Frontal and multi-frontal solvers: Reutilization

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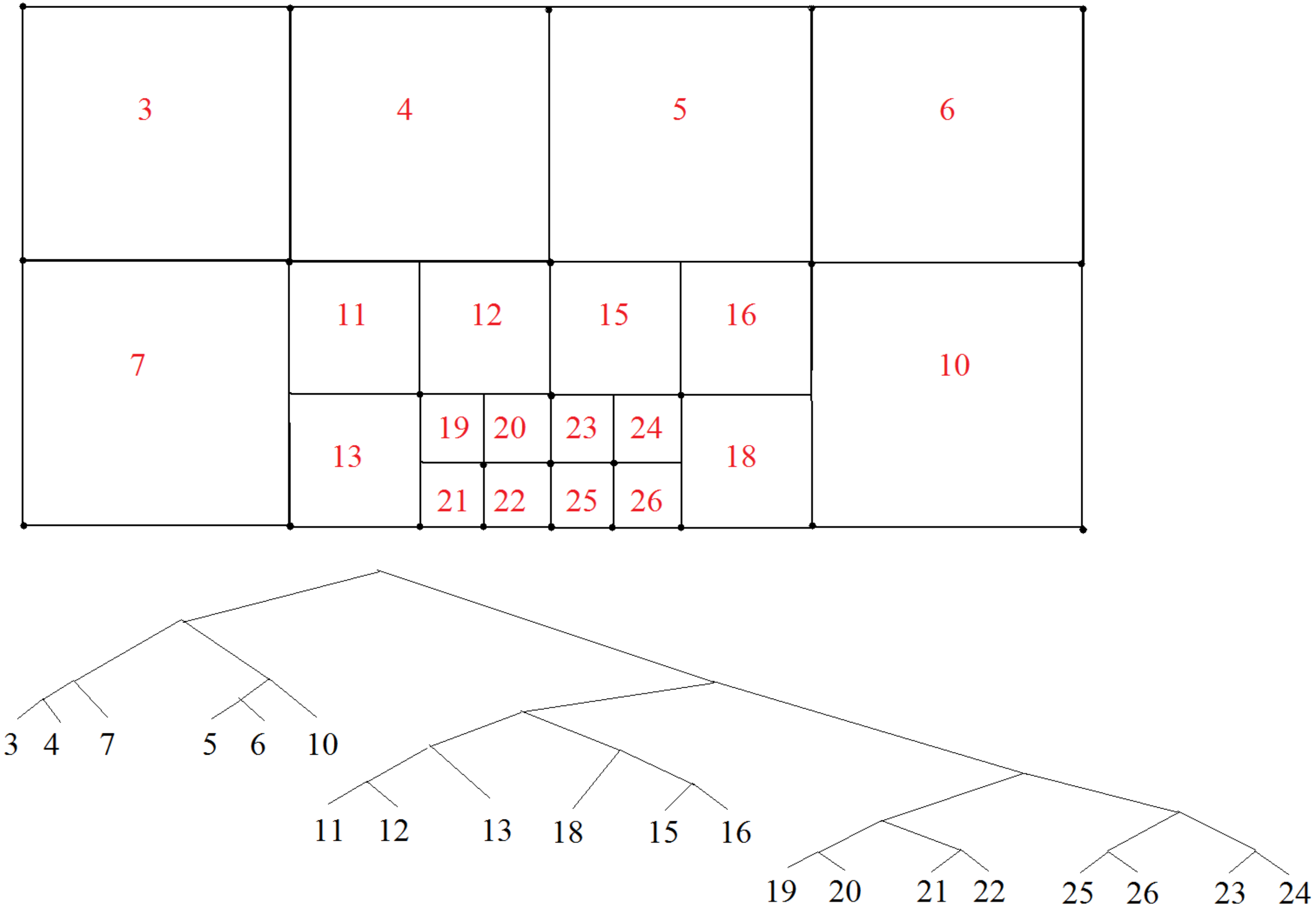
Leszek Demkowicz (ICES, UT)
David Pardo (IKERBASQUE)

OUTLINE

1. Introduction
   a) Frontal solver
   b) Multi-frontal solver
   c) Computational costs over regular grids
2. How *h* refined grids with point singularities imply linear computational cost O(N) and linear memory usage O(N) of a direct solver algorithm
3. How to find an optimal elimination tree in an automatic way
4. How to solve a sequence of computational grids refined towards point singularities with linear cost O(N) and memory O(N)

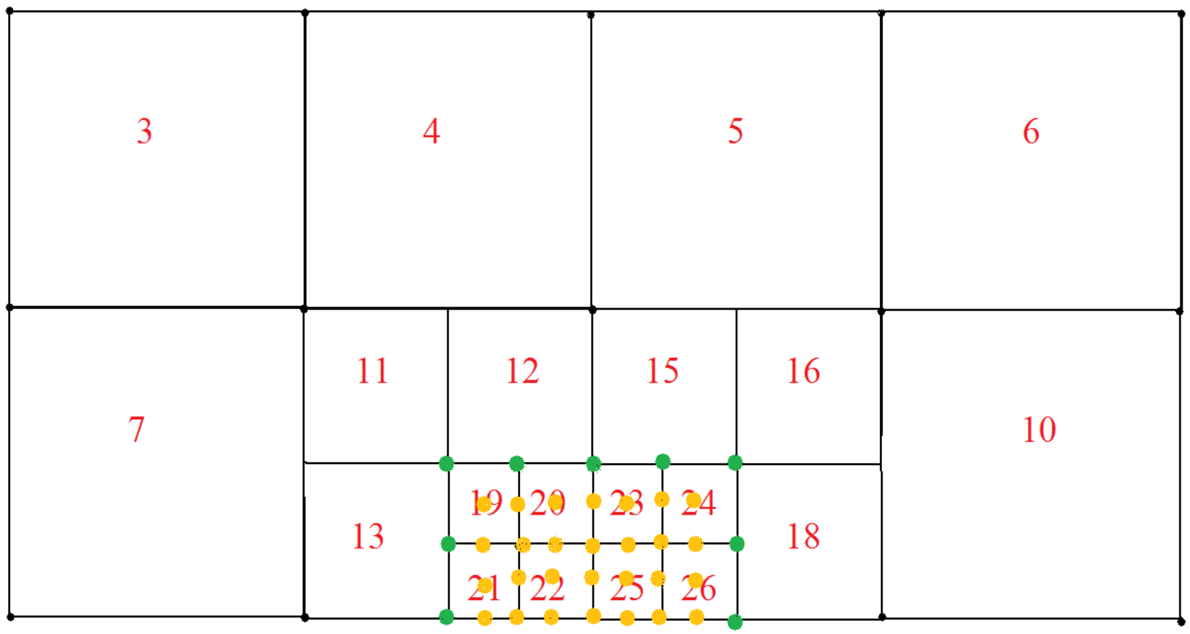
DIRECT SOLVER
WITH
LINEAR COMPUTATIONAL COST

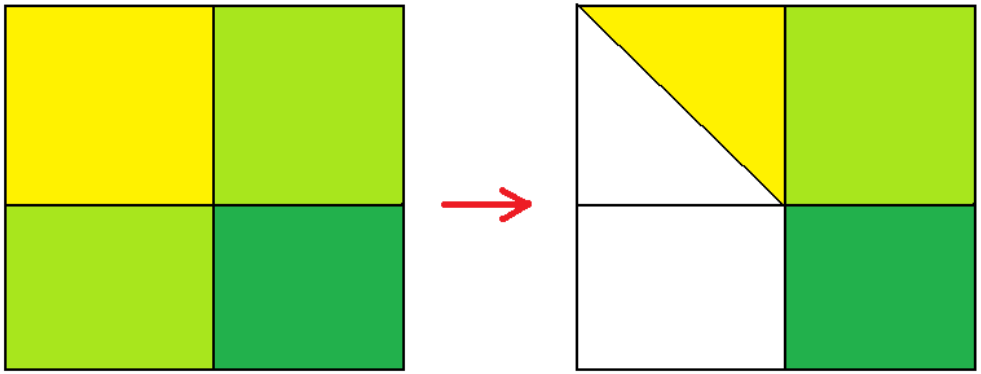
DIRECT SOLVER FOR RADICAL MESH



The elimination tree constructed based level by level ordering

DIRECT SOLVER FOR RADICAL MESH

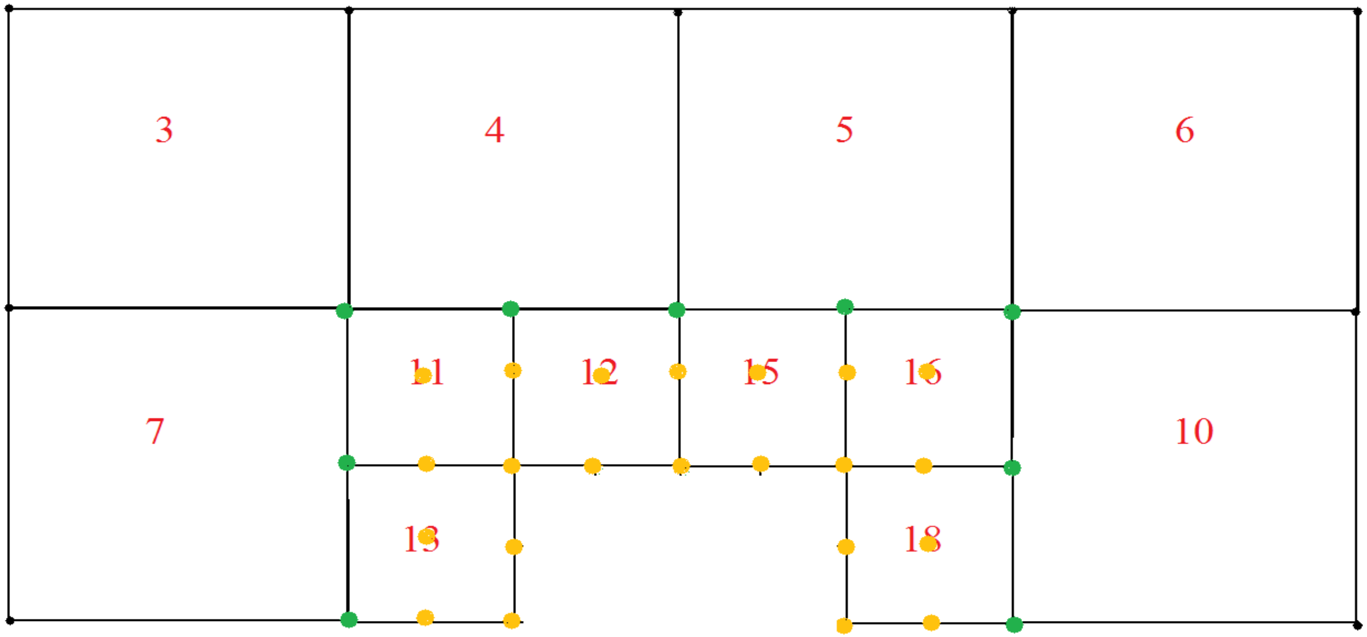


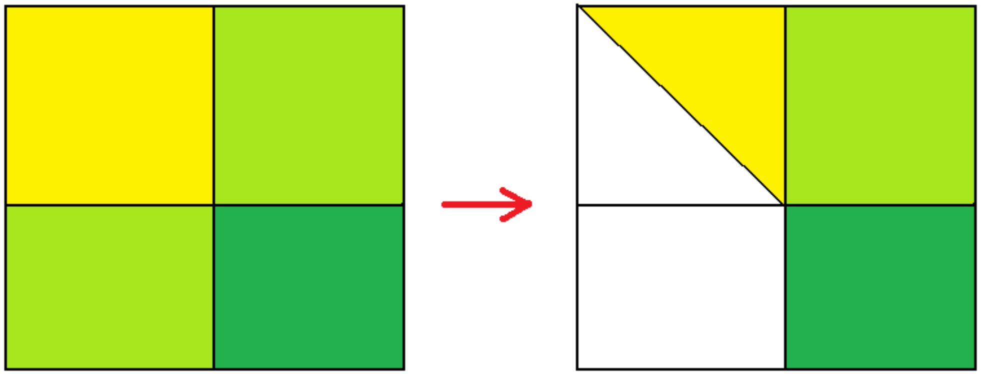


Interface size: green nodes

Constant matrix size → linear cost of the direct solver

DIRECT SOLVER FOR RADICAL MESH





Interface size: green nodes

Constant matrix size → linear cost of the direct solver

LINEAR COMPUTATIONAL COST

Lemma 2. Computational cost of the solver with respect to the number of degrees of freedom N and polynomial order of approximation is $T(p,N) = O(Np^3)$.

Proof.

$$T(p,k) = \frac{16p^6 + 96p^5 + 264p^4 + 864p^3 + 533p^2 + 93p}{6} + k\frac{12p^6 + 72p^5 + 198p^4 + 1558p^3 - 291p^2 - 157p - 223}{6}$$

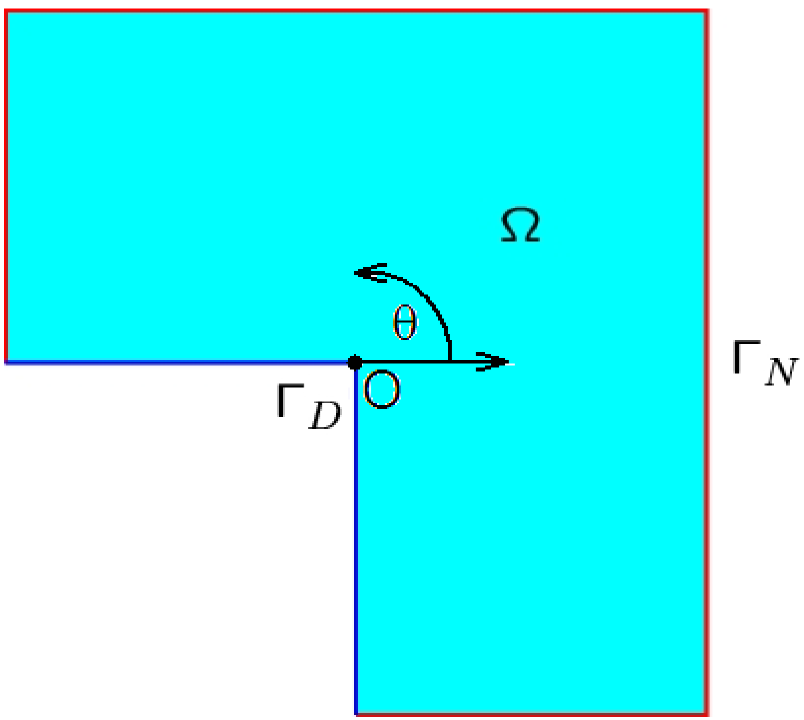
$$N = \; kp^3$$

$$T(p,N) = t_{comp}\Big(\frac{16p^6 + 96p^5 + 264p^4 + 864p^3 + 533p^2 + 93p}{6} + \frac{12Np^3 + 72Np^2 + 198Np + 1558N - 291\frac{N}{p} - 157\frac{N}{p^2} - 223\frac{N}{p^3}}{6}\Big) = O(Np^3)$$

L SHAPE DOMAIN PROBLEM

$$\begin{cases} \Delta u = 0 & in \quad \Omega \\ u = 0 & on \quad \Gamma_D \\ \dfrac{\partial u}{\partial n} = g & on \quad \Gamma_N \end{cases}$$

$$g(r,\theta) = r^{\frac{2}{3}} \sin \frac{2}{3}\left(\theta + \frac{\Pi}{2}\right)$$



