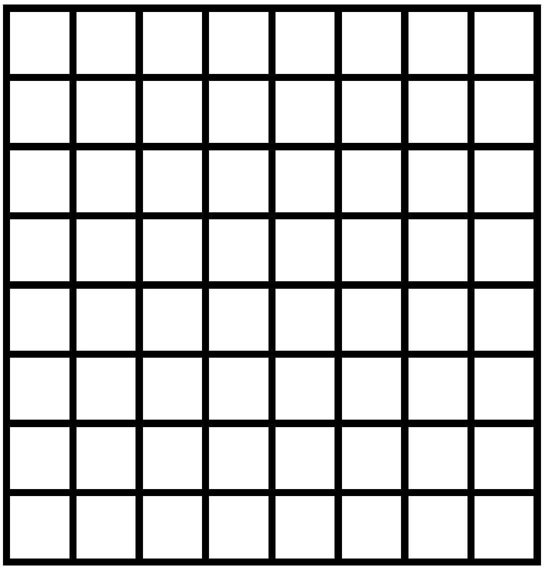
L2 Projection problem over two dimensional mesh



Find $u \in V$ such that

$$b(u,v) = l(v) \quad \forall v \in V$$

$$b(u,v) = \int_\Omega uv dx$$

$$l(v) = \int_\Omega fv dx$$

We use standard discretization

$$u \approx u_h = \sum_{i=1}^{N} a_i e_i$$

$$\sum_{i=1}^{N} a_i b(e_i, e_j) = l(e_j) \quad j = 1, ..., N$$

$$b(e_i, e_j) = \int_\Omega e_i e_j dx$$

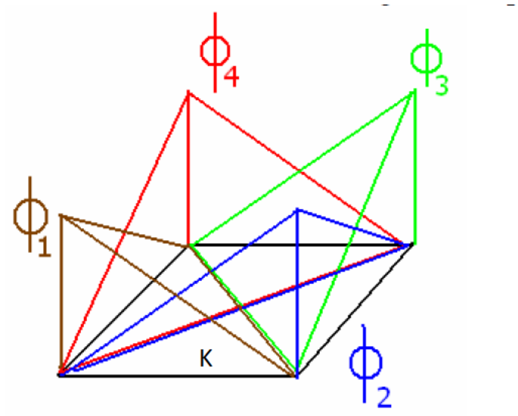
$$l(e_j) = \int_{\partial \Omega_N} f e_j dS$$

We span linear basis functions over the mesh

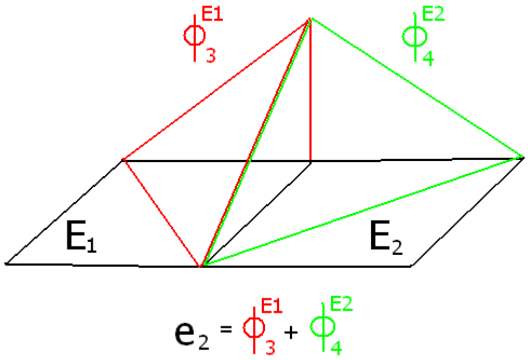
$\{e_i\}_{i=1,\ldots,N}$

$e_i|_K = \phi_l^K$

We introduce the following four shape functions over each element (here for p=1)



Each basis functions is a sum of one or two shape functions ϕ_i



(for example basis function e2 consists of the third shape function over element E1, and fourth basis function over element E2)

Over a master element [0,1]x[0,1] our shape functions are defined as follows:

$$\hat{\Phi}_1(\xi_1, \xi_2) = \hat{\chi}_1(\xi_1)\hat{\chi}_1(\xi_2) = (1 - \xi_1)(1 - \xi_2)$$
$$\hat{\Phi}_2(\xi_1, \xi_2) = \hat{\chi}_2(\xi_1)\hat{\chi}_1(\xi_2) = \xi_1(1 - \xi_2)$$
$$\hat{\Phi}_3(\xi_1, \xi_2) = \hat{\chi}_2(\xi_1)\hat{\chi}_2(\xi_2) = \xi_1\xi_2$$
$$\hat{\Phi}_4(\xi_1, \xi_2) = \hat{\chi}_1(\xi_1)\hat{\chi}_2(\xi_2) = (1 - \xi_1)\xi_2$$

We assume regular two-dimensional mesh, so each arbitrary rectangular element E can be defined by its location (b_1, b_2), length, and height (a_1, a_2).

