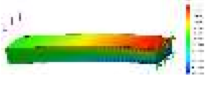
$$\vec{u} = \left(u_1, u_2, u_3\right)$$



$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \qquad \varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} \qquad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} \qquad \varepsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$$





## Cauchy equations in matrix form for plain strain

$$\{\varepsilon\} = [B]\{U\} \qquad [B] = \begin{bmatrix} \frac{\partial [N]}{\partial x} & 0 \\ 0 & \frac{\partial [N]}{\partial y} \\ \frac{\partial [N]}{\partial y} & \frac{\partial [N]}{\partial x} \end{bmatrix}$$

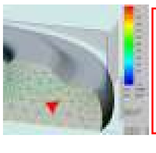


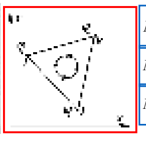
## Cauchy equations

$$\varepsilon_x = \frac{\partial U_x}{\partial X}$$

$$\varepsilon_y = \frac{\partial U_y}{\partial Y}$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial U_x}{\partial Y} + \frac{\partial U_y}{\partial X}\right)$$





$$N_i = \frac{1}{2A}\left(a_i + b_i X + c_i Y\right)$$

$$N_j = \frac{1}{2A}\left(a_j + b_j X + c_j Y\right)$$

$$N_k = \frac{1}{2A}\left(a_k + b_k X + c_k Y\right)$$

$$a_i = X_j Y_k - X_k Y_j \qquad b_i = Y_j - Y_k \qquad c_i = X_k - X_j$$

$$U_x = U_{xi} N_i + U_{xj} N_j + U_{xk} N_k$$

$$U_y = U_{yi} N_i + U_{yj} N_j + U_{yk} N_k$$

$$a_j = X_k Y_i - X_i Y_k \qquad b_j = Y_k - Y_i \qquad c_j = X_i - X_k$$

$$\varepsilon_x = \frac{\partial U_x}{\partial X} = U_{xi}\frac{\partial N_i}{\partial X} + U_{xj}\frac{\partial N_j}{\partial X} + U_{xk}\frac{\partial N_k}{\partial X} = \frac{1}{2A}\left(U_{xi} b_i + U_{xj} b_j + U_{xk} b_k\right)$$

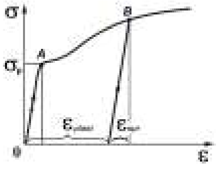
$$\varepsilon_y = \frac{\partial U_y}{\partial Y} = U_{yi}\frac{\partial N_i}{\partial Y} + U_{yj}\frac{\partial N_j}{\partial Y} + U_{yk}\frac{\partial N_k}{\partial Y} = \frac{1}{2A}\left(U_{yi} c_i + U_{yj} c_j + U_{yk} c_k\right)$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial U_x}{\partial Y} + \frac{\partial U_y}{\partial X}\right) = \frac{1}{2}\left(U_{xi}\frac{\partial N_i}{\partial Y} + U_{xj}\frac{\partial N_j}{\partial Y} + U_{xk}\frac{\partial N_k}{\partial Y} + U_{yi}\frac{\partial N_i}{\partial X} + U_{yj}\frac{\partial N_j}{\partial X} + U_{yk}\frac{\partial N_k}{\partial X}\right) =$$
$$\frac{1}{4A}\left(U_{xi} c_i + U_{xj} c_j + U_{xk} c_k + U_{yi} b_i + U_{yj} b_j + U_{yk} b_k\right)$$

$$a_k = X_i Y_j - X_j Y_i \qquad b_k = Y_i - Y_j \qquad c_k = X_j - X_i$$



## Young's modulus. Poisson's ratio.



$$E = \frac{\bar{\sigma}}{\bar{\varepsilon}}$$

$$\varepsilon_2 = \varepsilon_3 = -\varepsilon_1 \nu = -\nu\frac{\sigma_1}{E}$$

$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}$$

$$\nu = \left|\frac{\varepsilon_2}{\varepsilon_1}\right| = \left|\frac{\varepsilon_3}{\varepsilon_1}\right|$$

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## Hooke's law.