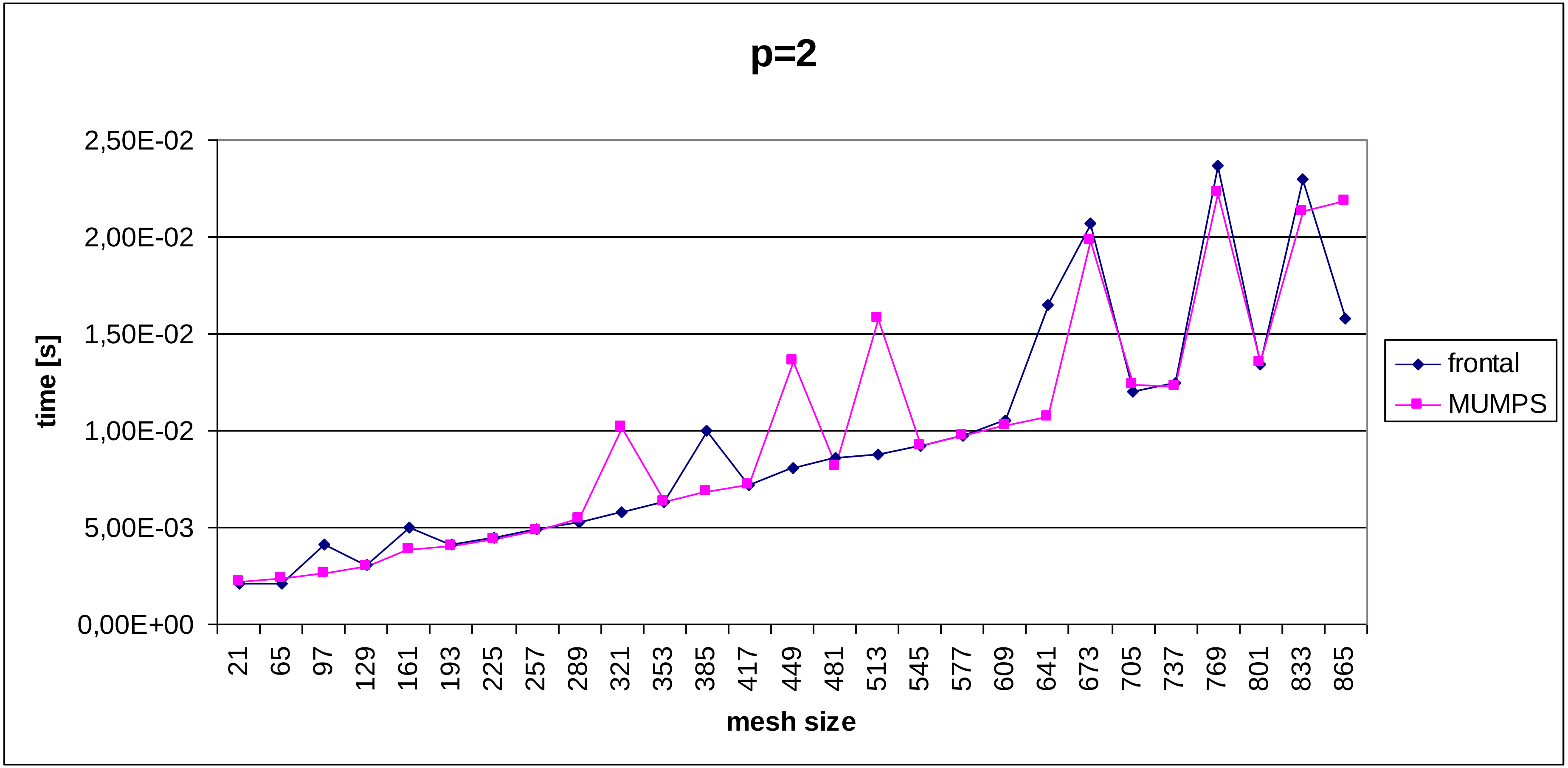
$$u \in V \subset H^1(\Omega)$$

$$b(u,v) = l(v) \quad \forall v \in V$$

$$b(u,v) = \int_\Omega \nabla u \nabla v dx$$

$$l(v) = \int_{\Gamma_C} gvdS$$

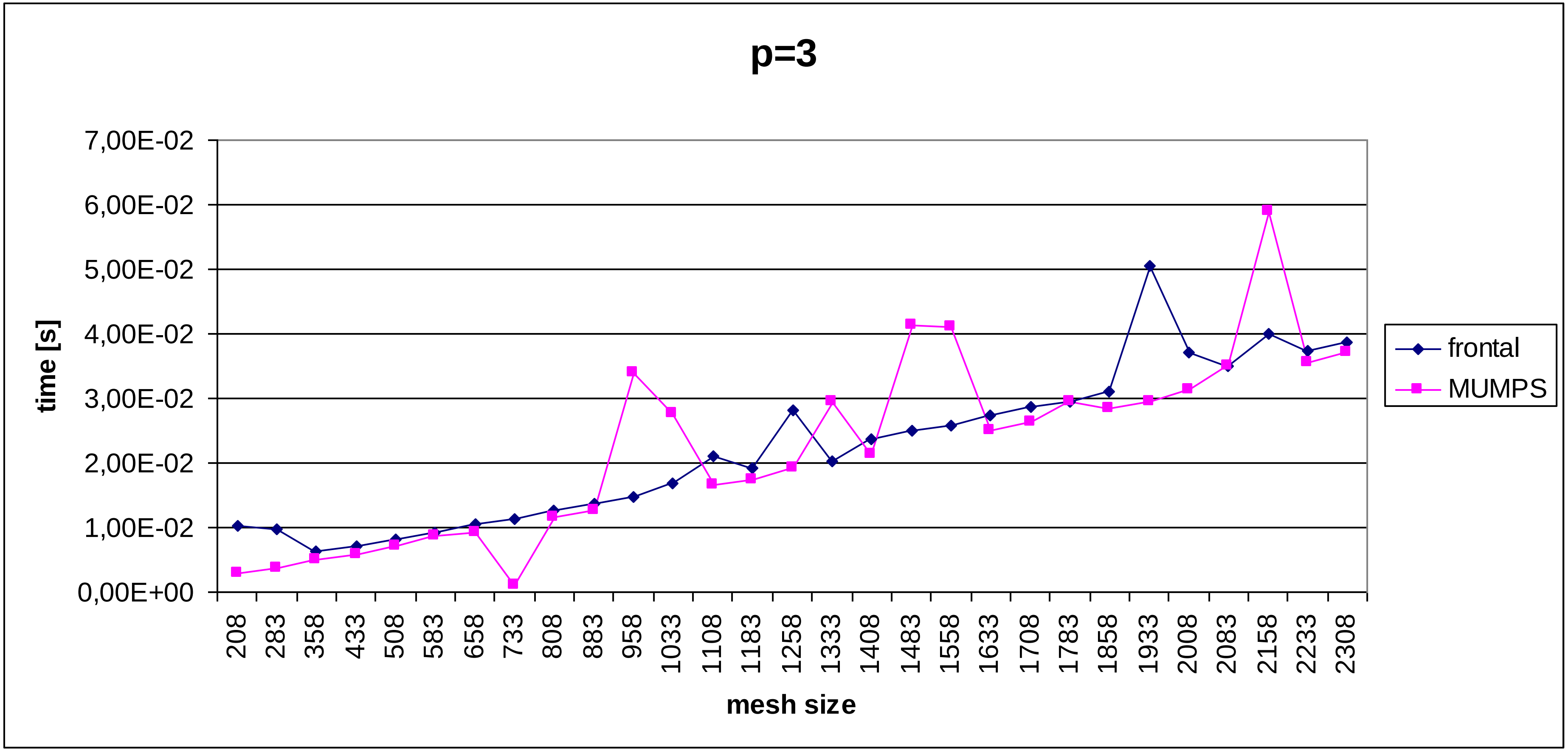
NUMERICAL RESULTS



2D Lshape problem

MUMPS vs frontal solver with ordering level by level

NUMERICAL RESULTS



2D Lshape problem

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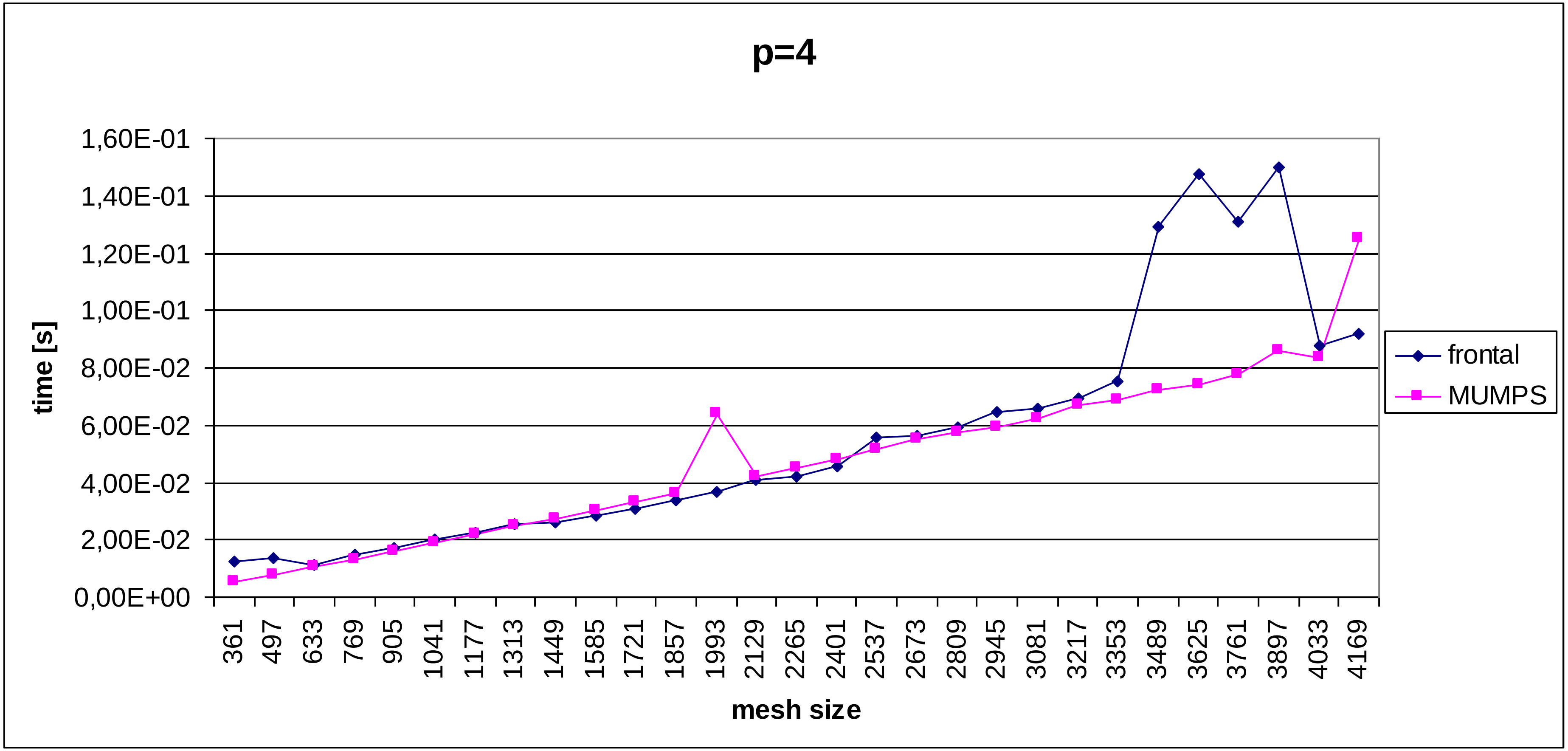
NUMERICAL RESULTS



2D Lshape problem

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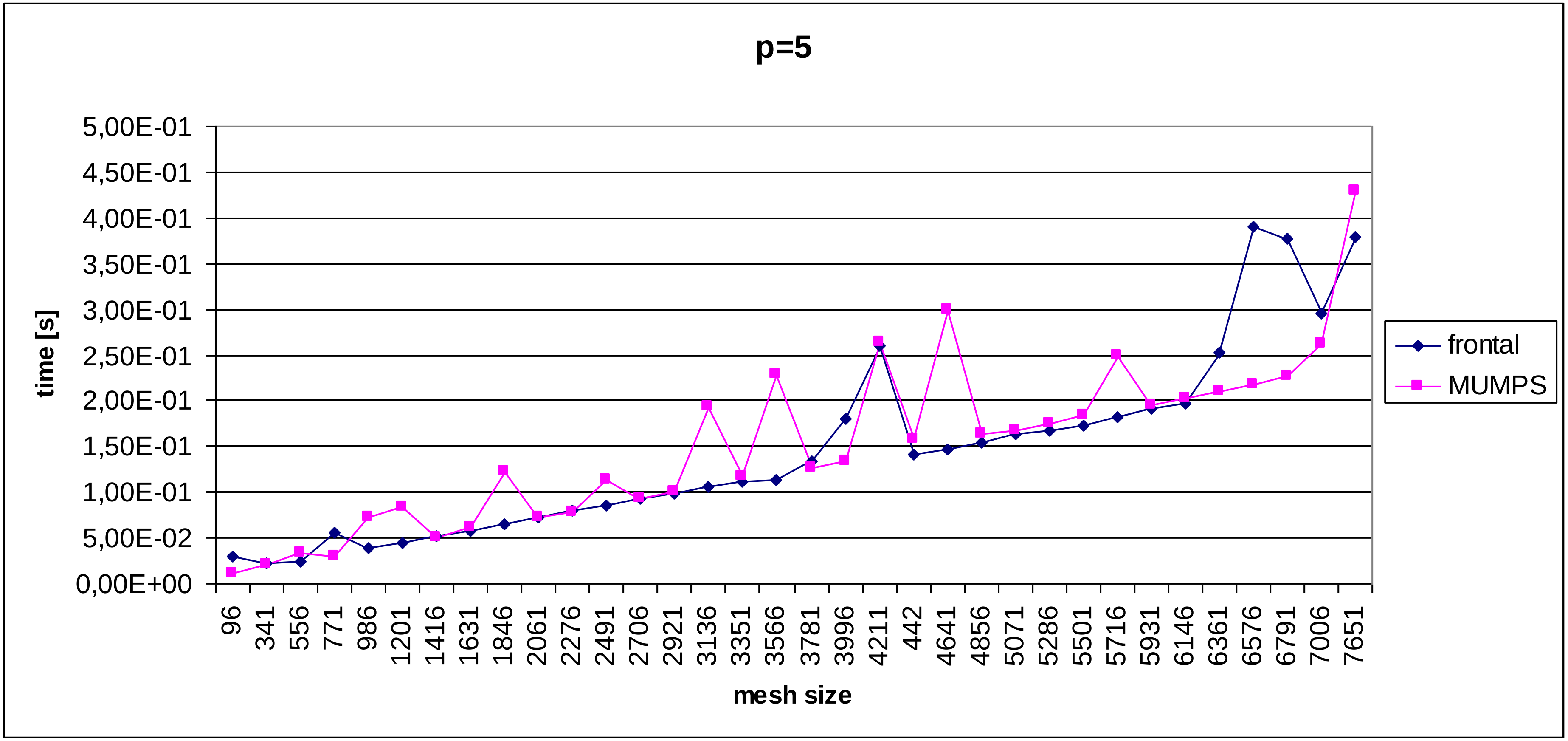
NUMERICAL RESULTS

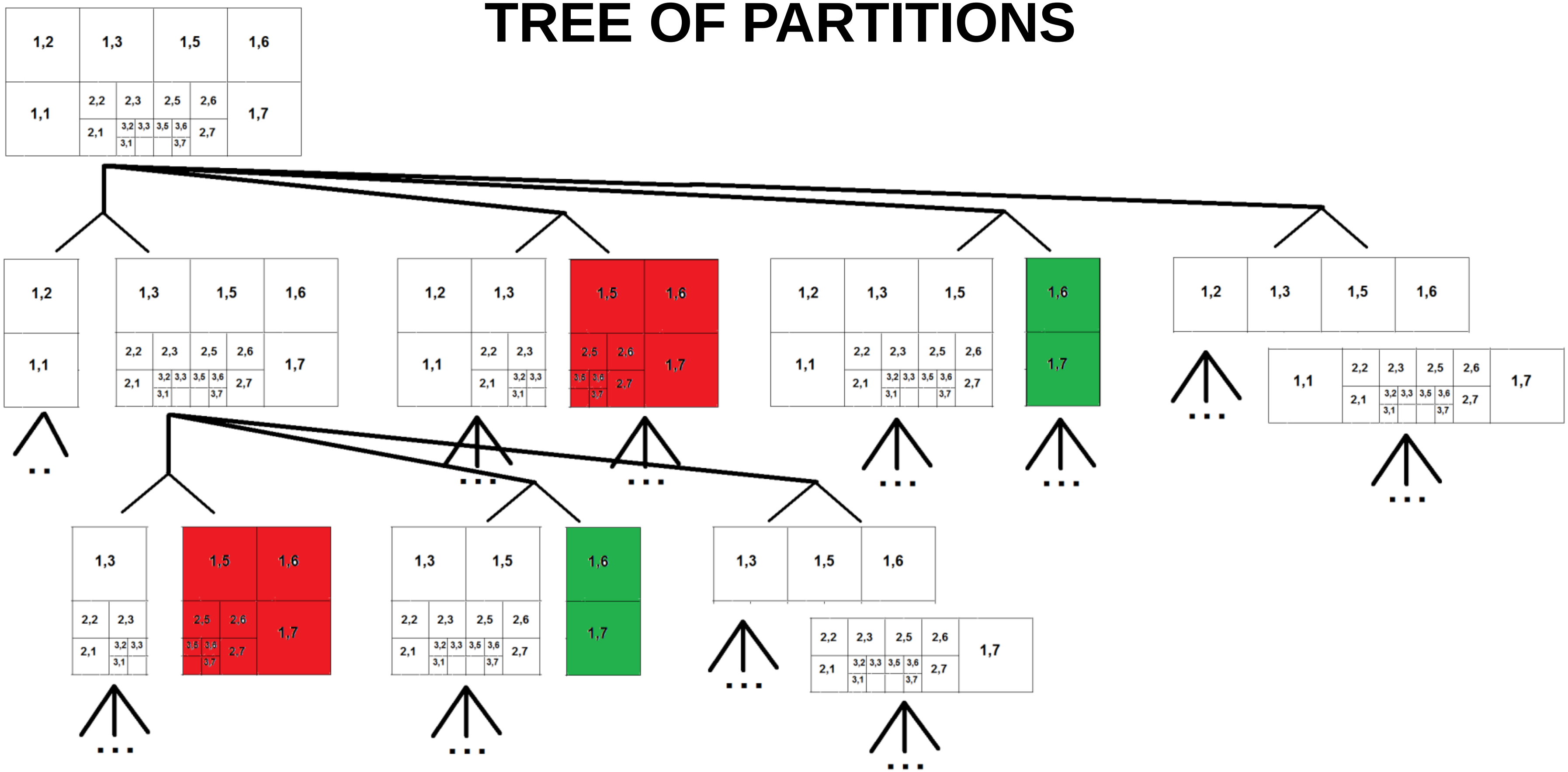


2D Lshape problem

MUMPS vs frontal solver with ordering level by level

AUTOMATIC WAY OF FINDIND OPTIMAL ELIMINATION TREES

DYNAMIC PROGRAMMING WITH TREE OF PARTITIONS

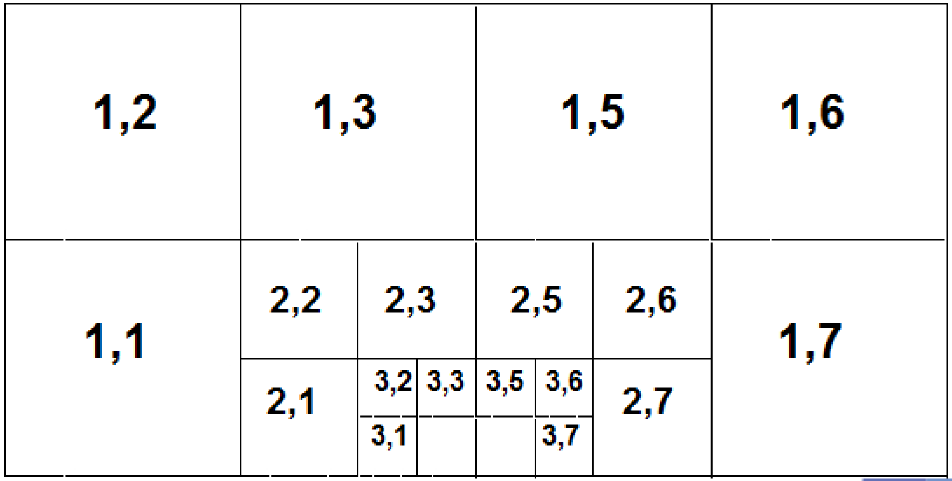


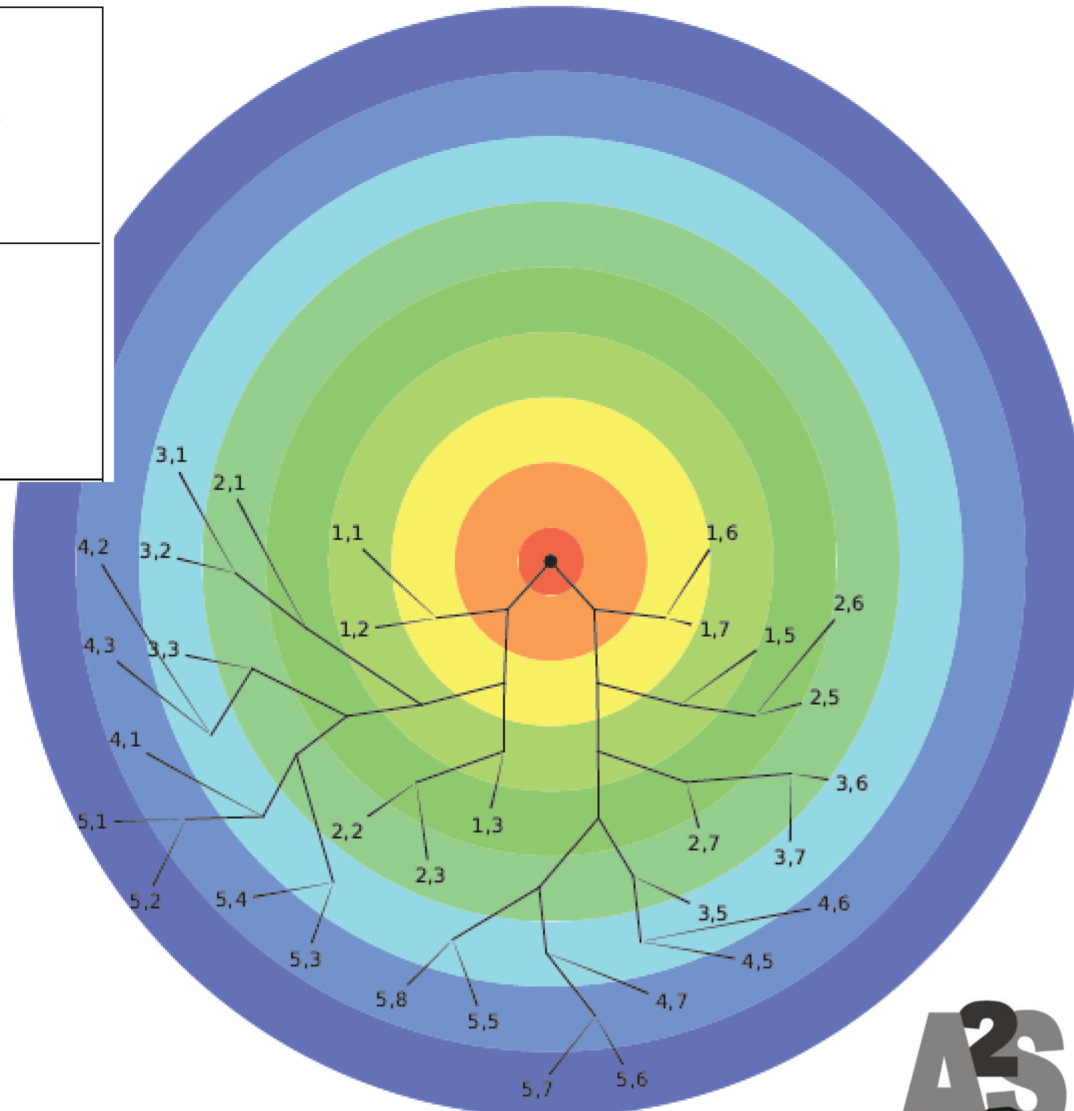
Computational cost for a leaf = cost of elimination of interior degrees of freedom

Computational cost for a node = cost for son1 + cost for son2 + cost of elimination of common interface