We define the map from the master element $\hat{E}$ into an arbitrary element E

$$\hat{E} \ni (\xi_1, \xi_2) \rightarrow x_{E_k}(\xi_1, \xi_2) = (b_1 + a_1\xi_1, b_2 + a_2\xi_2) = (x_1, x_2) \in E_k$$

We also define the inverse map from an arbitrary element E into the master element $\hat{E}$

$$E_k \ni (x_1, x_2) \to x_{E_k}^{-1}(x_1, x_2) = (\frac{x_1 - b_1}{a_1}, \frac{x_2 - b_2}{a_2}) = (\xi_1, \xi_2) \in \hat{E}$$

In other words,

$$x_1 = b_1 + a_1\xi_1 \qquad\qquad x_2 = b_2 + a_2\xi_2$$

and

$$\xi_1 = \frac{x_1 - b_1}{a_1} \qquad\qquad \xi_2 = \frac{x_2 - b_2}{a_2}.$$

We can now define four vertex shape functions $\Phi_i^k, i = 1, 2, 3, 4$ over an arbitrary element. We can do that by using our template shape functions and the inverse map x_E^{-1}:

$$\Phi_1^k(x_1, x_2) = \hat{\Phi}_1(x_{E_k}^{-1}(x_1, x_2)) = \hat{\Phi}_1(\frac{x_1 - b_1}{a_1}, \frac{x_2 - b_2}{a_2})$$
$$= \hat{\chi}_1(\frac{x_1 - b_1}{a_1})\hat{\chi}_1(\frac{x_2 - b_2}{a_2}) = (1 - \frac{x_1 - b_1}{a_1})(1 - \frac{x_2 - b_2}{a_2})$$

$$\Phi_2^k(x_1, x_2) = \hat{\Phi}_2(x_{E_k}^{-1}(x_1, x_2)) = \hat{\Phi}_2(\frac{x_1 - b_1}{a_1}, \frac{x_2 - b_2}{a_2})$$
$$= \hat{\chi}_2(\frac{x_1 - b_1}{a_1})\hat{\chi}_1(\frac{x_2 - b_2}{a_2}) = \frac{x_1 - b_1}{a_1}(1 - \frac{x_2 - b_2}{a_2})$$

$$\Phi_3^k(x_1, x_2) = \hat{\Phi}_3(x_{E_k}^{-1}(x_1, x_2)) = \hat{\Phi}_3(\frac{x_1 - b_1}{a_1}, \frac{x_2 - b_2}{a_2})$$
$$= \hat{\chi}_2(\frac{x_1 - b_1}{a_1})\hat{\chi}_2(\frac{x_2 - b_2}{a_2}) = \frac{x_1 - b_1}{a_1}\frac{x_2 - b_2}{a_2}$$

$$\Phi_4^k(x_1, x_2) = \hat{\Phi}_4(x_{E_k}^{-1}(x_1, x_2)) = \hat{\Phi}_4(\frac{x_1 - b_1}{a_1}, \frac{x_2 - b_2}{a_2})$$
$$= \hat{\chi}_1(\frac{x_1 - b_1}{a_1})\hat{\chi}_2(\frac{x_2 - b_2}{a_2}) = (1 - \frac{x_1 - b_1}{a_1})\frac{x_2 - b_2}{a_2}$$

We can compute the above integrals element-wise, since integral of the sum is the sum of integrals:

$$b(\Phi_i^k, \Phi_j^k) = \int_{E_k} \Phi_i^k(x_1, x_2)\Phi_j^k(x_1, x_2)dx_1dx_2$$