$$\varepsilon_{xx}=\frac{1}{E}\left[\sigma_{xx}-\nu\left(\sigma_{yy}+\sigma_{zz}\right)\right]$$

$$\varepsilon_{yy}=\frac{1}{E}\left[\sigma_{yy}-\nu\left(\sigma_{xx}+\sigma_{zz}\right)\right]$$

$$\varepsilon_{zz}=\frac{1}{E}\left[\sigma_{zz}-\nu\left(\sigma_{yy}+\sigma_{xx}\right)\right]$$

$$\varepsilon_{xy}=\frac{1+\nu}{E}\sigma_{xy}$$

$$\varepsilon_{xz}=\frac{1+\nu}{E}\sigma_{xz}$$

$$\varepsilon_{zy}=\frac{1+\nu}{E}\sigma_{zy}$$

## Hooke's law in matrix form.

$$\{\sigma\}=[D]\{\varepsilon\}$$

$$[D]=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & (1-2\nu)/2\end{bmatrix}$$



## Principle of virtual work in matrix form

$$W=\int_V \frac{1}{2}\left(\{\varepsilon\}^T\{\sigma\}\right)dV$$

$$[B]=\begin{bmatrix}\frac{\partial[N]}{\partial x} & 0\\ 0 & \frac{\partial[N]}{\partial y}\\ \frac{\partial[N]}{\partial y} & \frac{\partial[N]}{\partial x}\end{bmatrix}$$

$\{\sigma\}=[D]\{\varepsilon\}$ *Hooke`s law*

$\{\varepsilon\}=[B]\{U\}$ *Cauchy equations*

We substitute the result into first equation:

$$W_e=\int_{V_e}\frac{1}{2}\left(\{U\}^T[B]^T[D][B]\{U\}\right)dV$$



## Material properties in matrix form

Plain strain

$$[D]=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & (1-2\nu)/2\end{bmatrix}$$

Plain stress

$$[D]=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & (1-\nu)/2\end{bmatrix}$$

3d strain

$$[D]=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ & 1-\nu & \nu & 0 & 0 & 0\\ & & 1-\nu & 0 & 0 & 0\\ & & & (1-2\nu)/2 & 0 & 0\\ & & & & (1-2\nu)/2 & 0\\ sym & & & & & (1-2\nu)/2\end{bmatrix}$$



## Principle of virtual work in matrix form

$$W_s=\{U\}^T\{P\}=\{P\}^T\{U\} \qquad \{P\}=\begin{Bmatrix}\{P_x\}\\ \{P_y\}\end{Bmatrix}$$

$$W_b=\int_{V_e}\left(u_x m_x+u_y m_y\right)dV=\int_{V_e}\{U\}^T\left[\overline{N}\right]^T\{M\}dV \qquad \{M\}=\begin{Bmatrix}m_x\\ m_y\end{Bmatrix}$$

$$\left[\overline{N}\right]=\begin{bmatrix}N_1 & N_2\ldots & N_p & 0 & 0\ldots & 0\\ 0 & 0\ldots & 0 & N_1 & N_2\ldots & N_p\end{bmatrix}=\begin{bmatrix}[N] & 0\\ 0 & [N]\end{bmatrix}$$

$$W_p=\int_{S_e}\left(u_x p_x+u_y p_y\right)dS=\int_{S_e}\{U\}^T\left[\overline{N}\right]^T\{p\}dS \qquad \{p\}=\begin{Bmatrix}p_x\\ p_y\end{Bmatrix}$$

$$W_e=\int_{V_e}\frac{1}{2}\{U\}^T[B]^T[D][B]\{U\}dV-\int_{V_e}\{U\}^T\left[\overline{N}\right]^T\{M\}dV-\int_{S_e}\{U\}^T\left[\overline{N}\right]^T\{p\}dS-\{U\}^T\{P\}$$