Optimal elimination tree = binary subtree with minimal cost

SEARCH FOR OPTIMAL ELIMINATION TREE



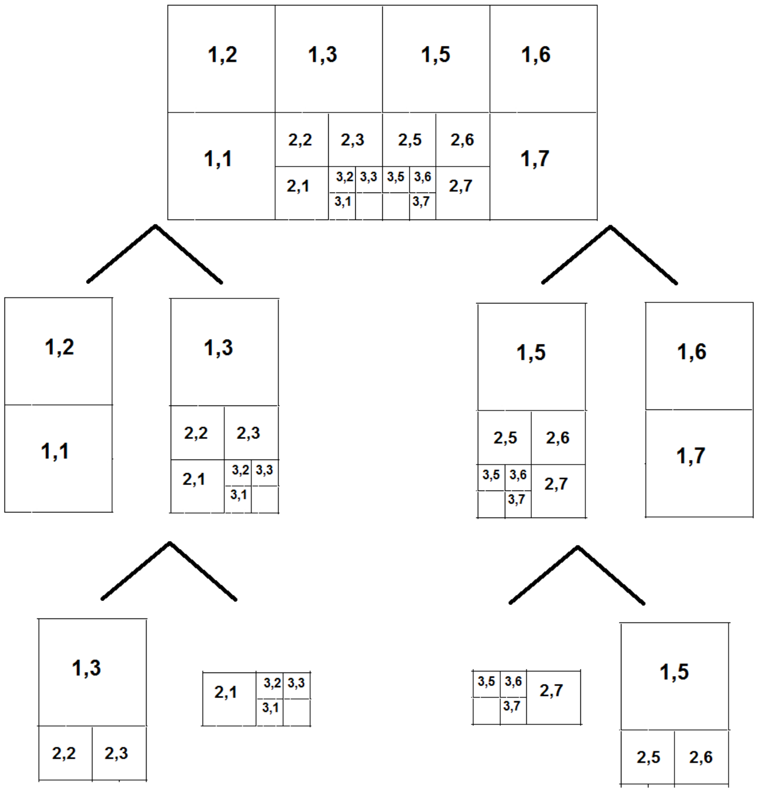


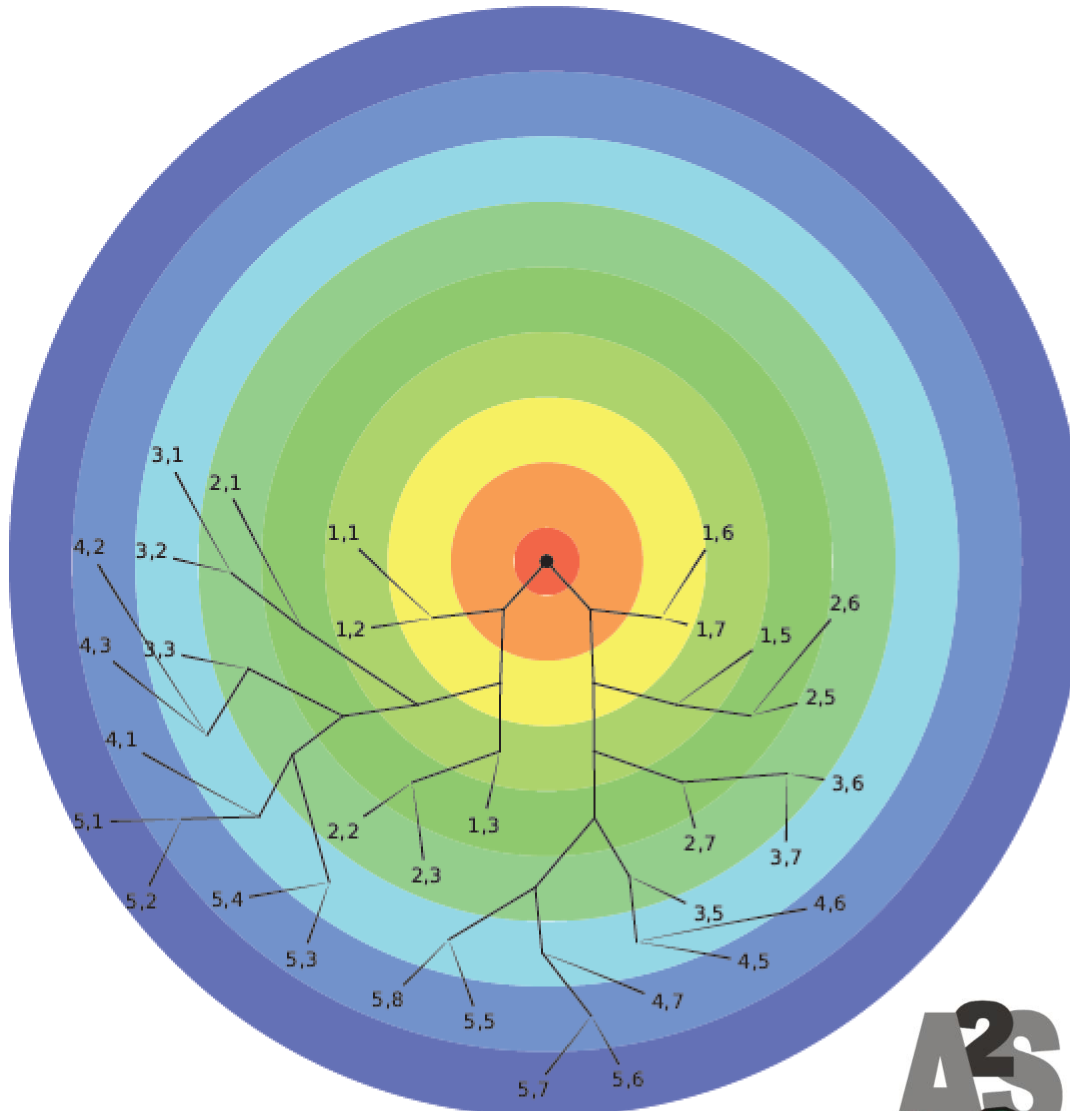
p=2, k=5

Optimal tree

Recursive bisections

SEARCH FOR OPTIMAL ELIMINATION TREE





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Optimal tree

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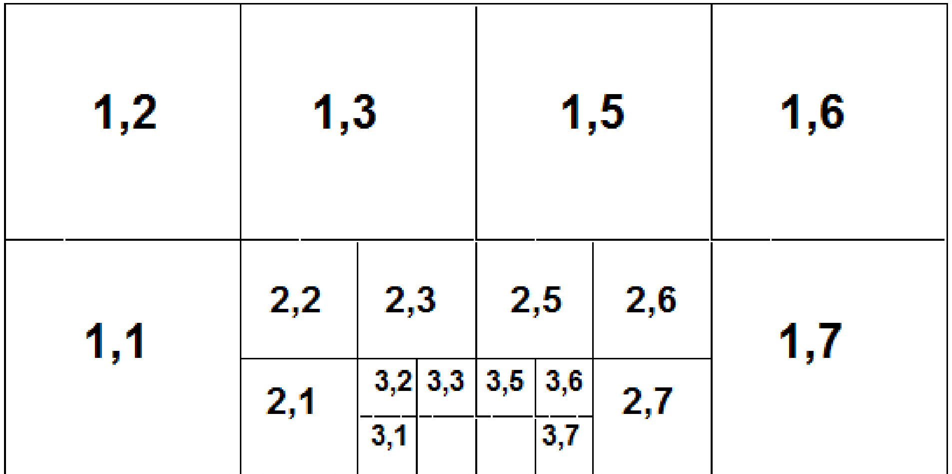
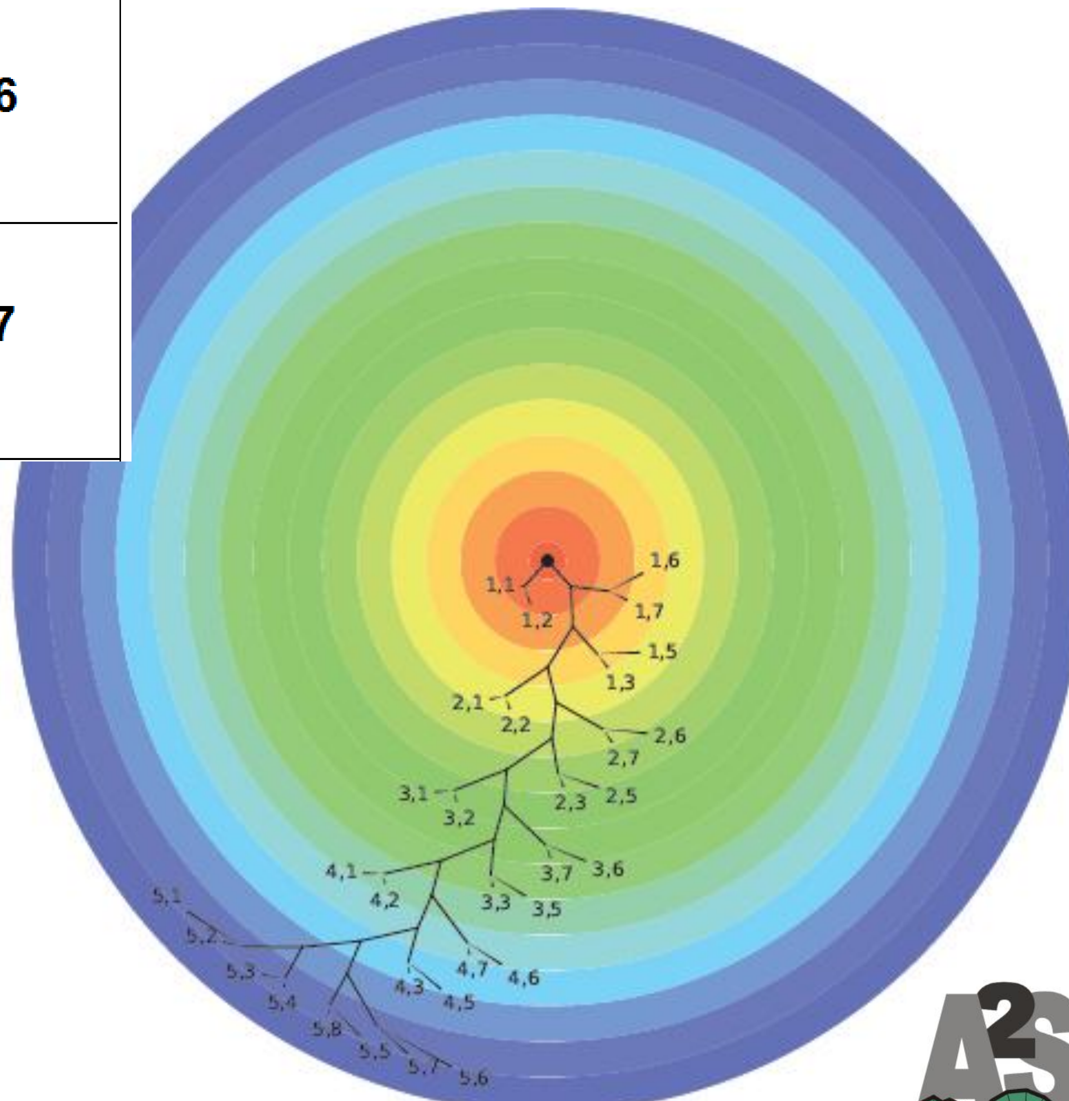
SEARCH FOR OPTIMAL ELIMINATION TREE



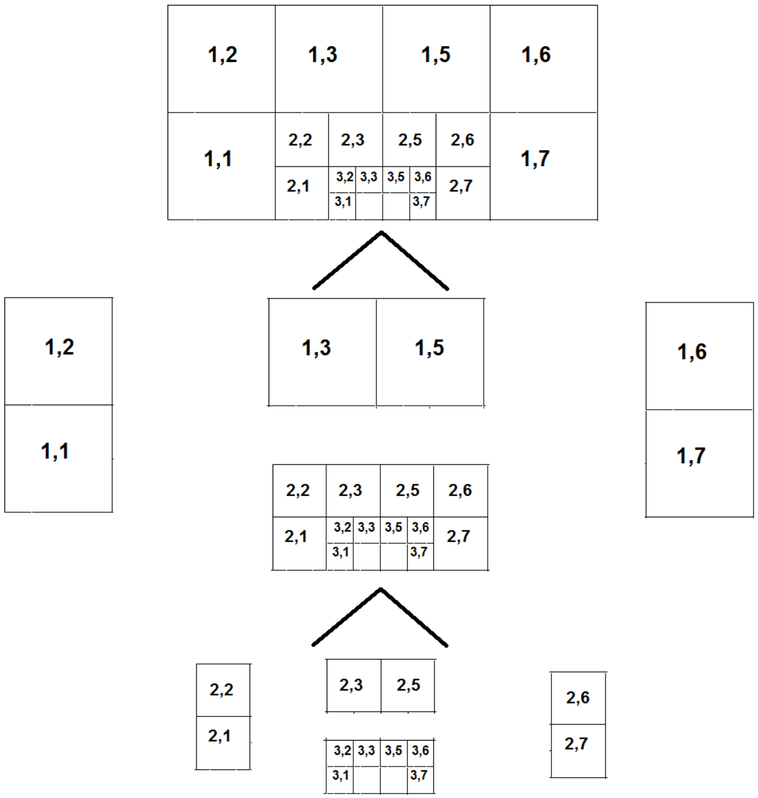


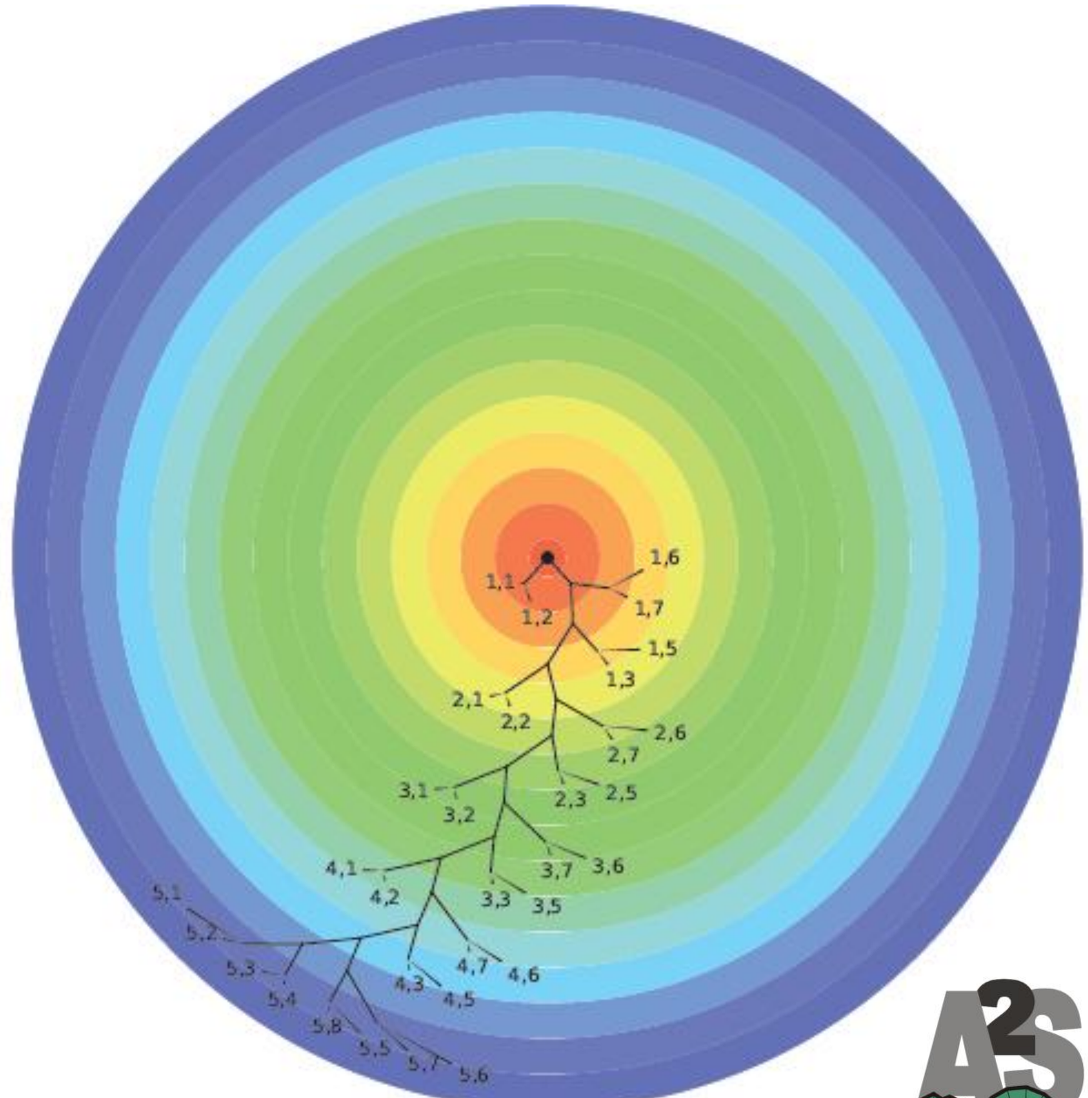
p=2, k=5

Heuristic tree

Downside up, level by level

SEARCH FOR OPTIMAL ELIMINATION TREE





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Heuristic tree

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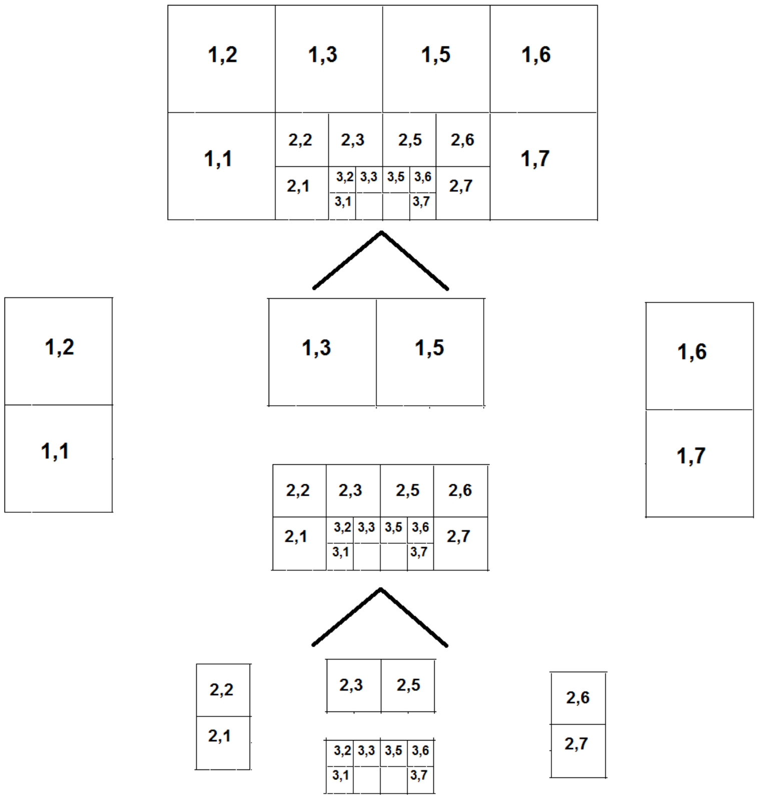
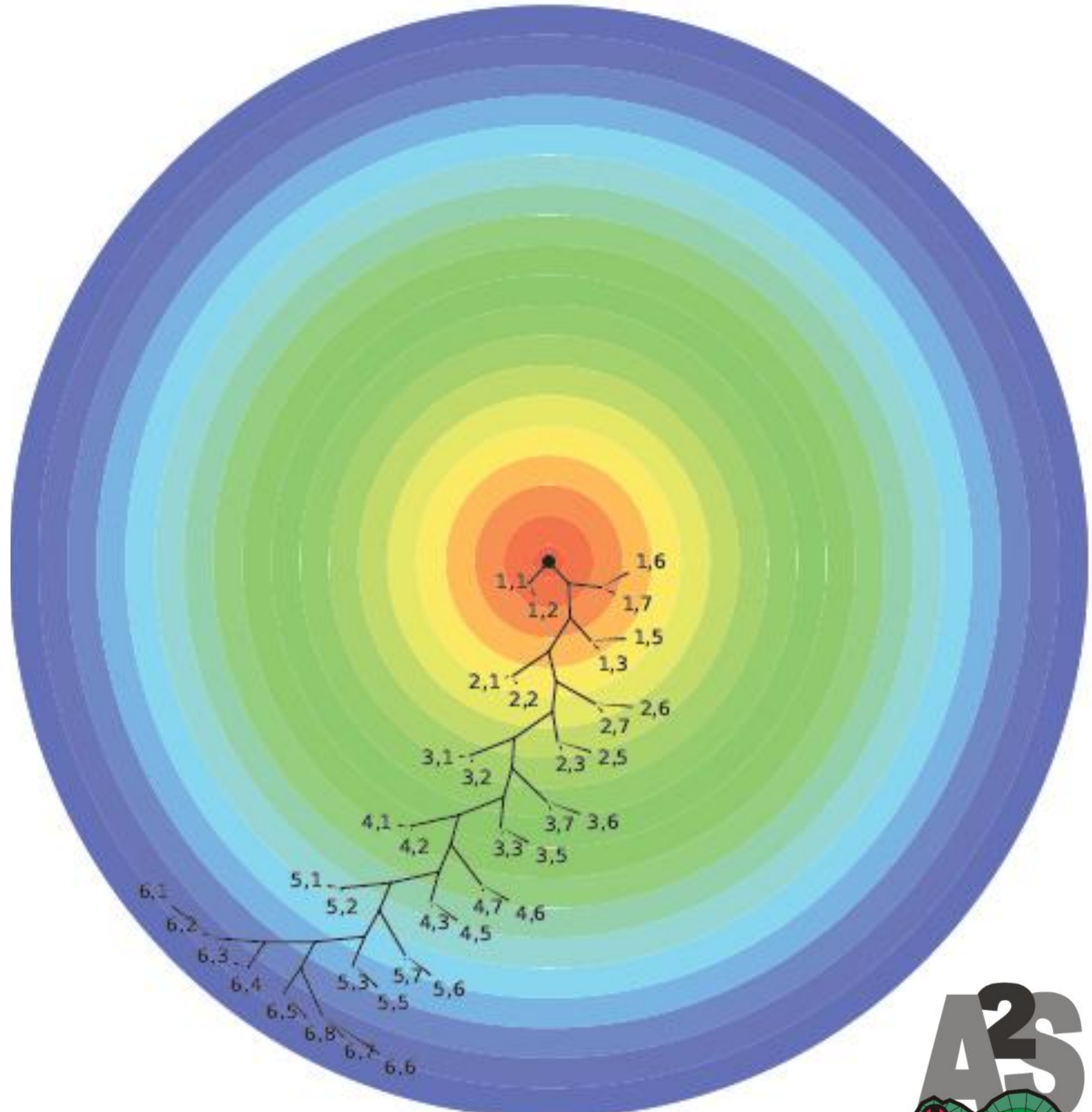
SEARCH FOR OPTIMAL ELIMINATION TREE





