Hypergraphs and Hypergraph Grammars on the example of hp-adaptive Finite Element Method
HYPERGRAPHS

• A hyperedge is an edge with sequences of nodes assigned to it.
• Each hypergraph is composed of a set of nodes and a set of hyperedges with sequences of nodes assigned to them.
• For each hypergraph a sequence of its external nodes is specified.

A hyperedge and an exemplary hypergraph
HYPERGRAPHICS

• To nodes and hyperedges the sets of attributes are assigned.

An exemplary attributed hypergraph
HYPERGRAPHS

A hypergraph of type $k$ is a hypergraph with $k$ external nodes,

Exemplary hypergraphs with type 2
HYPERGRAPHS

Let \( C \) be a fixed alphabet of labels for nodes and hyperedges. Let \( A \) be a set of hypergraph attributes.

Definition 1.
An undirected attributed labelled hypergraph over \( C \) and \( A \) is a system \( G = (V, E, t, l, at) \),
where:
• \( V \) is a finite set of nodes,
• \( E \) is a finite set of hyperedges,
• \( t: E \rightarrow V^* \) is a mapping assigning sequences of target nodes to hyperedges of \( E \),
• \( l: V \cup E \rightarrow C \) is a node and hyperedge labelling function,
• \( at: V \cup E \rightarrow 2^A \) is a node and hyperedge attributing function.
HYPERGRAPH PRODUCTION

The hypergraph production:

\[ p = (L, R) \]

Exemplary hypergraph production:

Left-hand side graph and right-hand side graph should be of the same type
HYPERGRAPH GRAMMAR

The application of a production $p = (L, R)$ to a hypergraph $H$ consist of replacing a subhypergraph of $H$ isomorphic with $L$ by a hypergraph $R$ and replacing nodes of the removed subhypergraph isomorphic with external nodes of $L$ by the corresponding external nodes of $R$.

An initial graph; A hypergraph production; Final hypergraph

A production $p=(L,R)$ can be applied to a hypergraph $H$ if $H$ contains a subhypergraph isomorphic with $L$. 
HYPERGRAPH GRAMMAR FOR FINITE ELEMENT METHOD

- mesh generation process,
- mesh refinements,
- translation of the mesh into the sequence of matrices for the solver algorithm

are expressed by graph grammar productions.
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The hypergraph corresponding to one element mesh
HYPERGRAPH GRAMMAR FOR FINITE ELEMENT METHOD

- attribute $p$ - the polynomial order of approximation.
- attributes $ph$ and $pv$ - polynomial order of approximation in the horizontal and vertical direction, respectively.
- attributes $m$, $s$ and $err$ - the matrix, the solution and the vector of error for the corresponding element.

The hypergraph corresponding to one element mesh
• attribute $hp$ - the kind of $hp$-adaptation (values from $\{0, \ldots, 24\}$). The attribute value $hp=0$ means no adaptation.
• The meaning of other values of attribute $hp$ is presented
The hypergraph grammar for hp-adaptive finite element method is composed of four groups of productions:

1. productions describing the **generation of the initial mesh**

2. productions for calculating the **matrix, the solution and error** for each finite element, as well as **maximum relative error**

3. productions for **virtual hp-adaptation**

4. productions performing **hp-adaptation**
HYPERGRAPH GRAMMAR PRODUCTION FOR GENERATION OF THE INITIAL MESH
The production is applied to all elements
(the applicability predicate is m==null)
HYPERGRAPH GRAMMAR PRODUCTION
FOR CALCULATING ERROR AND SOLUTION VECTOR

The production is applied to all elements
(the applicability predicate is: s==null;err=null; m!=null)
The production is applied to all elements
(the applicability predicate is: \( \text{hp}==\text{null}; \text{err}!=\text{null}; \))

The decision about the kind of the adaptation is based on the error vector \( \text{err} \) of the element and is denoted by \( \text{hp}=f(\text{err}) \).
In order to follow the **1-irregularity rule** (a finite element can be broken only once without breaking the adjacent large elements), after performing virtual hp-adaptation for each element, several **additional refinements** on large adjacent elements may be required.

A hypergraph production for propagating the adaptation
HYPERGRAPH GRAMMAR PRODUCTION FOR PROPAGATING OF THE ADAPTATION

The function \( f(x, y) \) decides about the kind of the adaptation:

• If the value for \( hp \) attribute for bigger element equals 0 (no adaptation) the new value for \( hp \) attribute for the bigger element is the same as for the smaller element \( (f(x, y) = x) \).

• In other case \( f(x, y) \) works like logical \( OR \), for example if \( x \) denotes h-adaptation in one direction and \( y \) in other direction, then \( f(x, y) \) denotes h-adaptation in both directions.

A hypergraph production for propagating the adaptation.
The **hp-adaptation** is modeled by:

- breaking interiors of elements and assigning values for attribute $p$ for newly created interiors according the value of attribute $hp$ of the parental interior,

- breaking the edges,

- performing the minimum rule in order to calculate the polynomial order of approximation for edges (the value of attribute $p$).
HYPERGRAPH GRAMMAR PRODUCTION FOR HP-ADAPTATION – BREAKING INTERIORS

Production for performing hp-adaptation, where \( hp \) equals to 12
After breaking interiors of elements, all boundary edges adjacent to one broken element have to be broken.

In the similar way all edges adjacent to two broken elements have to be broken.
HYPERGRAPH GRAMMAR PRODUCTION FOR HP-ADAPTATION – BREAKING OF SHARED EDGES

After breaking interiors of elements, all edges adjacent to two broken elements have to be broken.

Hypergraph productions for breaking shared edges
HYPERGRAPH GRAMMAR PRODUCTION
FOR HP-ADAPTATION – PERFORMING THE MINIMUM RULE

Minumum rule:
The value for attribute p for the edge is the minimum from the p values for adjacent interiors

A hypergraph production for assigning appropriate polynomial orders of approximation for edges
A hypergraph resulting after using production generating the mesh (once) and production breaking mesh elements to all hyperedges of the hypergraph.
HYPERGRAPH PRODUCTION FOR SOLVER, based on \( h \)-adaptation,

If during \( h \)-adaptation an element was broken, then during the solver algorithm matrices of „son” elements should be merged.
BIBLIOGRAPHY