## Stiffness matrix

$$\frac{\partial W_e}{\partial\{U\}}=\{U\}\int_{V_e}[B]^T[D][B]dV-\int_{V_e}\left[\overline{N}\right]^T\{M\}dV-\int_{S_e}\left[\overline{N}\right]^T\{p\}dS-\{P\}=0$$

$$\frac{\partial W_e}{\partial\{U\}}=[K_e]\{U\}+\{F_e\}=0$$

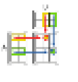
$$[K_e]=\int_{V_e}[B]^T[D][B]dV \qquad \{F_e\}=-\int_{V_e}\left[\overline{N}\right]^T\{M\}dV-\int_{S_e}\left[\overline{N}\right]^T\{p\}dS-\{P\}$$

$$[K]\{U\}+\{F\}=0$$

$$[K]=\sum_{e=1}^{n_e}[K_e] \qquad \{F\}=\sum_{e=1}^{n_e}\{F_e\}$$



## Stiffness matrix



$$[K]^e=\int_{V_e}[B]^T[D][B]dV$$

$$[B]^T=\begin{bmatrix}\frac{\partial[N]^T}{\partial x} & 0 & \frac{\partial[N]^T}{\partial y}\\ 0 & \frac{\partial[N]^T}{\partial y} & \frac{\partial[N]^T}{\partial x}\end{bmatrix} \qquad [B]=\begin{bmatrix}\frac{\partial[N]}{\partial x} & 0\\ 0 & \frac{\partial[N]}{\partial y}\\ \frac{\partial[N]}{\partial y} & \frac{\partial[N]}{\partial x}\end{bmatrix}$$

$$[D]=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & (1-2\nu)/2\end{bmatrix}$$

## Stiffness matrix for plain strain problem

$$\int_{V_e} [B]^T [D][B] dV = \int \frac{E}{(1+\nu)(1-2\nu)} \left[ \begin{array}{c|c} (1-\nu)\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial x} + \frac{(1-2\nu)}{2}\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y} & \nu\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial y} + \frac{(1-2\nu)}{2}\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial y} \\ \hline \nu\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial x} + \frac{(1-2\nu)}{2}\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial x} & (1-\nu)\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y} + \frac{(1-2\nu)}{2}\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial x} \end{array} \right] dV$$

$$\{F_e\} = -\int_{V_e} [\overline{N}]^T \{M\} dV - \int_{S_e} [\overline{N}]^T \{p\} dS - \{P\}$$

Finite Element Analysis for Metal Forming and Material Engineering, A.Milenin

## Example of FEM code for solving plain strain problem by FEM.

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Lection 5.
Solution of rigid plastic problems by FEM.

- Theory of plasticity on non compressible materials.
- Variation principle.
- Equations for stiffness matrix and load vector.
- Example of FEM code for simulation of plain strain problem in flow formulation.
- Analogy between flow dynamic and theory of plasticity in flow formulation.

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## The theoretical foundations of the theory of plastic flow of incompressible materials

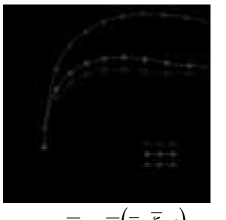
$$\xi_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i}\right)$$

$$\sigma_{ij,j} = 0$$

$$\sigma_{ij} = \delta_{ij}\sigma_0 + \frac{2\overline{\sigma}}{3\overline{\xi}}\xi_{ij} \qquad s_{ij} = \frac{2\overline{\sigma}}{3\overline{\xi}}\xi_{ij} = 2\mu\xi_{ij}$$

$$\xi_0 = 0$$

$$\xi_0 = \frac{1}{3}div(\vec{v}) = 0$$



$$\overline{\sigma} = \overline{\sigma}\left(\overline{\varepsilon}, \overline{\xi}, t\right)$$

$$\mu = \frac{\overline{\sigma}}{3\overline{\xi}}$$

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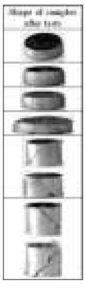
## Mechanical properties of the workable metal



$$\overline{\sigma} = \overline{\sigma}\left(\overline{\varepsilon}, \overline{\xi}, t\right)$$

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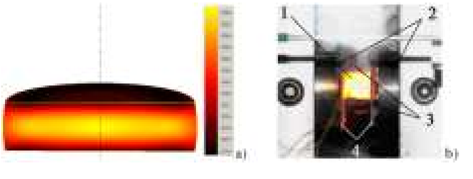
## Problems:




$$\delta = \sum_{m}^{m_{test}} \sum_{n=1}^{n_{pnt}} \left(P_{mn}^{calc} - P_{mn}^{exp}\right)^2$$

$$\overline{\sigma} = \overline{\sigma}\left(\overline{\varepsilon}, \overline{\xi}, t\right)$$

$$\overline{\sigma} = A\overline{\varepsilon}^n \exp\left(-B\overline{\varepsilon}\right)\overline{\xi}^m \exp\left(-Ct\right)$$