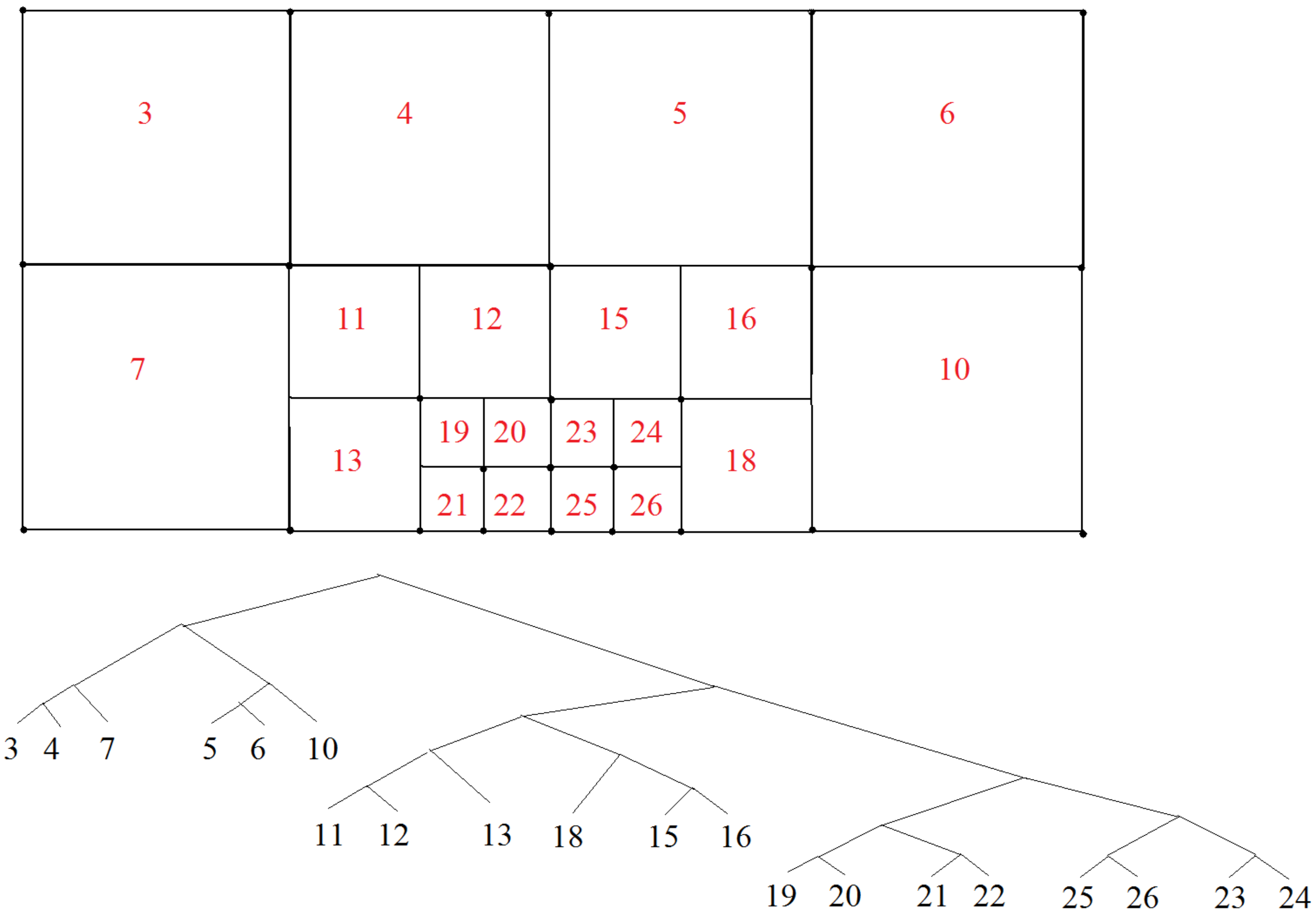
p=2, k=6

Heuristic tree equivalent to optimal tree

Downside up, level by level

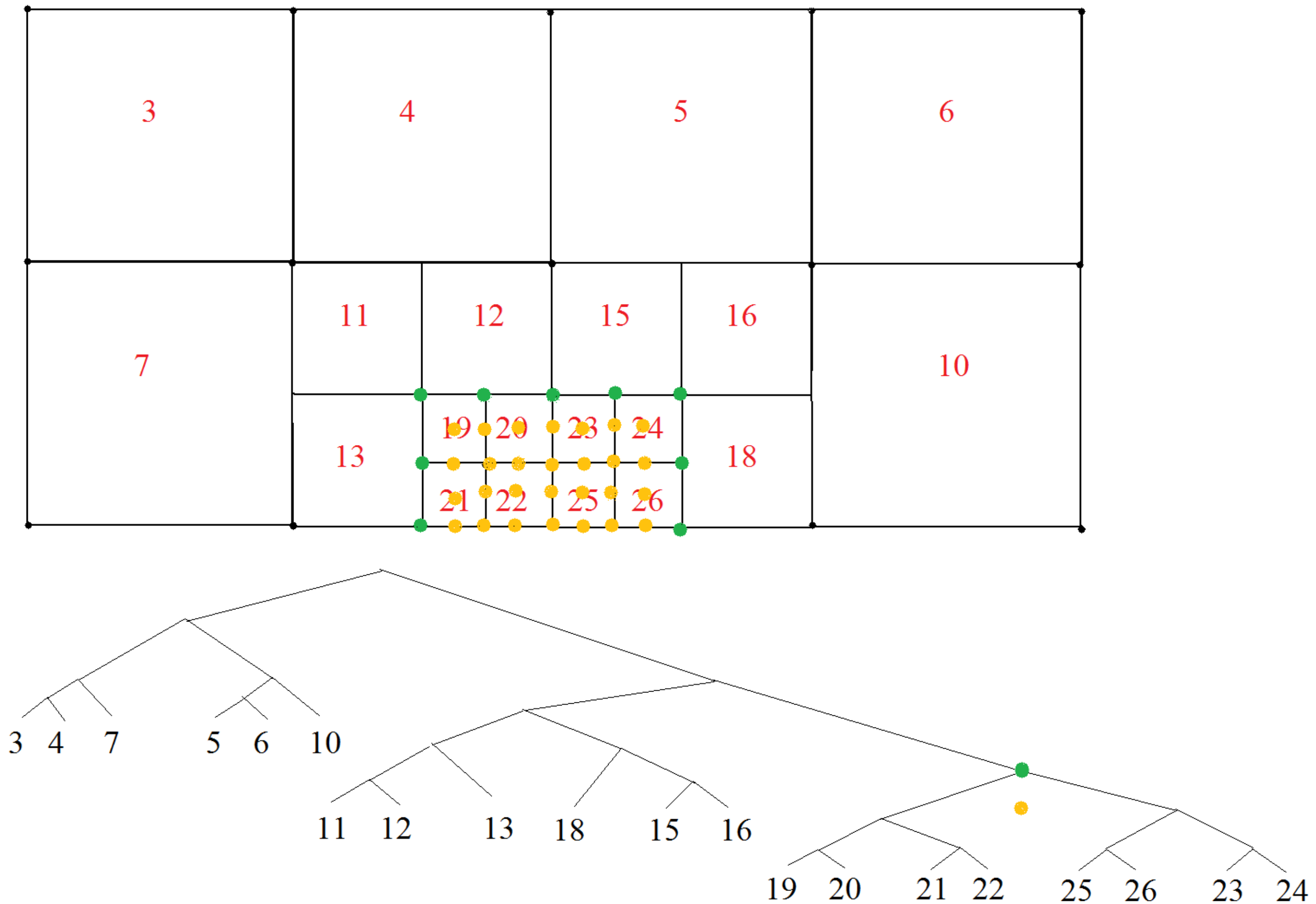
REUTILIZATION

DOWNSIDE-UP ORDERING



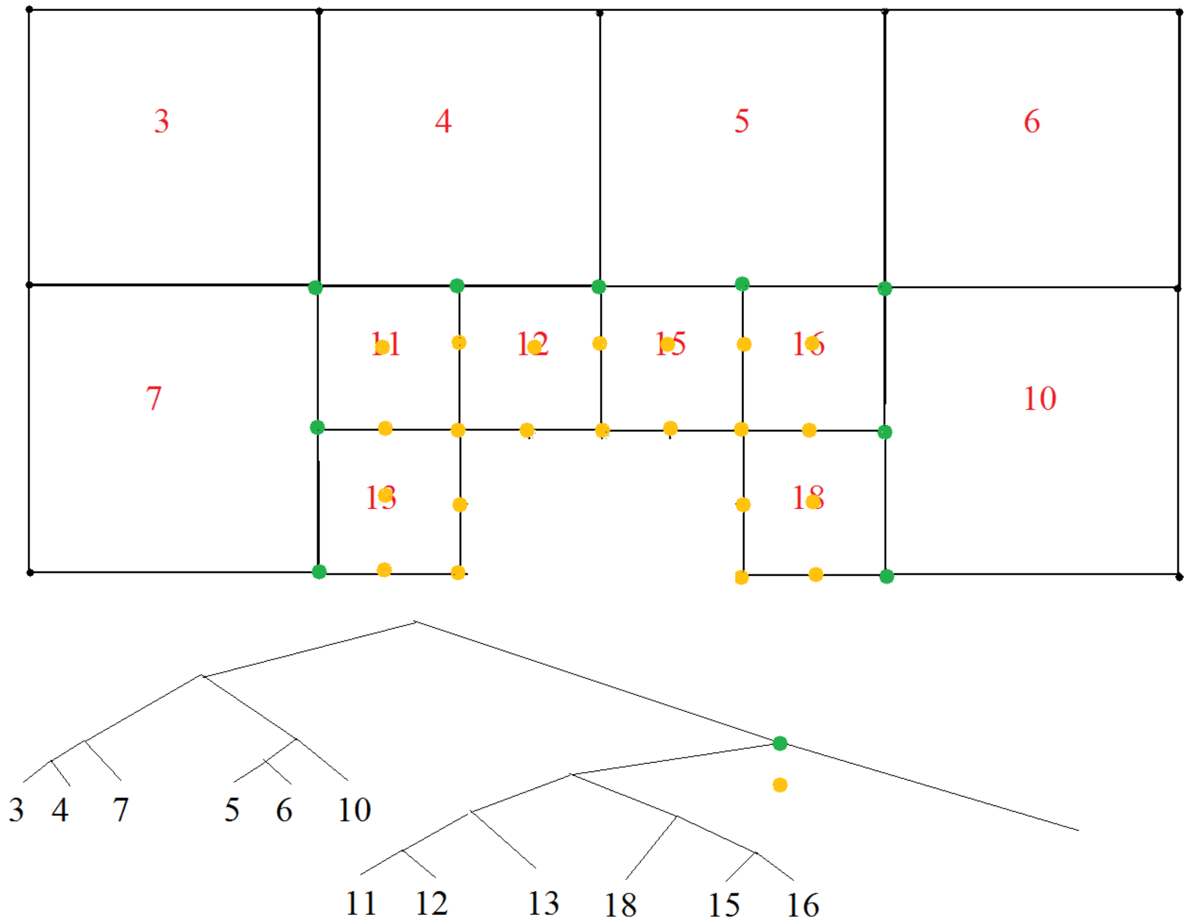
The elimination tree constructed based on the unrefinement algorithm

DOWNSIDE-UP ORDERING



Interface size: green (nodes)

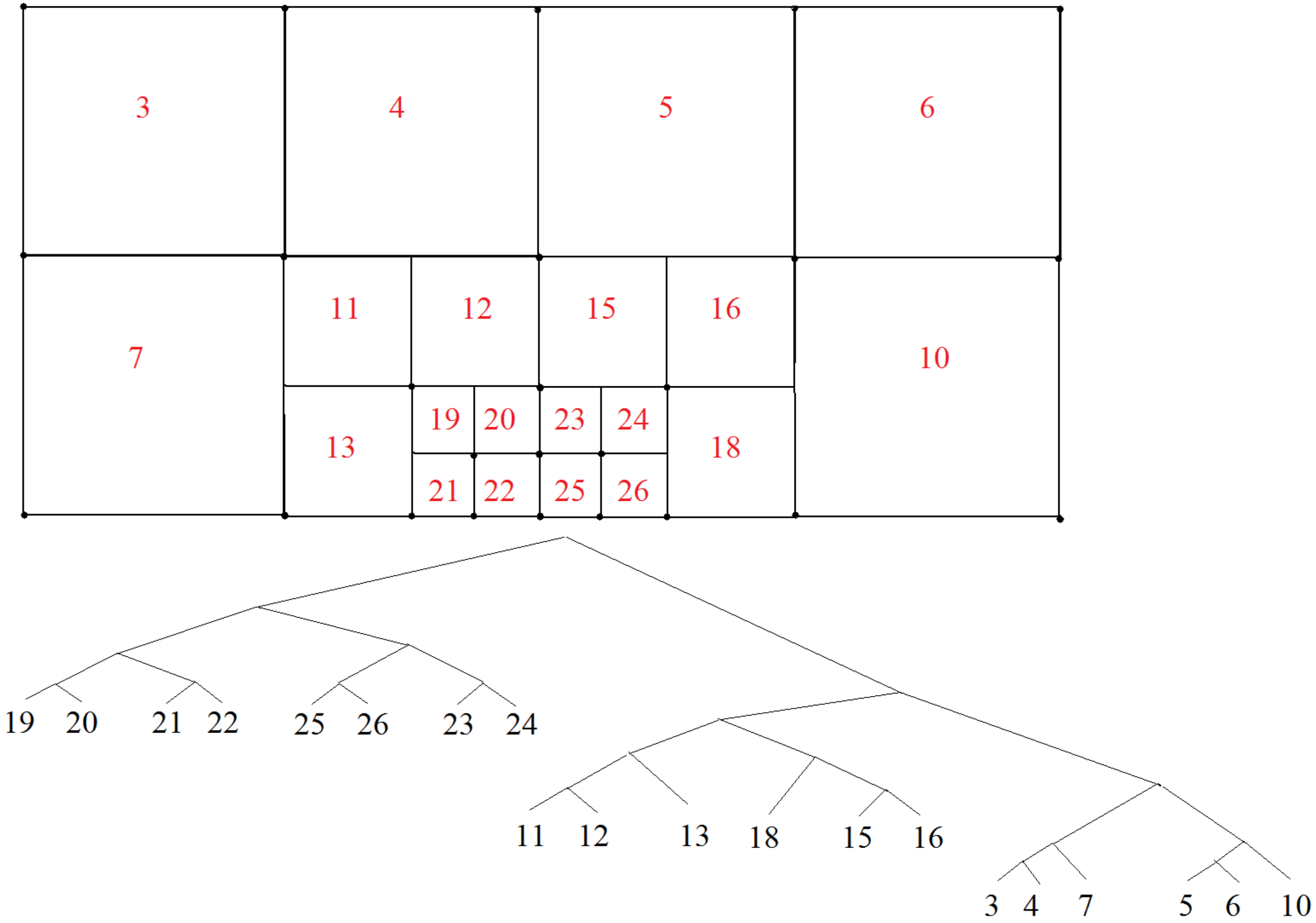
DOWNSIDE-UP ORDERING



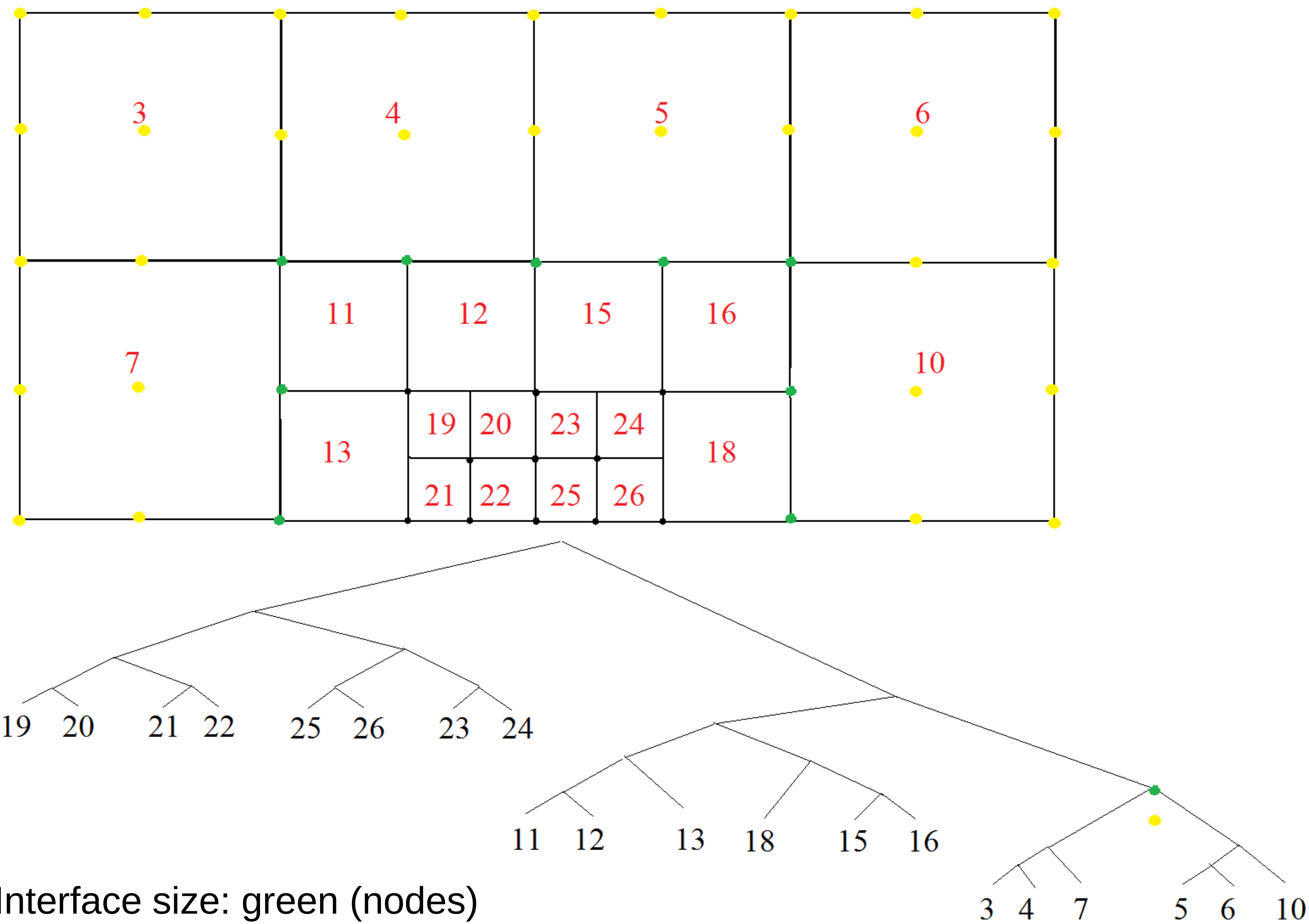
Interface size: green (nodes)

Constant !!!

UPSIDE-DOWN ORDERING

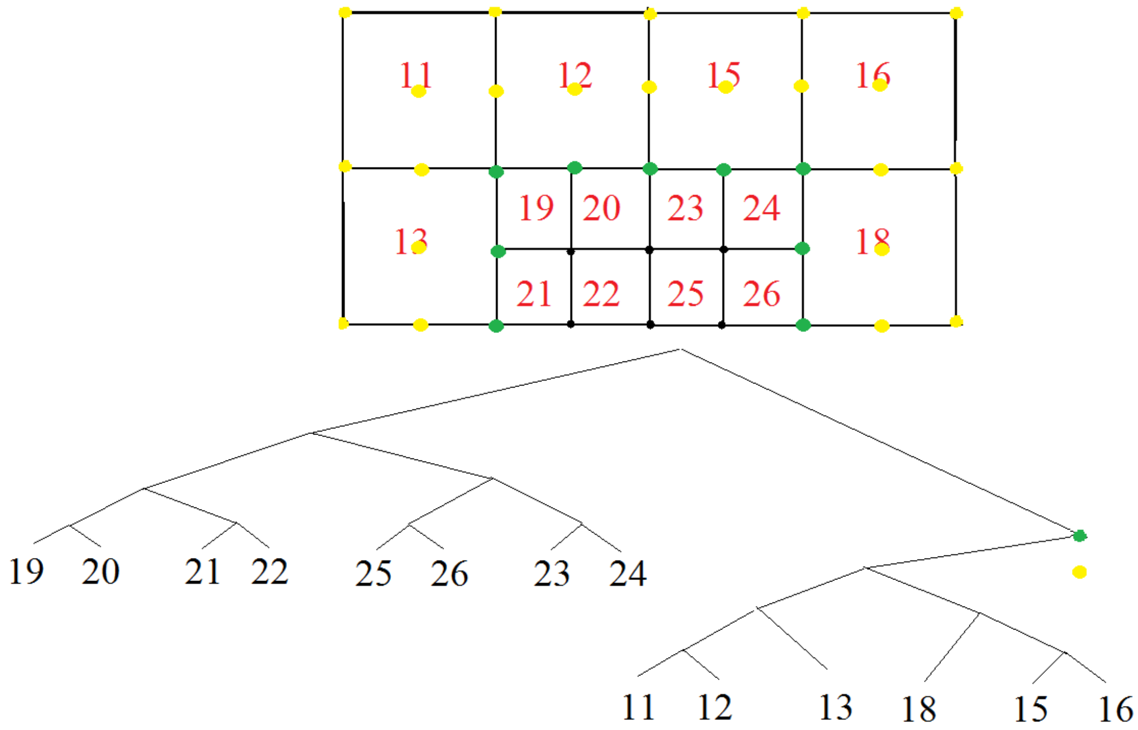


UPSIDE-DOWN ORDERING



Interface size: green (nodes)

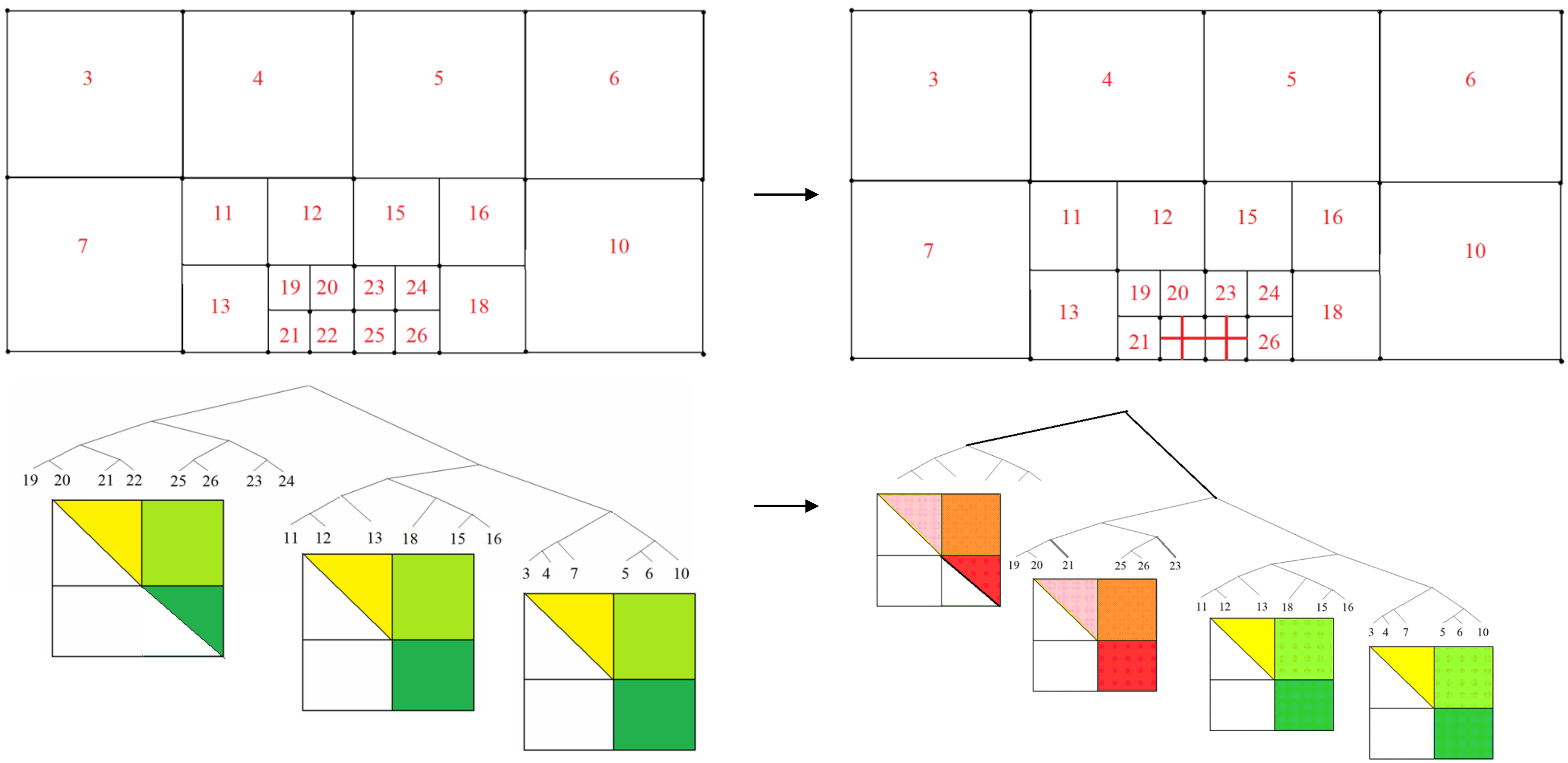
UPSIDE-DOWN ORDERING



Interface size: green (nodes)

Constant size!!!