Let us compute the integrals at the local system of equations over the master elements $\hat{E}$:

$$\begin{bmatrix} b\left(\hat{\Phi}_1, \hat{\Phi}_1\right) & b\left(\hat{\Phi}_1, \hat{\Phi}_2\right) & b\left(\hat{\Phi}_1, \hat{\Phi}_3\right) & b\left(\hat{\Phi}_1, \hat{\Phi}_4\right) \\ b\left(\hat{\Phi}_2, \hat{\Phi}_1\right) & b\left(\hat{\Phi}_2, \hat{\Phi}_2\right) & b\left(\hat{\Phi}_2, \hat{\Phi}_3\right) & b\left(\hat{\Phi}_2, \hat{\Phi}_4\right) \\ b\left(\hat{\Phi}_3, \hat{\Phi}_1\right) & b\left(\hat{\Phi}_3, \hat{\Phi}_2\right) & b\left(\hat{\Phi}_3, \hat{\Phi}_3\right) & b\left(\hat{\Phi}_3, \hat{\Phi}_4\right) \\ b\left(\hat{\Phi}_4, \hat{\Phi}_1\right) & b\left(\hat{\Phi}_4, \hat{\Phi}_2\right) & b\left(\hat{\Phi}_4, \hat{\Phi}_3\right) & b\left(\hat{\Phi}_4, \hat{\Phi}_4\right) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} \int_{\hat{E}} f\hat{\Phi}_1 dx_1dx_2 \\ \int_{\hat{E}} f\hat{\Phi}_2 dx_1dx_2 \\ \int_{\hat{E}} f\hat{\Phi}_3 dx_1dx_2 \\ \int_{\hat{E}} f\hat{\Phi}_4 dx_1dx_2 \end{bmatrix}$$

The shape functions are symmetric over the master element, so we can distinguish the three cases to compute

– The case where there are two identical shape functions

$$b(\hat{\Phi}_1, \hat{\Phi}_1) = b(\hat{\Phi}_2, \hat{\Phi}_2) = b(\hat{\Phi}_3, \hat{\Phi}_3) = b(\hat{\Phi}_4, \hat{\Phi}_4) =$$
$$\int_{[0,1]^2} (x_1x_2)^2 dx_1dx_2 = \int_{[0,1]} (x_1)^2 dx1 \int_{[0,1]} (x_2)^2 dx_2 = \frac{1}{3}\frac{1}{3} = \frac{1}{9}$$

– The case where there are two shape functions $\hat{\Phi}_i$ and $\hat{\Phi}_{i+1}$

$$b(\hat{\Phi}_1, \hat{\Phi}_2) = b(\hat{\Phi}_2, \hat{\Phi}_3) = b(\hat{\Phi}_3, \hat{\Phi}_4) = b(\hat{\Phi}_4, \hat{\Phi}_1) =$$
$$b(\hat{\Phi}_2, \hat{\Phi}_1) = b(\hat{\Phi}_3, \hat{\Phi}_2) = b(\hat{\Phi}_4, \hat{\Phi}_3) = b(\hat{\Phi}_1, \hat{\Phi}_4) =$$
$$\int_{[0,1]^2} x_1x_2(1-x_1)x_2dx_1dx_2 = \int_{[0,1]^2} x_1x_2^2 - x_1^2x_2^2dx_1dx_2 =$$
$$\int_{[0,1]} x_1dx1 \int_{[0,1]} (x_2)^2dx_2 - \int_{[0,1]} (x_1)^2dx1 \int_{[0,1]} (x_2)^2dx_2 =$$
$$\frac{1}{2}\frac{1}{3} - \frac{1}{3}\frac{1}{3} = -\frac{1}{18}$$