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## Features of using FEM to solve problems in the theory of plastic flow

$$\{\sigma\}=\sigma_0[I]^T+\{s\}=\sigma_0[I]^T+[D]\{\xi\} \qquad [I]=[1 \;\; 1 \;\; 0]$$

$$\{\xi\}=\begin{Bmatrix}\xi_{xx}\\ \xi_{yy}\\ 2\xi_{xy}\end{Bmatrix} \qquad \xi_0=\frac{1}{3}[I]\{\xi\}=\frac{1}{3}[1 \;\; 1 \;\; 0]\begin{Bmatrix}\xi_{xx}\\ \xi_{yy}\\ 2\xi_{xy}\end{Bmatrix} \qquad \sigma_0=\frac{1}{3}[I]\{\sigma\}=\frac{1}{3}[1 \;\; 1 \;\; 0]\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ 2\sigma_{xy}\end{Bmatrix}$$

$$W=\int_V\frac{1}{2}\{\xi\}^T\{s\}dV+\int_V\sigma_0[I]\{\xi\}dV$$

$$W_p=\int_S\{v\}^T\{p\}dS$$

$$J=\int_V\frac{1}{2}\{\xi\}^T\{s\}dV+\int_V\sigma_0[I]\{\xi\}dV-\int_S\{v\}^T\{p\}dS$$



$$J=\int_V\frac{1}{2}\{\xi\}^T\{s\}dV+\int_V\sigma_0[I]\{\xi\}dV-\int_S\{v\}^T\{p\}dS$$

$$\{s\}=[D]\{\xi\}=\begin{Bmatrix}2\mu\xi_{xx}\\ 2\mu\xi_{yy}\\ \mu 2\xi_{xy}\end{Bmatrix} \qquad \{\xi\}=\begin{Bmatrix}\xi_{xx}\\ \xi_{yy}\\ 2\xi_{xy}\end{Bmatrix}$$

$$\sigma_x=\sigma_0+2\mu\xi_x$$
$$\sigma_y=\sigma_0+2\mu\xi_y$$
$$\sigma_{xy}=\mu 2\xi_{xy}$$

$$[D]=\begin{bmatrix}2\mu & 0 & 0\\ 0 & 2\mu & 0\\ 0 & 0 & \mu\end{bmatrix} \qquad \{s\}=\begin{Bmatrix}2\mu\xi_{xx}\\ 2\mu\xi_{yy}\\ \mu 2\xi_{xy}\end{Bmatrix}$$

$$J=\int_V\frac{1}{2}\{\xi\}^T[D]\{\xi\}dV+\int_V\sigma_0[I]\{\xi\}dV-\int_S\{v\}^T\{p\}dS$$



$$\{\xi\}=\begin{Bmatrix}\xi_x\\ \xi_y\\ 2\xi_{xy}\end{Bmatrix}=\begin{bmatrix}\frac{\partial}{\partial x} & 0\\ 0 & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x}\end{bmatrix}\begin{Bmatrix}v_x\\ v_y\end{Bmatrix}=\begin{bmatrix}\frac{\partial}{\partial x} & 0\\ 0 & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x}\end{bmatrix}\begin{Bmatrix}[N]\{\mathrm{v}_x\}\\ [N]\{\mathrm{v}_y\}\end{Bmatrix}=\begin{bmatrix}\frac{\partial[N]}{\partial x} & 0\\ 0 & \frac{\partial[N]}{\partial y}\\ \frac{\partial[N]}{\partial y} & \frac{\partial[N]}{\partial x}\end{bmatrix}\begin{Bmatrix}\{\mathrm{v}_x\}\\ \{\mathrm{v}_y\}\end{Bmatrix}=[B]\{\mathrm{v}\}$$

$$v_x=[N]\{\mathrm{v}_x\}$$
$$v_y=[N]\{\mathrm{v}_y\}$$

$$[B]=\begin{bmatrix}\frac{\partial[N]}{\partial x} & 0\\ 0 & \frac{\partial[N]}{\partial y}\\ \frac{\partial[N]}{\partial y} & \frac{\partial[N]}{\partial x}\end{bmatrix}$$