REUTILIZATION OF PARTIAL LU FACTORIZATIONS



Interface size: green (nodes)

Constant size!!!

REDUCTION OF COMPUTATIONAL COST FOR A SQUENCE OF H REFINED GRIDS $O(N^2) \rightarrow O(N)$

Lemma 1.3

Computational cost for the solution over a sequence of h refined grids without the reutilization

$$\sum_{l=1}^{L} c(l) = \sum_{l=1}^{L} (a + n(l)d) = La + \sum_{l=1}^{L} n(l)d = La + d\sum_{l=1}^{L} (e + fl) = La + Lde + df\sum_{l=1}^{L} l = La + Lde + df\left(\frac{L(L+1)}{2}\right) =$$

$$O\left(L(a + de + df) + L^2 df\right) = O\left(L + L^2\right) = O\left(N^2\right)$$

Computational cost for the solution over a sequence of h refined grids with the reutilization

The reutilization implies $d=0$ and we get

$$\sum_{l=1}^{L} c(l) = \sum_{l=1}^{L} a = La = O(N)$$

We conclude that the reutilization reduces the computational cost from $O(N^2)$ down to $O(N)$.

NUMERICAL RESULTS – RADICAL GRID

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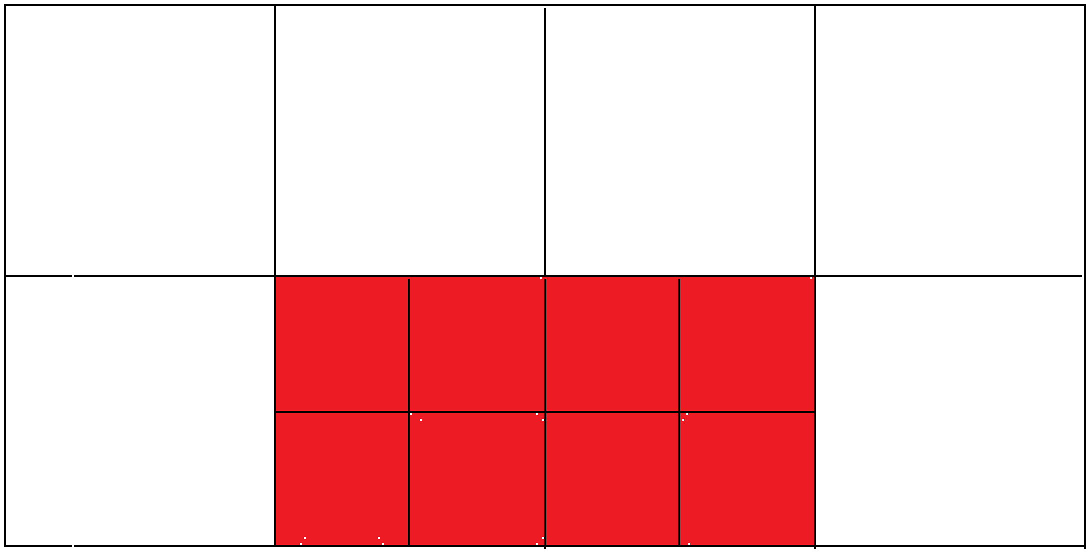
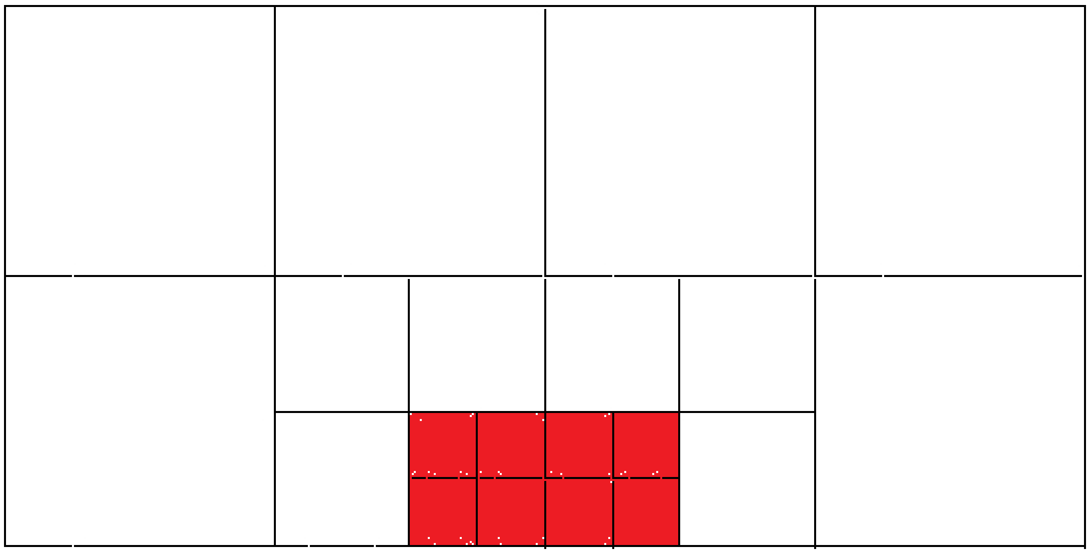
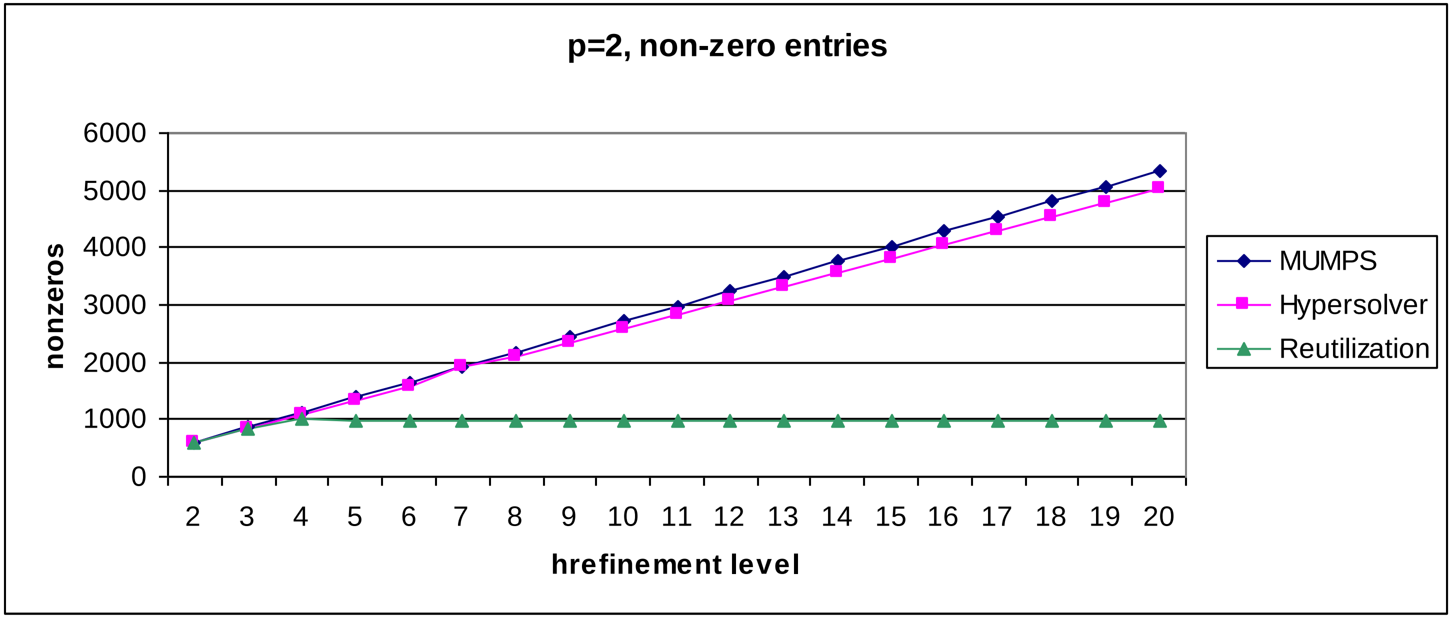
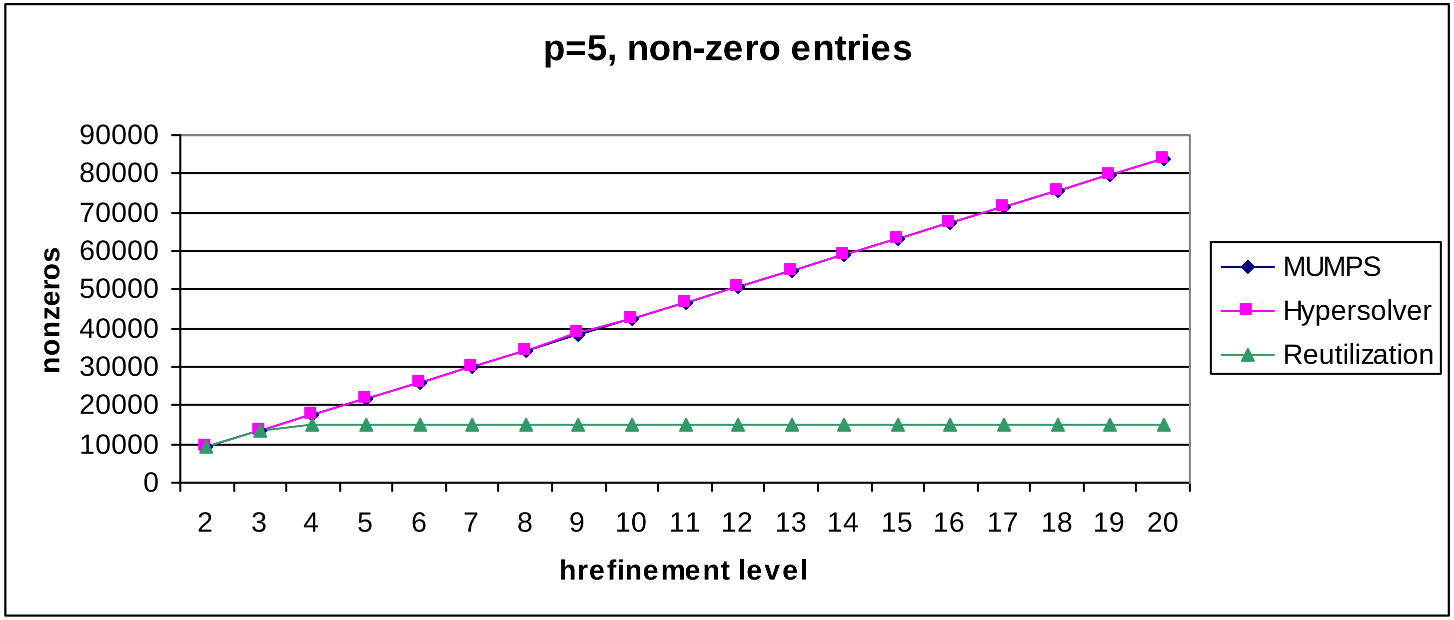