– The case where there are two shape functions $\hat{\Phi}_i$ and $\hat{\Phi}_{i+2}$

$$b(\hat{\Phi}_1, \hat{\Phi}_3) = b(\hat{\Phi}_2, \hat{\Phi}_4) = b(\hat{\Phi}_3, \hat{\Phi}_1) = b(\hat{\Phi}_4, \hat{\Phi}_2) =$$
$$\int_{[0,1]^2} x_1 x_2 (1 - x_1)(1 - x_2) dx_1 dx_2 = \int_{[0,1]^2} x_1 x_2 - x_1 x_2^2 - x_1^2 x_2 + x_1^2 x_2^2 dx_1 dx_2 =$$
$$\int_{[0,1]} x_1 dx_1 \int_{[0,1]} x_2 dx_2 - \int_{[0,1]} x_1 dx_1 \int_{[0,1]} x_2^2 dx_2 -$$
$$\int_{[0,1]} x_1^2 dx_1 \int_{[0,1]} x_2 dx_2 + \int_{[0,1]} x_1^2 dx_1 \int_{[0,1]} x_2^2 dx_2 =$$
$$\frac{1}{2}\frac{1}{2} - \frac{1}{2}\frac{1}{3} - \frac{1}{3}\frac{1}{2} + \frac{1}{3}\frac{1}{3} = \frac{1}{36}$$

Summing up, the system of linear equations over the master element $\hat{E}$ looks like this:

$$\begin{bmatrix} \frac{1}{9} & -\frac{1}{18} & \frac{1}{36} & -\frac{1}{18} \\ -\frac{1}{18} & \frac{1}{9} & -\frac{1}{18} & \frac{1}{36} \\ \frac{1}{36} & -\frac{1}{18} & \frac{1}{9} & -\frac{1}{18} \\ -\frac{1}{18} & \frac{1}{36} & -\frac{1}{18} & \frac{1}{9} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} \int_{\hat{E}} f \hat{\Phi}_1 dx_1 dx_2 \\ \int_{\hat{E}} f \hat{\Phi}_2 dx_1 dx_2 \\ \int_{\hat{E}} f \hat{\Phi}_3 dx_1 dx_2 \\ \int_{\hat{E}} f \hat{\Phi}_4 dx_1 dx_2 \end{bmatrix}$$

Let us now compute the integral over an arbitrary element with location (b_1, b_2), length and height (a_1, a_2).

$$\begin{bmatrix} b(\Phi_1, \Phi_1) & b(\Phi_1, \Phi_2) & b(\Phi_1, \Phi_3) & b(\Phi_1, \Phi_4) \\ b(\Phi_2, \Phi_1) & b(\Phi_2, \Phi_2) & b(\Phi_2, \Phi_3) & b(\Phi_2, \Phi_4) \\ b(\Phi_3, \Phi_1) & b(\Phi_3, \Phi_2) & b(\Phi_3, \Phi_3) & b(\Phi_3, \Phi_4) \\ b(\Phi_4, \Phi_1) & b(\Phi_4, \Phi_2) & b(\Phi_4, \Phi_3) & b(\Phi_4, \Phi_4) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} \int_E f \Phi_1 dx_1 dx_2 \\ \int_E f \Phi_2 dx_1 dx_2 \\ \int_E f \Phi_3 dx_1 dx_2 \\ \int_E f \Phi_4 dx_1 dx_2 \end{bmatrix}$$

Any integral over an arbitrary element E^k can be computed by changing variables into the master element $\hat{E}$ and incorporating the Jacobian.

$$b(\Phi_i^k, \Phi_j^k) = \int_{E^k} \Phi_i^k \Phi_j^k dx_1 dx_2 = \int_{\hat{E}} \hat{\Phi}_i \hat{\Phi}_j |jac\left(x_{E^k}^{-1}\right)| dx_1 dx_2$$