$$\{\xi\}=[B]\{\mathrm{v}\}$$





$$\{v\}=\begin{Bmatrix}v_x\\ v_y\end{Bmatrix}=\begin{bmatrix}[N] & 0\\ 0 & [N]\end{bmatrix}\begin{Bmatrix}\{\mathrm{v}_x\}\\ \{\mathrm{v}_y\}\end{Bmatrix}=[\overline{N}]\{\mathrm{v}\}$$

$$[\overline{N}]=\begin{bmatrix}N_1 & N_2\ldots & N_p & 0 & 0\ldots & 0\\ 0 & 0\ldots & 0 & N_1 & N_2\ldots & N_p\end{bmatrix}=\begin{bmatrix}[N] & 0\\ 0 & [N]\end{bmatrix}$$

Approximation of mean stress: $\sigma_0=[N]\{\sigma_0\}$ $\qquad \sigma_0=[H]\{\sigma_0\}$

$$\xi_0=[I]\{\xi\}=[1 \;\; 1 \;\; 0]\begin{Bmatrix}\xi_{xx}\\ \xi_{yy}\\ 2\xi_{xy}\end{Bmatrix}=[I][B]\{\mathrm{v}\}=[E]\{\mathrm{v}\}$$

$$[E]=[I][B]=\begin{bmatrix}\frac{\partial[N]}{\partial x} & \frac{\partial[N]}{\partial y}\end{bmatrix}$$

$$\{s\}=[D][B]\{\mathrm{v}\}$$





$$J=\int_V\frac{1}{2}\{\mathrm{v}\}^T[B]^T[D][B]\{\mathrm{v}\}dV+\int_V\sigma_0[E]\{\mathrm{v}\}dV-\int_S\{\mathrm{v}\}^T[\overline{N}]^T\{p\}dS=0$$

$$J=\int_V\frac{1}{2}\{\mathrm{v}\}^T[B]^T[D][B]\{\mathrm{v}\}dV+\int_V[H]\{\sigma_0\}[E]\{\mathrm{v}\}dV-\int_S\{\mathrm{v}\}^T[\overline{N}]^T\{p\}dS=0$$

$$\frac{\partial J}{\partial\{\mathrm{v}\}}=\left(\int_V[B]^T[D][B]dV\right)\{\mathrm{v}\}+\left(\int_V[E]^T[H]dV\right)\{\sigma_0\}-\int_S[\overline{N}]^T\{p\}dS=0$$

$$\frac{\partial J}{\partial\{\sigma_0\}}=\left(\int_V[H]^T[E]dV\right)\{\mathrm{v}\}=0$$

$$[K]\{\mathrm{v},\sigma_0\}+\{F\}=0$$





## Stiffness matrix [K]:

$$\frac{\partial J}{\partial\{\mathrm{v}\}}=\left(\int_V[B]^T[D][B]dV\right)\{\mathrm{v}\}+\left(\int_V[E]^T[H]dV\right)\{\sigma_0\}-\int_S[\overline{N}]^T\{p\}dS=0$$

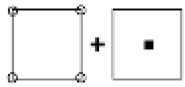
$$\frac{\partial J}{\partial\{\sigma_0\}}=\left(\int_V[H]^T[E]dV\right)\{\mathrm{v}\}=0$$

$$[K]=\int_V\begin{bmatrix}2\mu\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial x}+\mu\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y} & \mu\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial x} & \frac{\partial[N]^T}{\partial x}[H]\\ \mu\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial y} & 2\mu\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y}+\mu\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y} & \frac{\partial[N]^T}{\partial y}[H]\\ [H]^T\frac{\partial[N]}{\partial x} & [H]^T\frac{\partial[N]}{\partial y} & 0\end{bmatrix}dV.$$