NUMERICAL RESULTS – RADICAL GRID

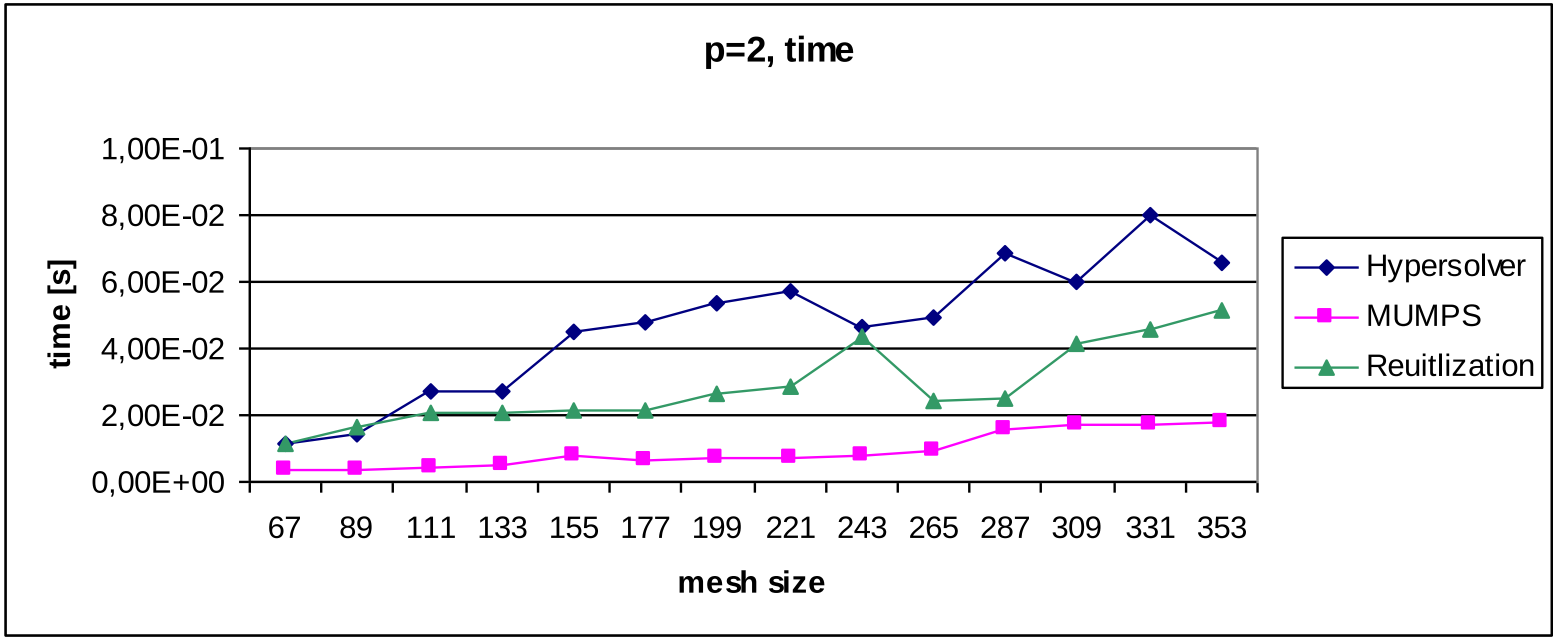


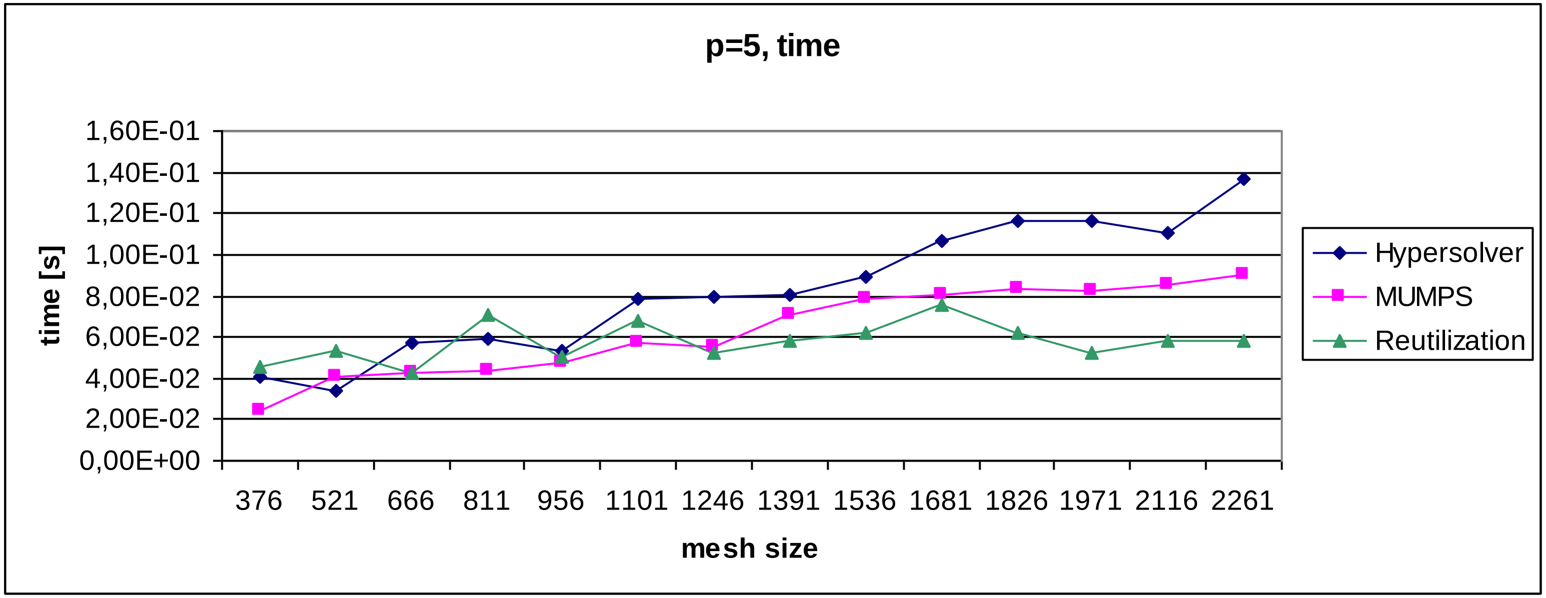
NON-ZERO ENTRIES



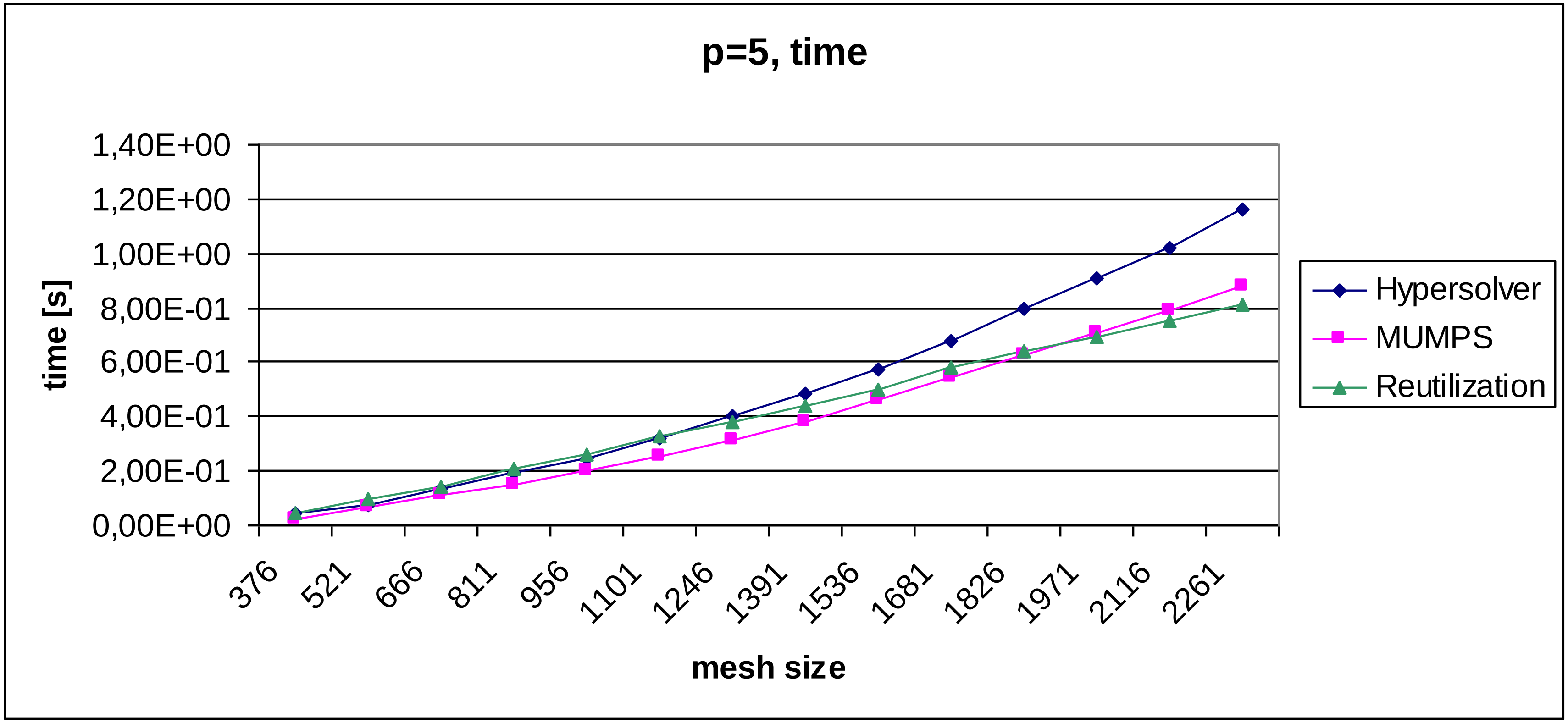


EXECUTION TIME





REUTILIZATION vs CLASSICAL APPROACH



NUMERICAL RESULTS – L SHAPE DOMAIN

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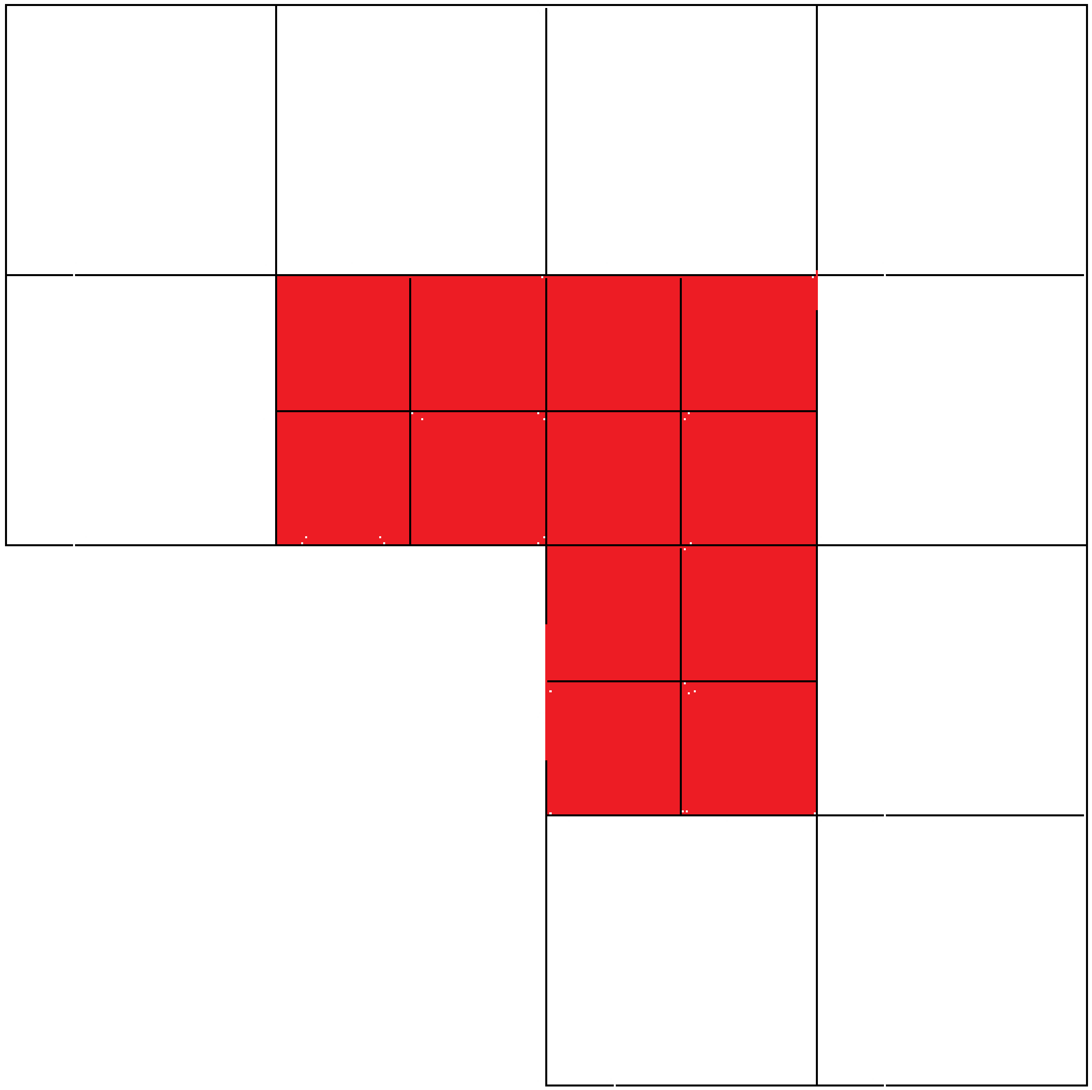
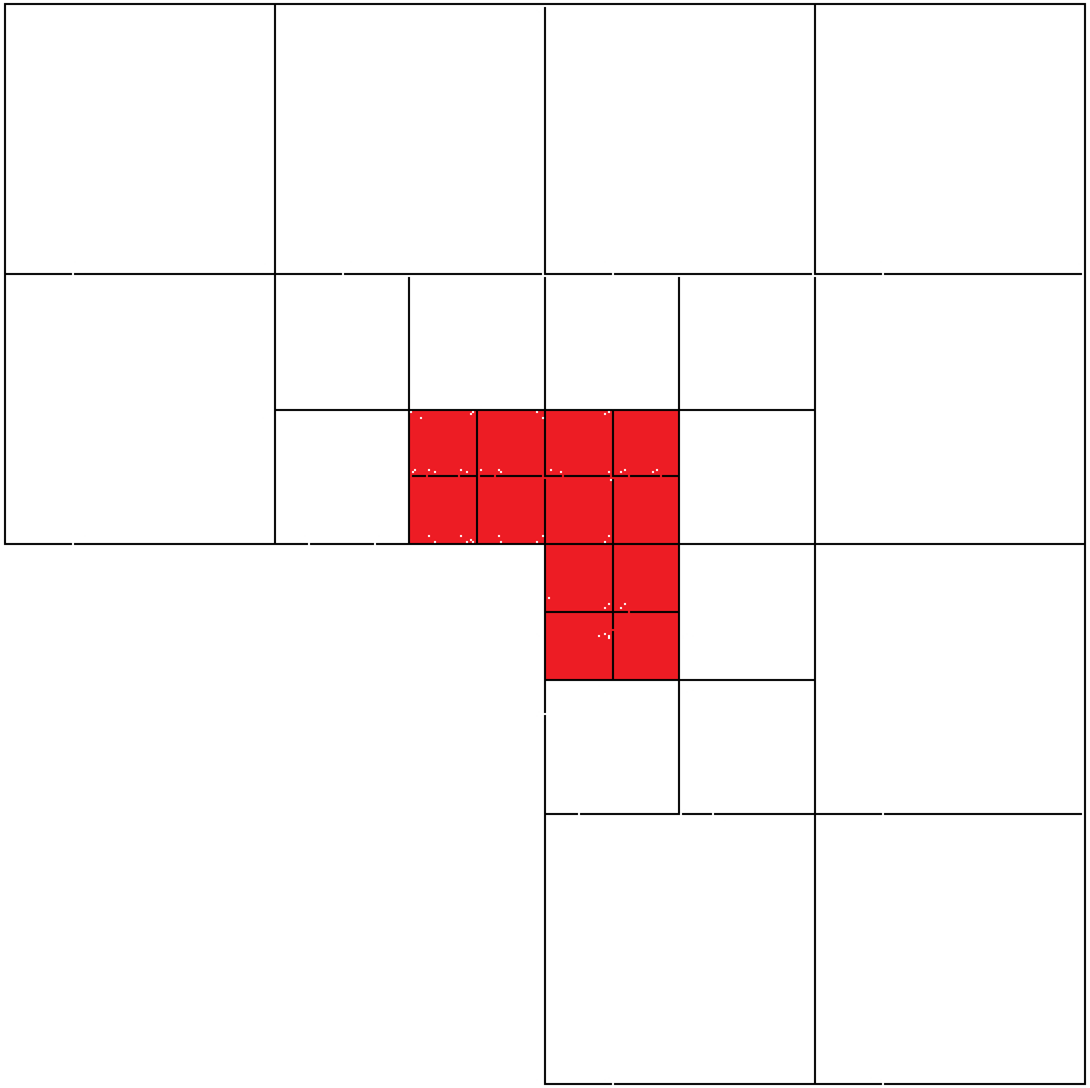
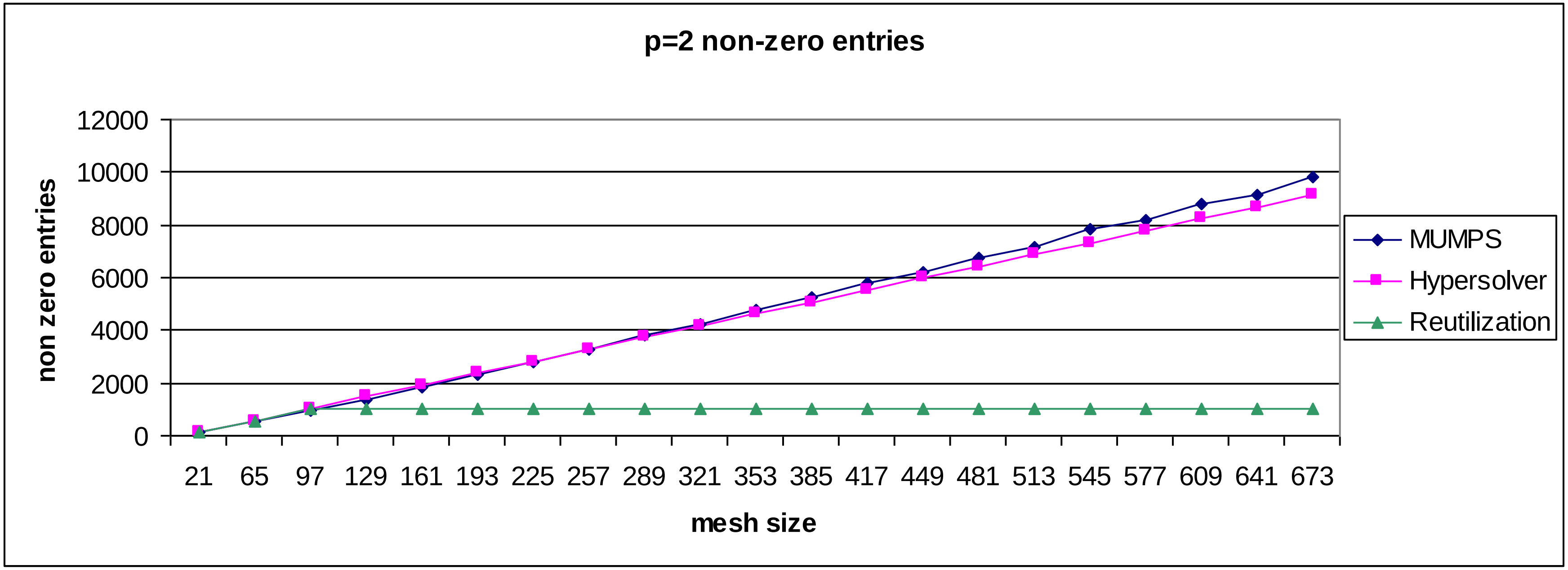
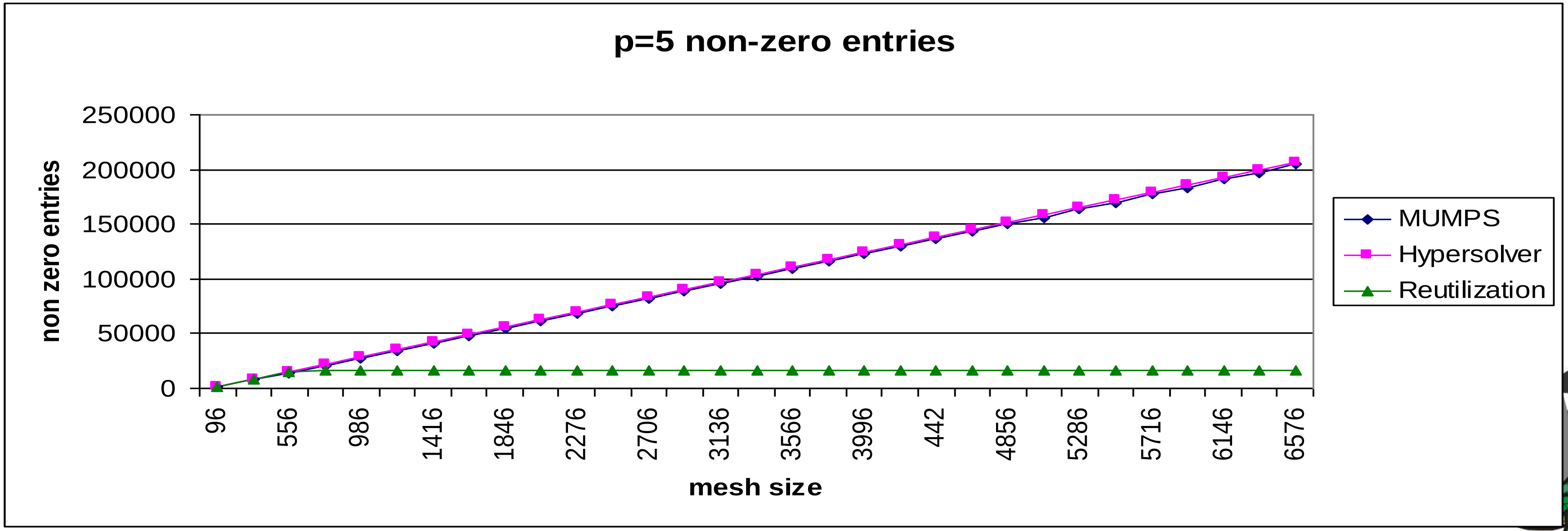