The Jacobian itself

$$|jac\left(x_{E^k}^{-1}\right)| = |\begin{bmatrix} \frac{1}{a_1} & 0 \\ 0 & \frac{1}{a_2} \end{bmatrix}| = \frac{1}{a_1 a_2}$$

thus

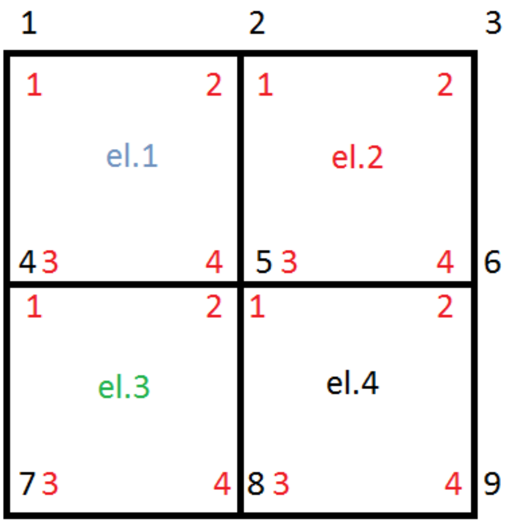
$$b(\Phi_i^k, \Phi_j^k) = \int_{\hat{E}} \frac{\hat{\Phi}_i^k \hat{\Phi}_j^k}{a_1 a_2} dx_1 dx_2$$

and the system of linear equations over an element E looks like this:

$$\frac{1}{a_1 a_2} \begin{bmatrix} \frac{1}{9} & -\frac{1}{18} & \frac{1}{36} & -\frac{1}{18} \\ -\frac{1}{18} & \frac{1}{9} & -\frac{1}{18} & \frac{1}{36} \\ \frac{1}{36} & -\frac{1}{18} & \frac{1}{9} & -\frac{1}{18} \\ -\frac{1}{18} & \frac{1}{36} & -\frac{1}{18} & \frac{1}{9} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \frac{1}{a_1 a_2} \begin{bmatrix} \int_{\hat{E}} f \hat{\Phi}_1 dx_1 dx_2 \\ \int_{\hat{E}} f \hat{\Phi}_2 dx_1 dx_2 \\ \int_{\hat{E}} f \hat{\Phi}_3 dx_1 dx_2 \\ \int_{\hat{E}} f \hat{\Phi}_4 dx_1 dx_2 \end{bmatrix} \quad (1)$$

We need to merge the element local matrices into global matrix

e.g.



$$\begin{bmatrix} \frac{1}{9} & -\frac{1}{18} & \frac{1}{36} & -\frac{1}{18} \\ -\frac{1}{18} & \frac{1}{9} & -\frac{1}{18} & \frac{1}{36} \\ \frac{1}{36} & -\frac{1}{18} & \frac{1}{9} & -\frac{1}{18} \\ -\frac{1}{18} & \frac{1}{36} & -\frac{1}{18} & \frac{1}{9} \end{bmatrix}$$

	1	2	3	4	5	6	7	8	9
1	1/9	-1/18		1/36	-1/18				
2	-1/18	1/9+1/9	-1/18	-1/18	1/36+1/36	-1/18			
3		-1/18	1/9		-1/18	1/36			
4	1/36	-1/18		1/9+1/9	-1/18 -1/18		1/36	-1/18	
5	-1/18	1/36+1/36	-1/18	-1/18 -1/18	1/9+1/9 +1/9+1/9	-1/18 -1/18	-1/18	1/36 +1/36	-1/18
6		-1/18	1/36		-1/18 -1/18	1/9+1/9		-1/18	1/36
7				1/36	-1/18		1/9	-1/18	
8				-1/18	1/36+1/36	-1/18	-1/18	1/9+1/9	-1/18
9					-1/18	1/36		-1/18	1/9

Task:

INPUT: Mesh size Ne x Ne, bitmap

1. Generate element matrices (1) over each element

2. Compute the right hand sides over each element

3. Merge element local matrices into one large global matrix

4. Factorize the matrix (Gaussian elimination is the simplest choice, although very not efficient)

5. Plot the solution over the mesh (just plot u_i at vertices of the mesh)