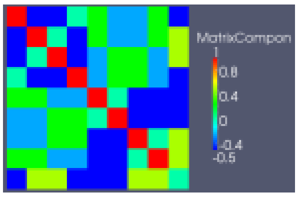
Load vector:

$$\{F\}=\int_S\begin{Bmatrix}[N]^T p_x\\ [N]^T p_y\\ 0\end{Bmatrix}dS$$

## Graphic interpretation of local stiffness matrix [K]:



$$[K]=\int_V \begin{bmatrix} 2\mu\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial x}+\mu\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y} & \mu\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial x} & \frac{\partial[N]^T}{\partial x}[H] \\ \mu\frac{\partial[N]^T}{\partial x}\frac{\partial[N]}{\partial y} & 2\mu\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y}+\mu\frac{\partial[N]^T}{\partial y}\frac{\partial[N]}{\partial y} & \frac{\partial[N]^T}{\partial y}[H] \\ [H]^T\frac{\partial[N]}{\partial x} & [H]^T\frac{\partial[N]}{\partial y} & 0 \end{bmatrix} dV.$$

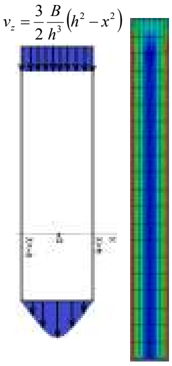


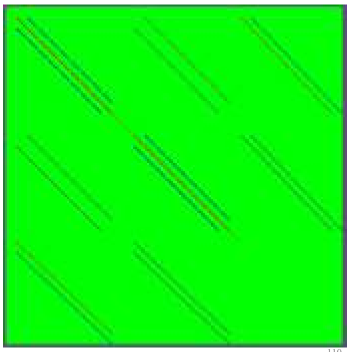





## Graphic interpretation of global stiffness matrix [K]:

$$v_z=\frac{3}{2}\frac{B}{h^3}\left(h^2-x^2\right)$$

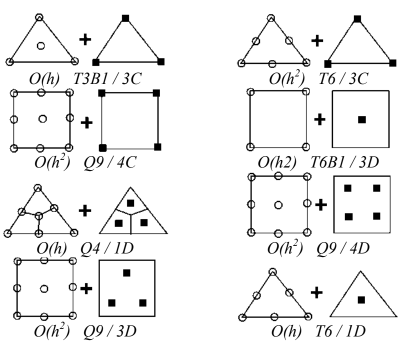








## Stabile interpolations $N$ (❍) and $H$ (■)



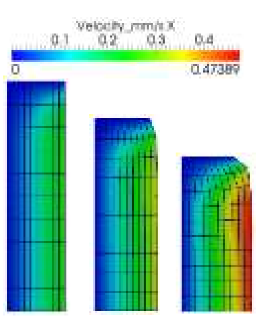





## Example of FORTRAN code

```
        DO P=1, ELSlv%N_p
                DO N=1,NBN
                        Row1 = N;
                        Row2 = NBN + N;
                        Row3 = 2*NBN + N;
                        DO I=1,NBN
                                C1 = I;
                                C2 = NBN + I;
                                C3 = 2*NBN + I;
feSM(Row1,C1)=feSM(Row1,C1) + m*(2*Ndx(N)*Ndx(i)+Ndy(N)*Ndy(i))*DetJ;
feSM(Row1,C2)=feSM(Row1,C2) + m*Ndx(i)*Ndy(N)*DetJ;

if (i<=NBNp) feSM(Row1,C3)=feSM(Row1,C3) + Ndx(N)*detJ*Hk(i,p);

feSM(Row2,C1)=feSM(Row2,C1) + m*Ndx(N)*Ndy(i)*DetJ;
feSM(Row2,C2)=feSM(Row2,C2) + m*(2*Ndy(N)*Ndy(i)+Ndx(N)*Ndx(i))*DetJ

if (i<=NBNp) feSM(Row2,C3) = feSM(Row2,C3) + Ndy(N)*detJ*Hk(i,p);

                if (N<=NBNp) then
feSM(Row3,C1) = feSM(Row3,C1) + Ndx(i)*detJ*Hk(n,p);
feSM(Row3,C2) = feSM(Row3,C2) + Ndy(i)*detJ*Hk(n,p);
                 end if
                        END DO
                END DO
        END DO
```










Lection 6.
Commercial FEM code Qform for simulation of hot metal forming processes.
Theoretical basics of Qform FEM program.
Structure and interface of Qform.
Simulation of forging, shape rolling and extrusion in Qform.
Implementation of advanced material models in Qform.
Lua scripts in Qform.
Implementation of flow stress and fracture models in Qform.