NUMERICAL RESULTS – LSHAPE DOMAIN

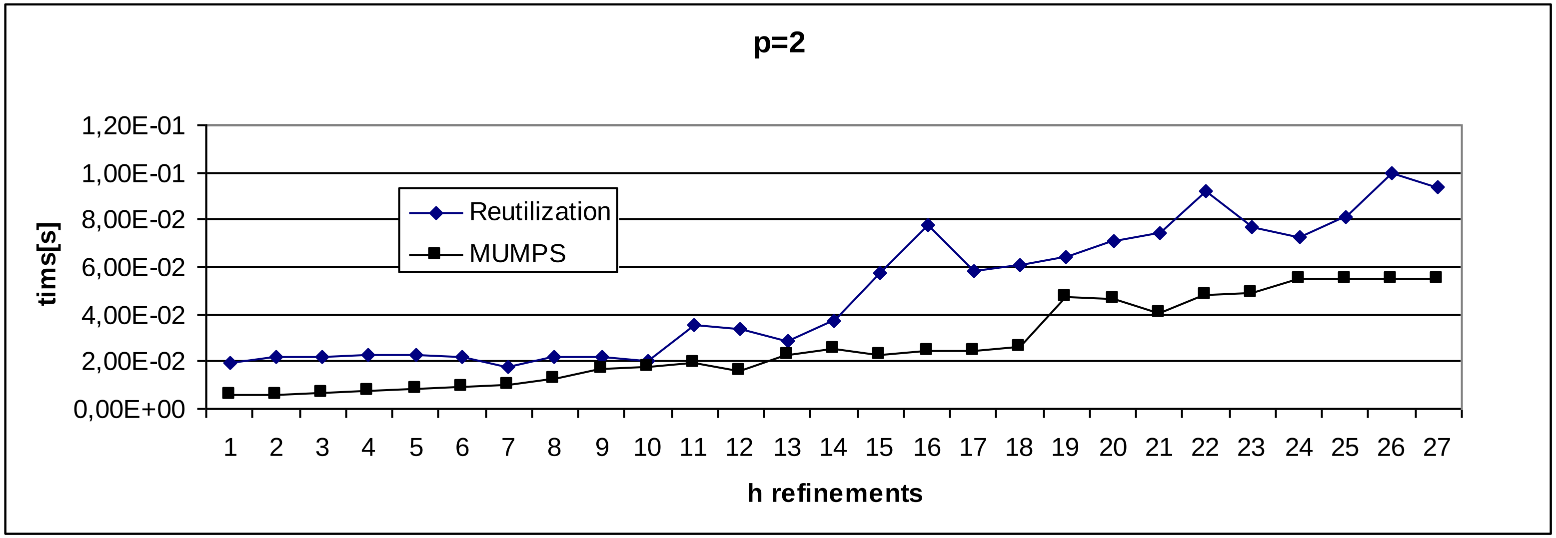


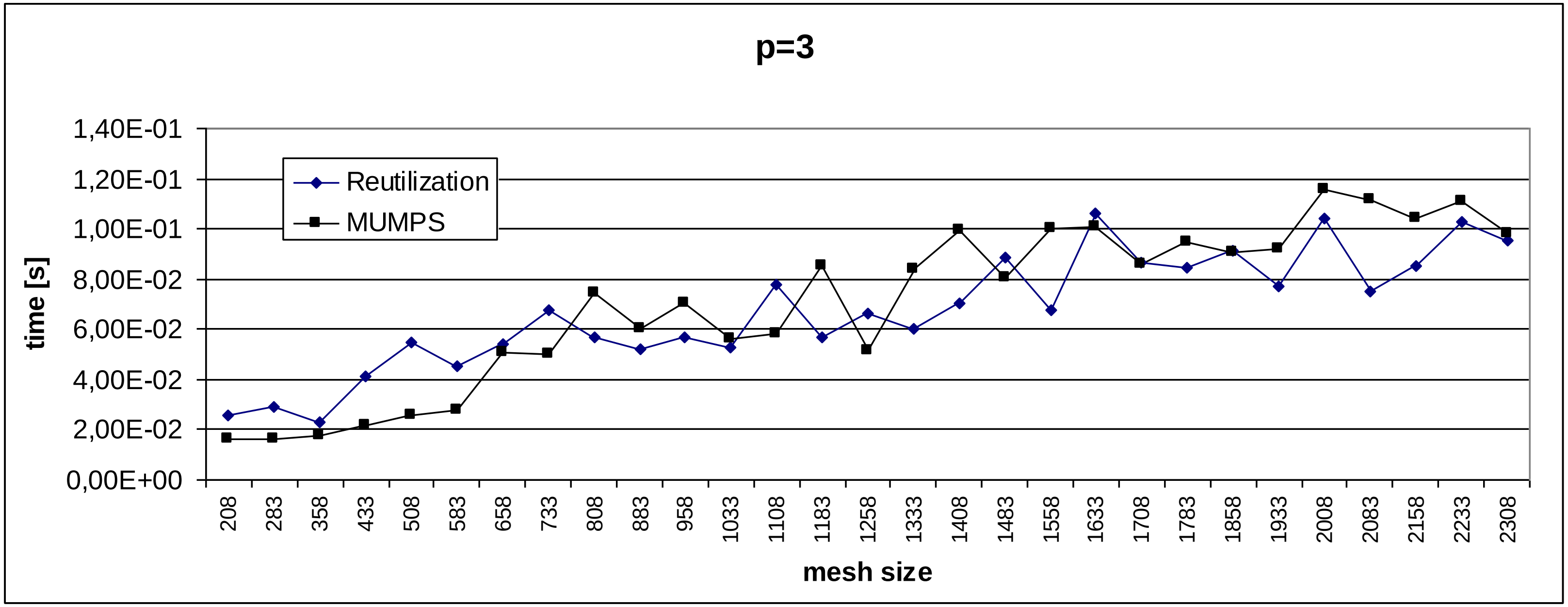
NON-ZERO ENTRIES





EXECUTION TIME





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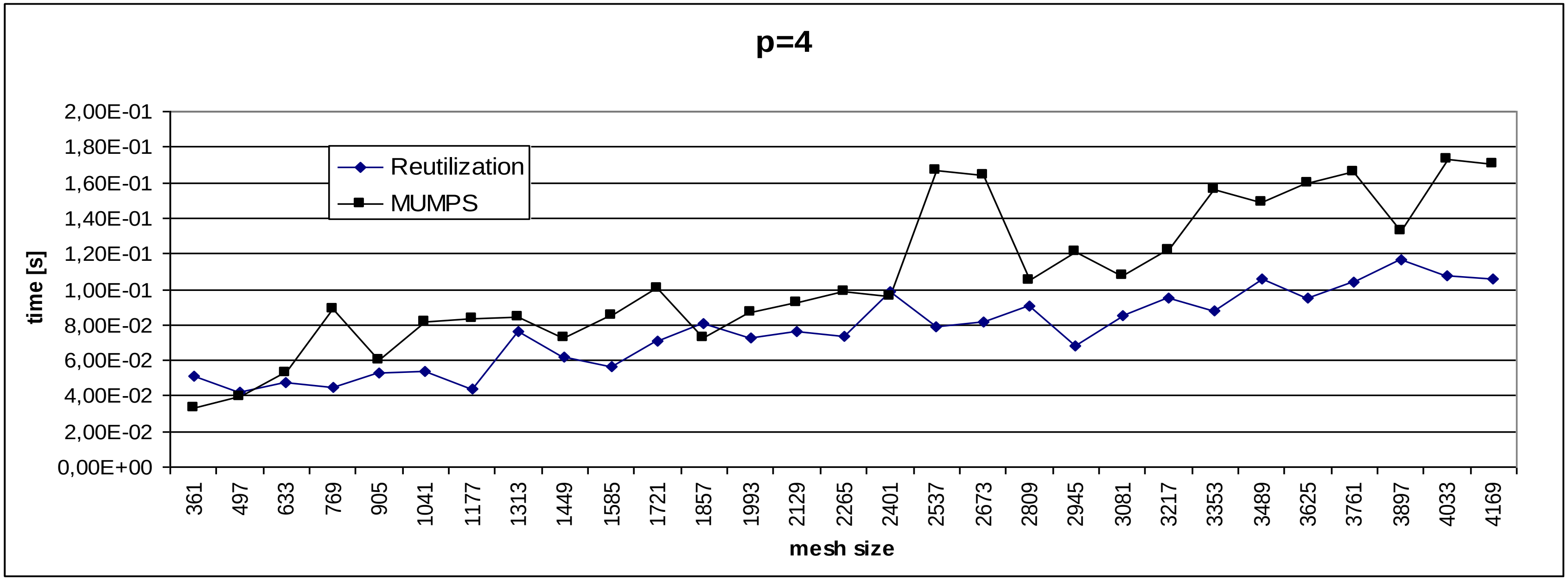
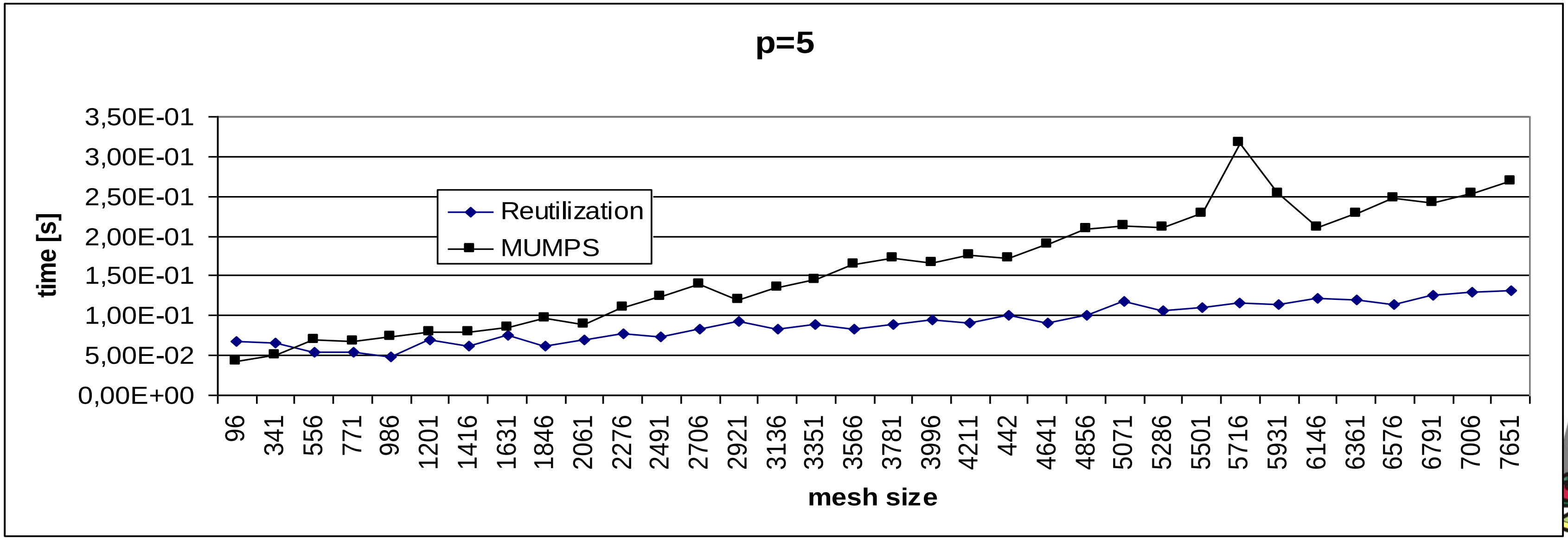
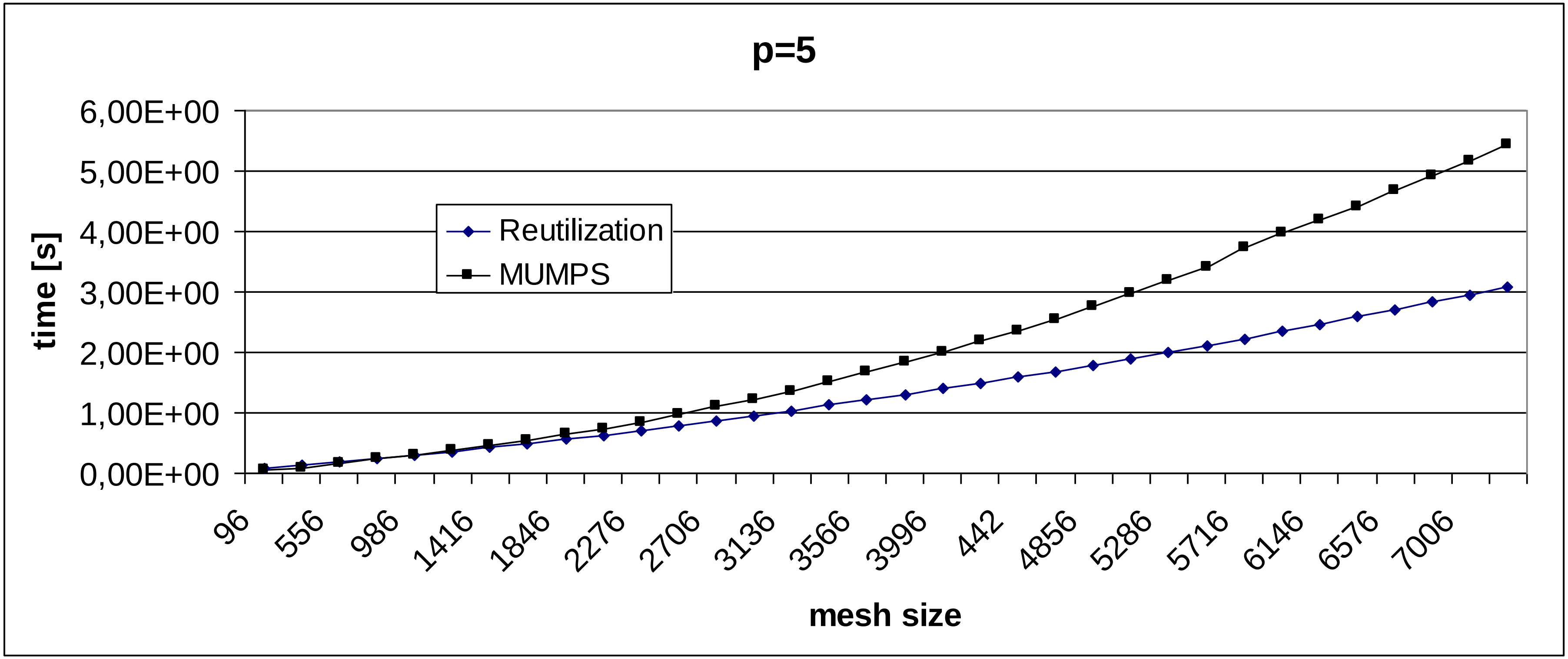
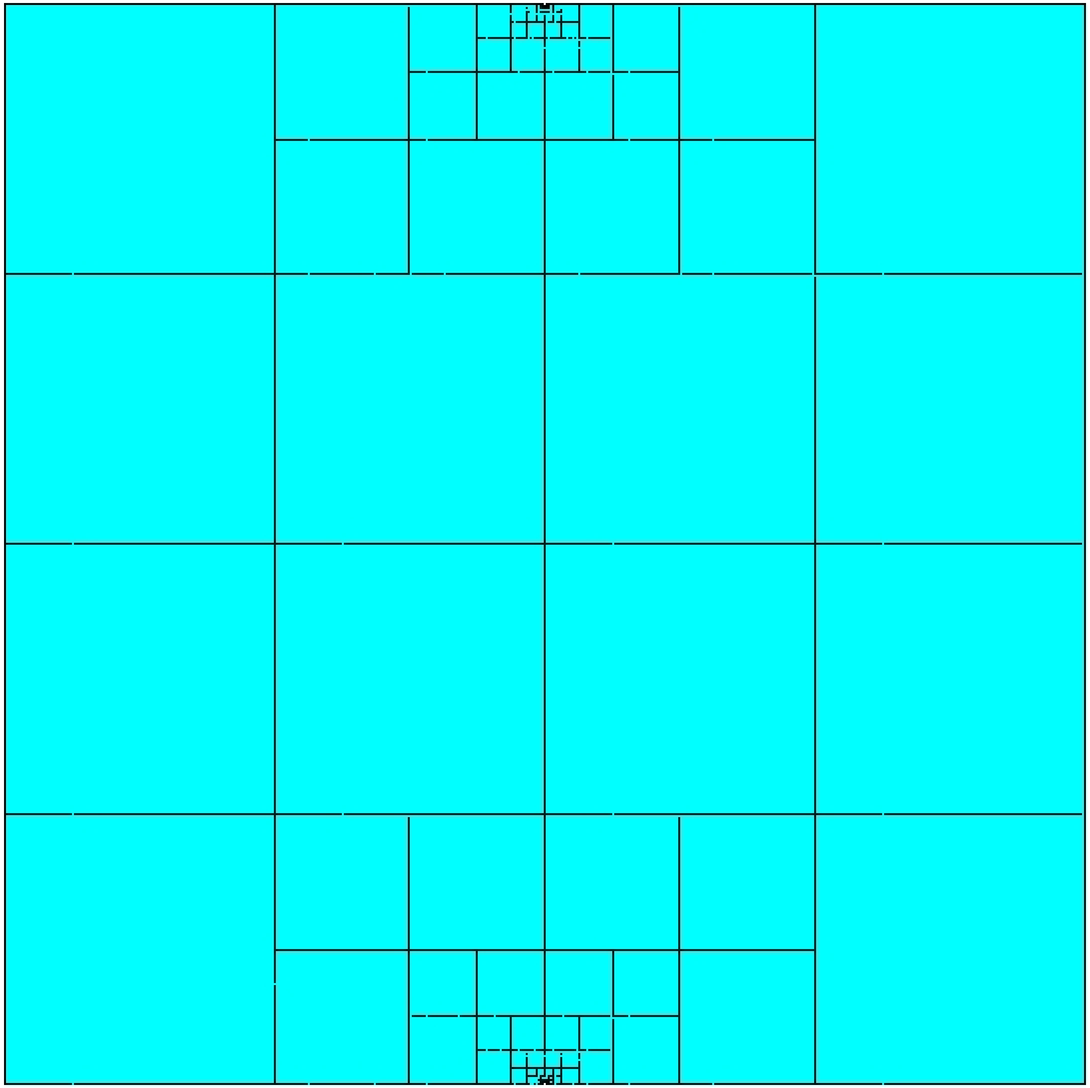


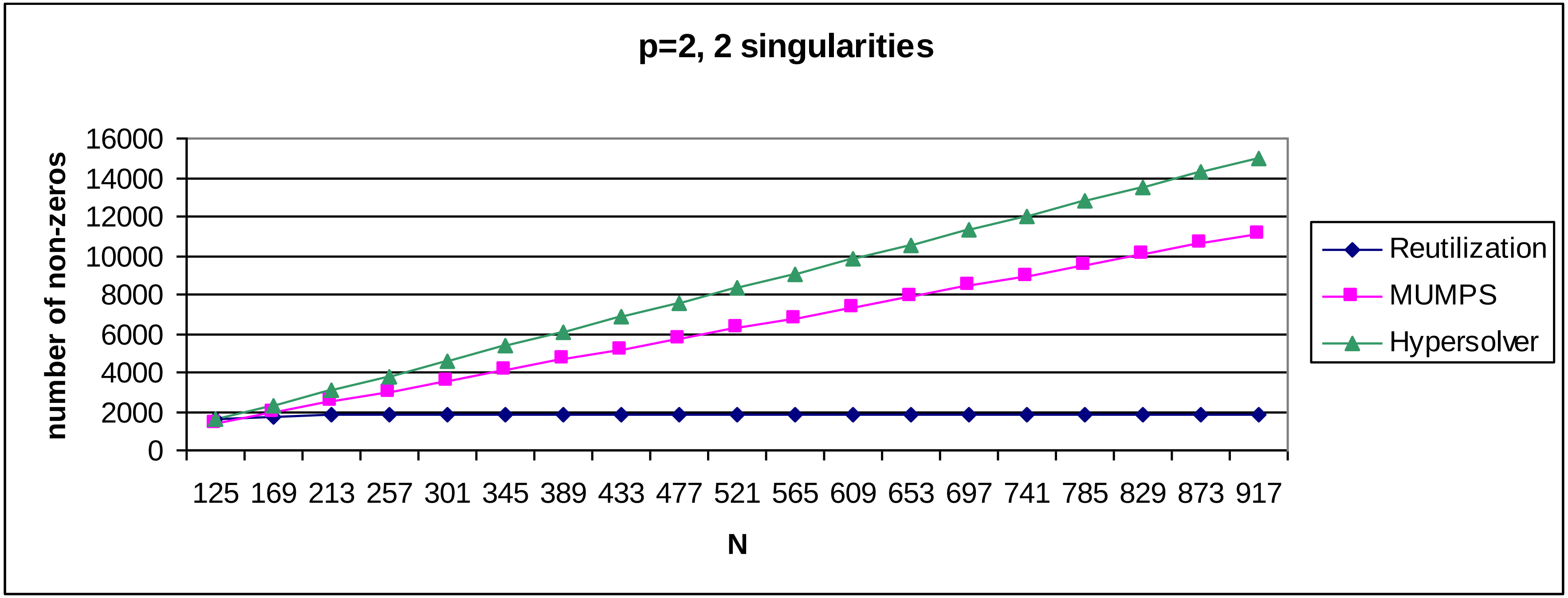


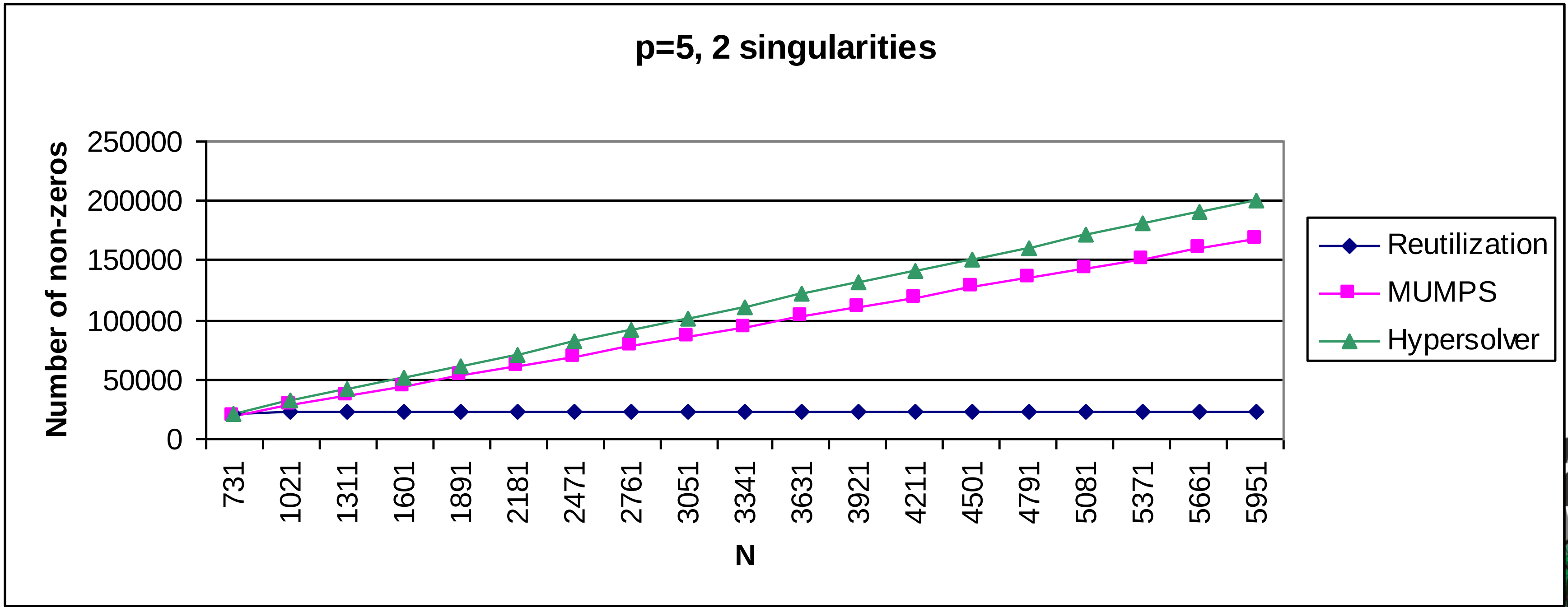
REUTILIZATION vs CLASSICAL APPROACH

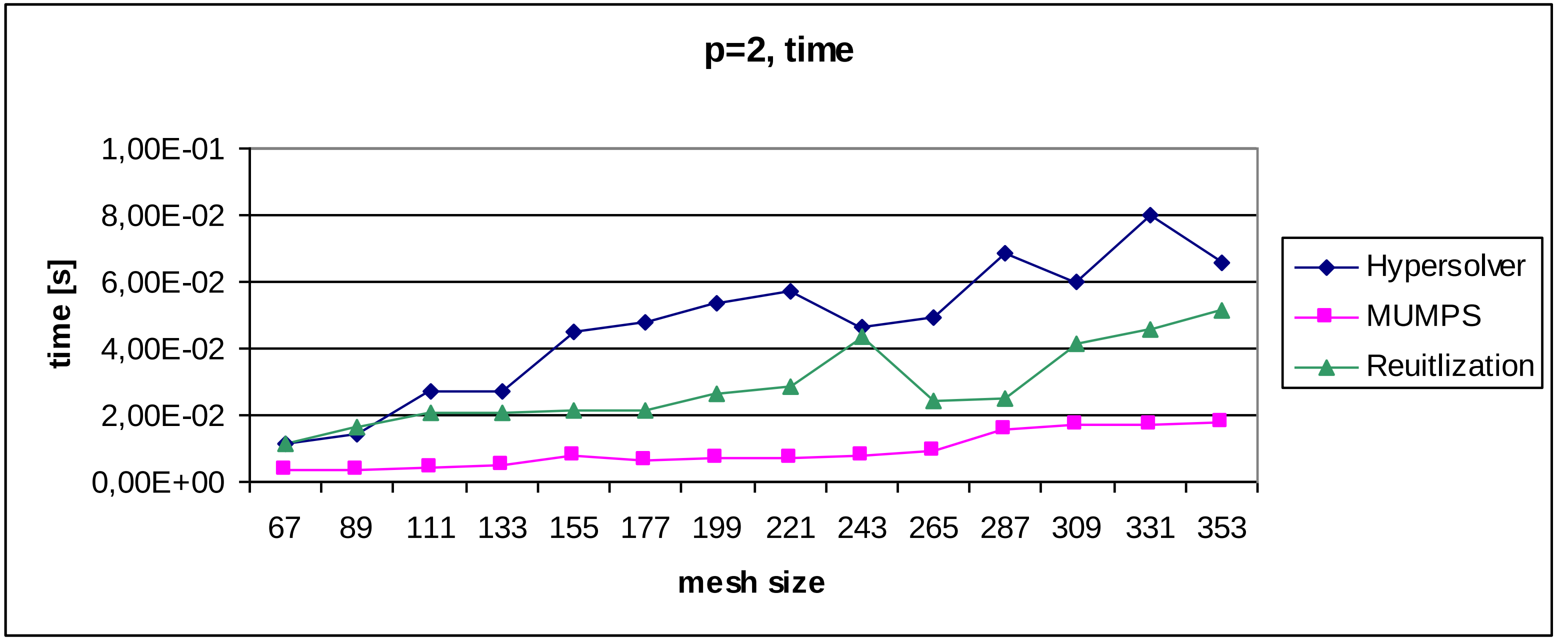


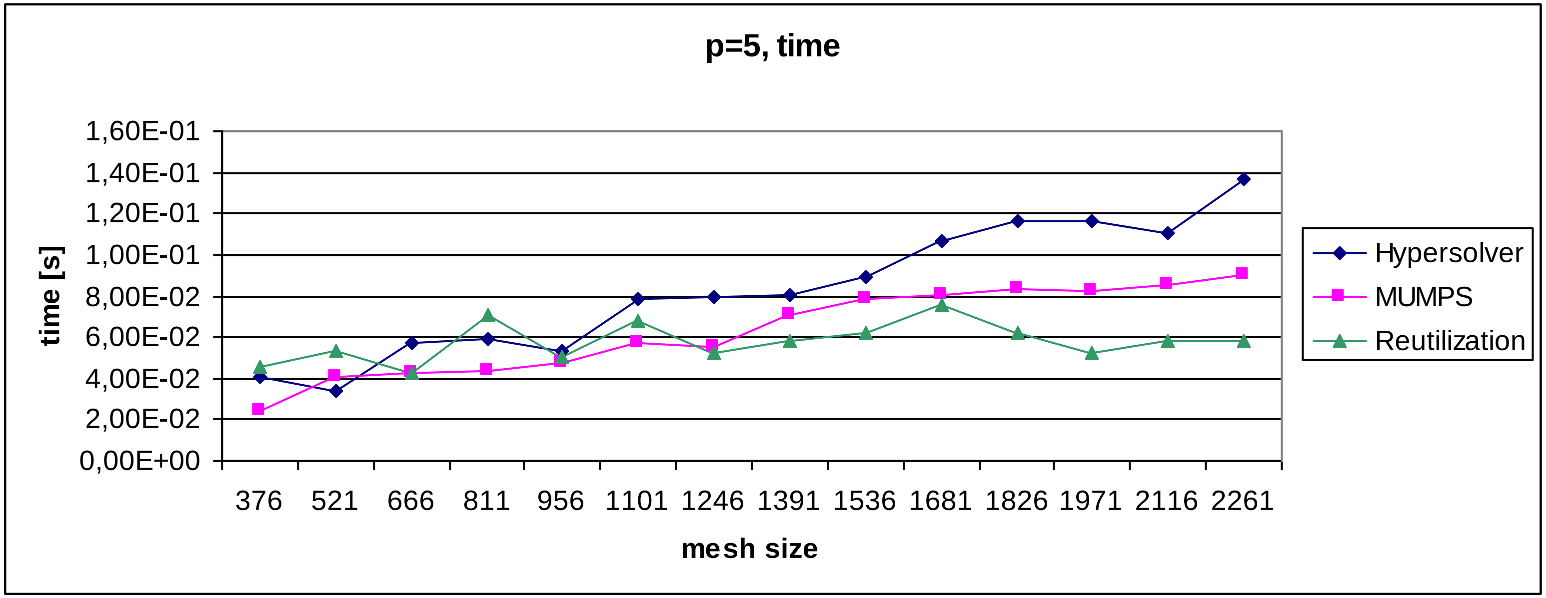
NUMERICAL RESULTS – TWO SINGULARITIES



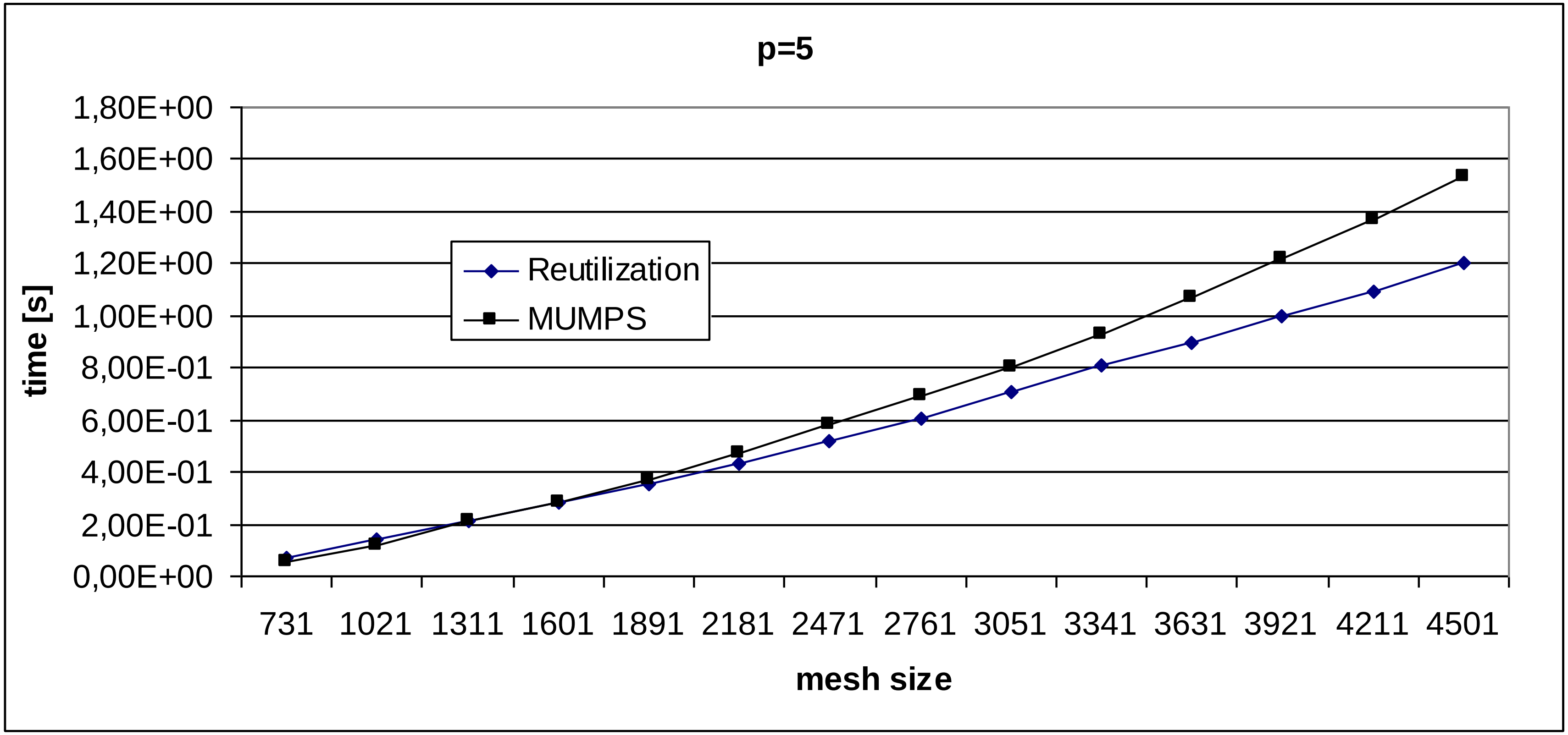

p=2, 2 singularities
number of non-zeros
16000
14000
12000
10000
8000
6000
4000
2000
0
125 169 213 257 301 345 389 433 477 521 565 609 653 697 741 785 829 873 917
N
Reutilization
MUMPS
Hypersolver


p=5, 2 singularities
Number of non-zeros
250000
200000
150000
100000
50000
0
731 1021 1311 1601 1891 2181 2471 2761 3051 3341 3631 3921 4211 4501 4791 5081 5371 5661 5951
N
Reutilization
MUMPS
Hypersolver


p=2, time
time [s]
1,00E-01
8,00E-02
6,00E-02
4,00E-02
2,00E-02
0,00E+00
67
89
111
133
155
177
199
221
243
265
287
309
331
353
mesh size
Hypersolver
MUMPS
Reuitlization


p=5, time
time [s]
1,60E-01
1,40E-01
1,20E-01
1,00E-01
8,00E-02
6,00E-02
4,00E-02
2,00E-02
0,00E+00
376
521
666
811
956
1101
1246
1391
1536
1681
1826
1971
2116
2261
mesh size
Hypersolver
MUMPS
Reutilization

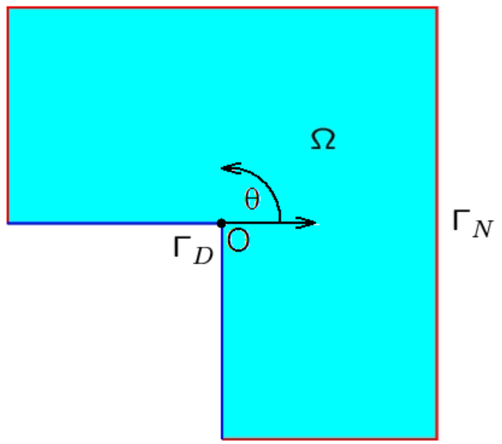
REUTILIZATION vs CLASSICAL APPROACH



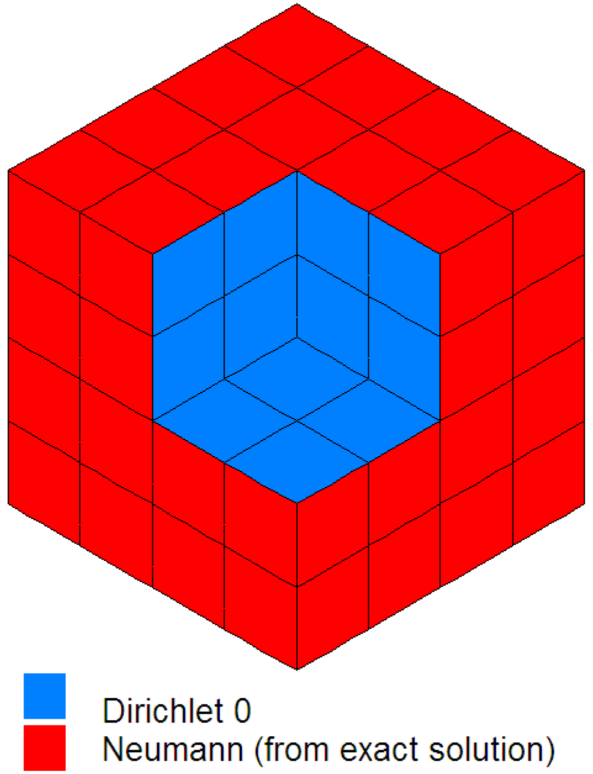
3D Fichera problem

$$\begin{cases} \Delta u = 0 & w & \Omega \\ u = 0 & na & \Gamma_D \\ \dfrac{\partial u}{\partial n} = g & na & \Gamma_N \end{cases}$$

$$g(r,\theta) = r^{\frac{2}{3}} \sin \frac{2}{3}\left(\theta + \frac{\Pi}{2}\right)$$

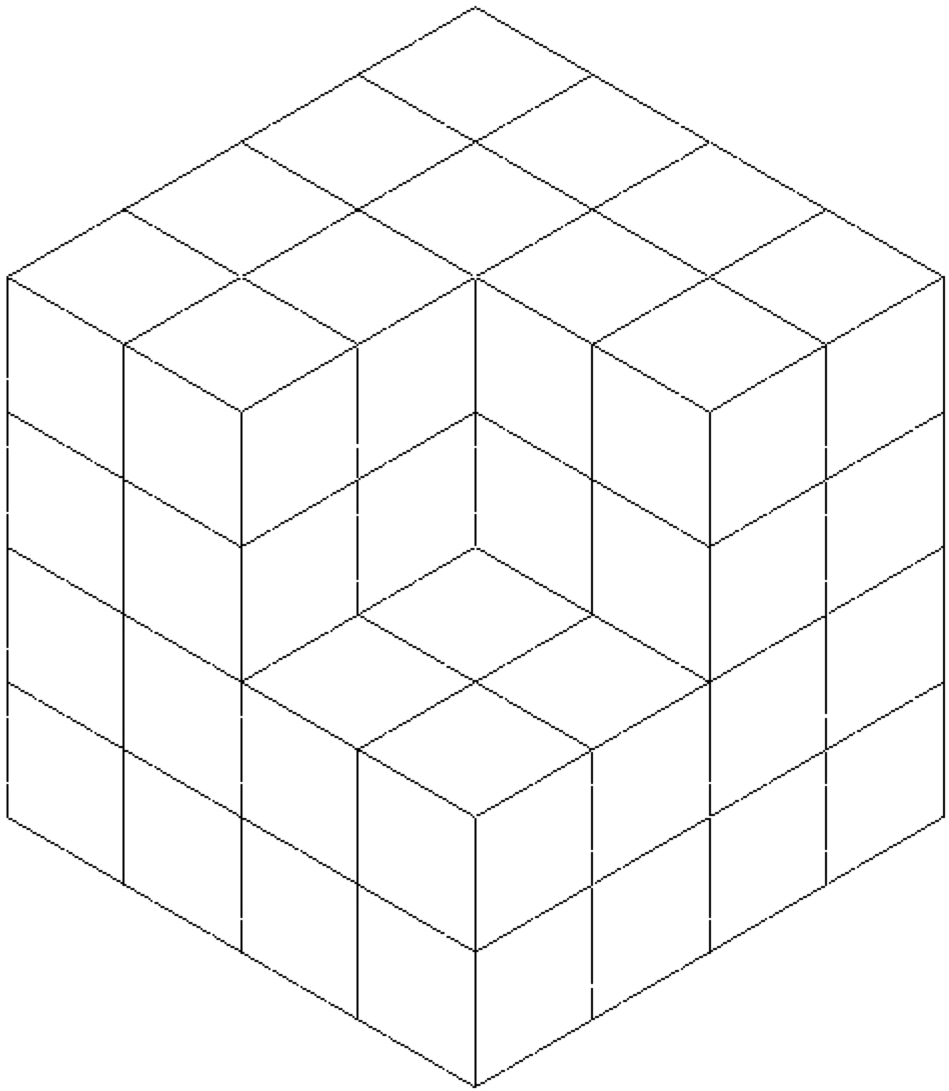


$$\begin{cases} \Delta u = 0 & w & \Omega \\ u = 0 & na & \Gamma_D \\ \dfrac{\partial u}{\partial n} = g & na & \Gamma_N \end{cases}$$

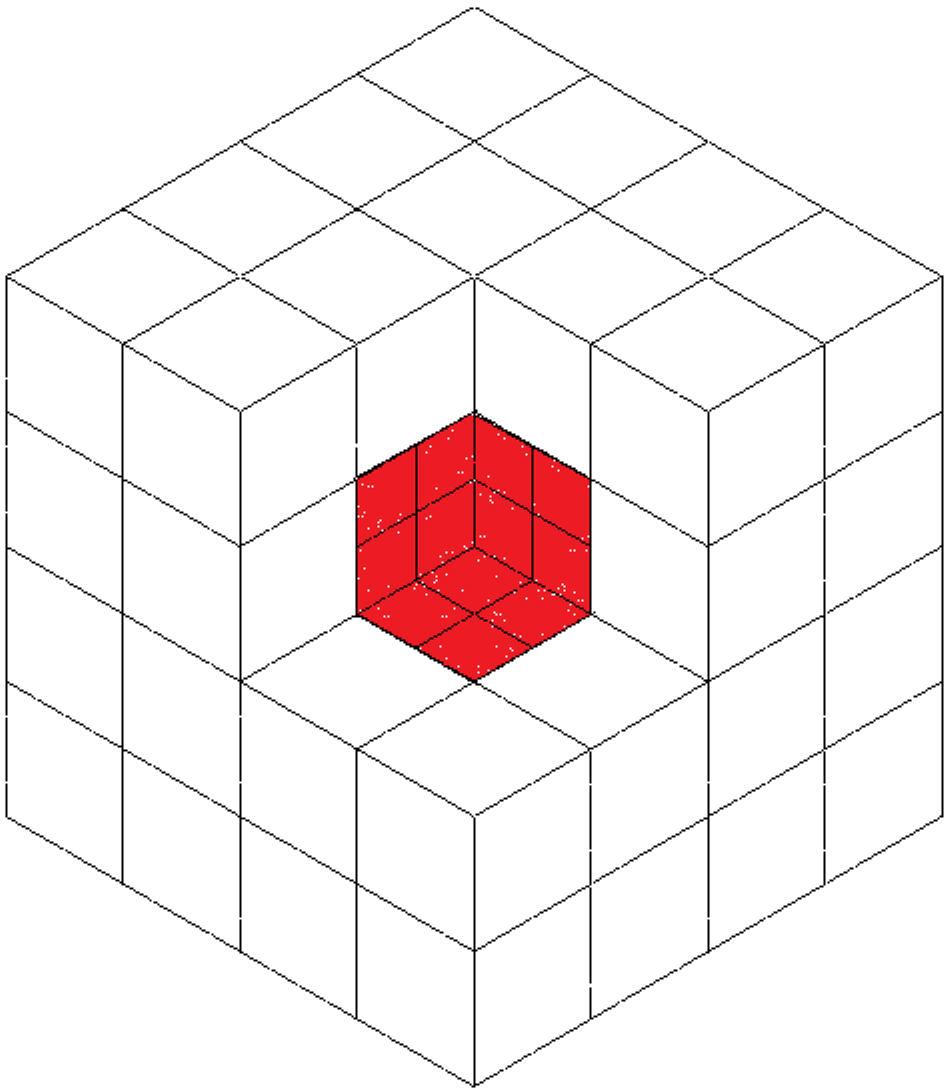


Laplace equation

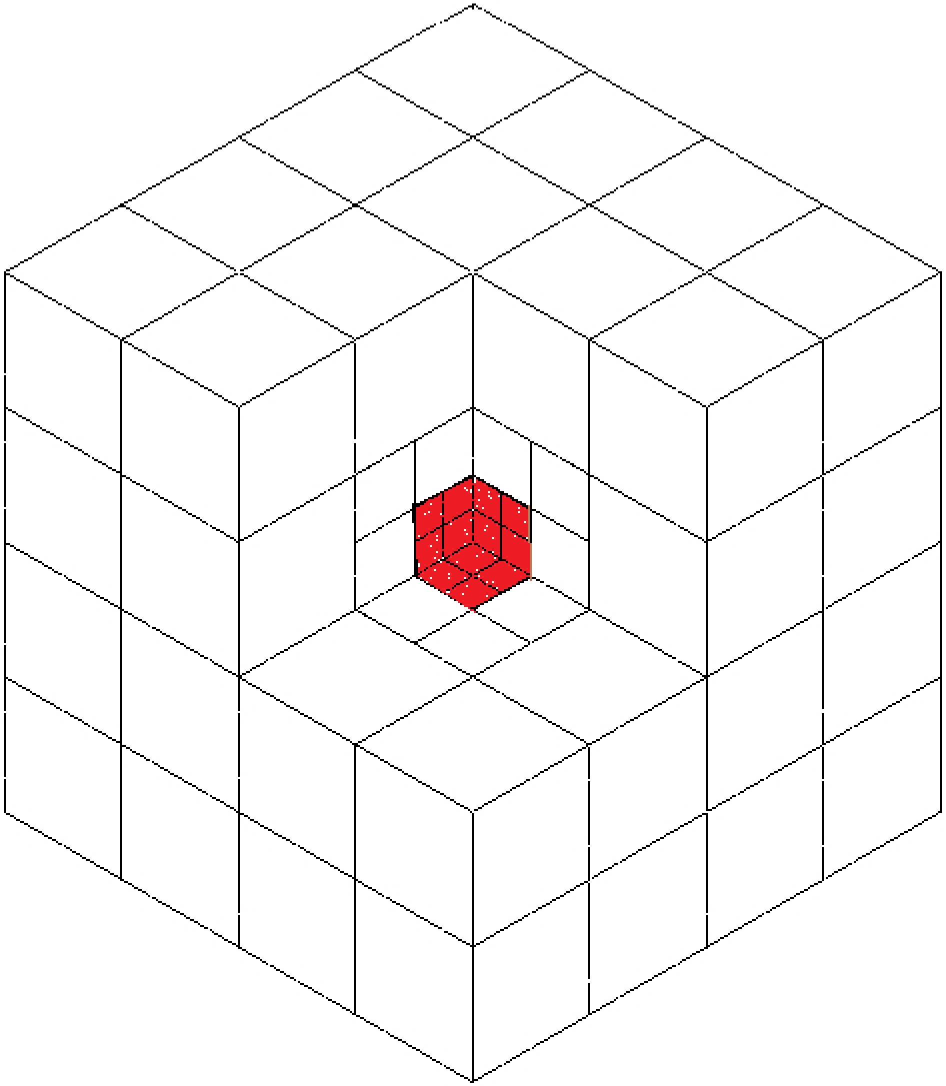
NUMERICAL RESULTS - FICHERA



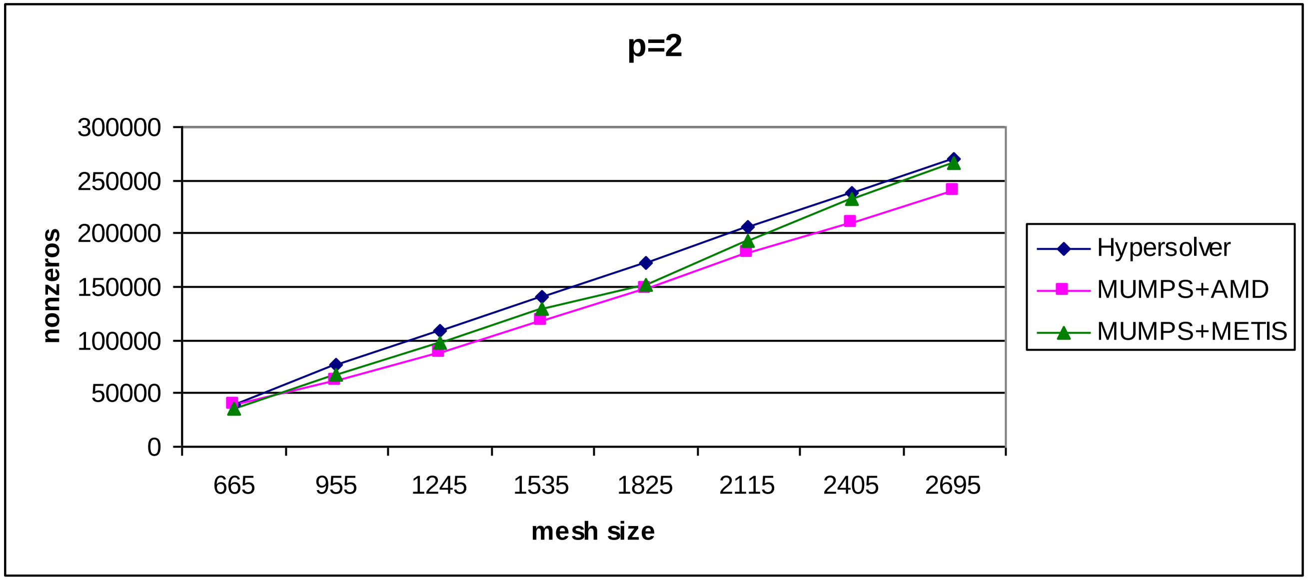
NUMERICAL RESULTS – FICHERA

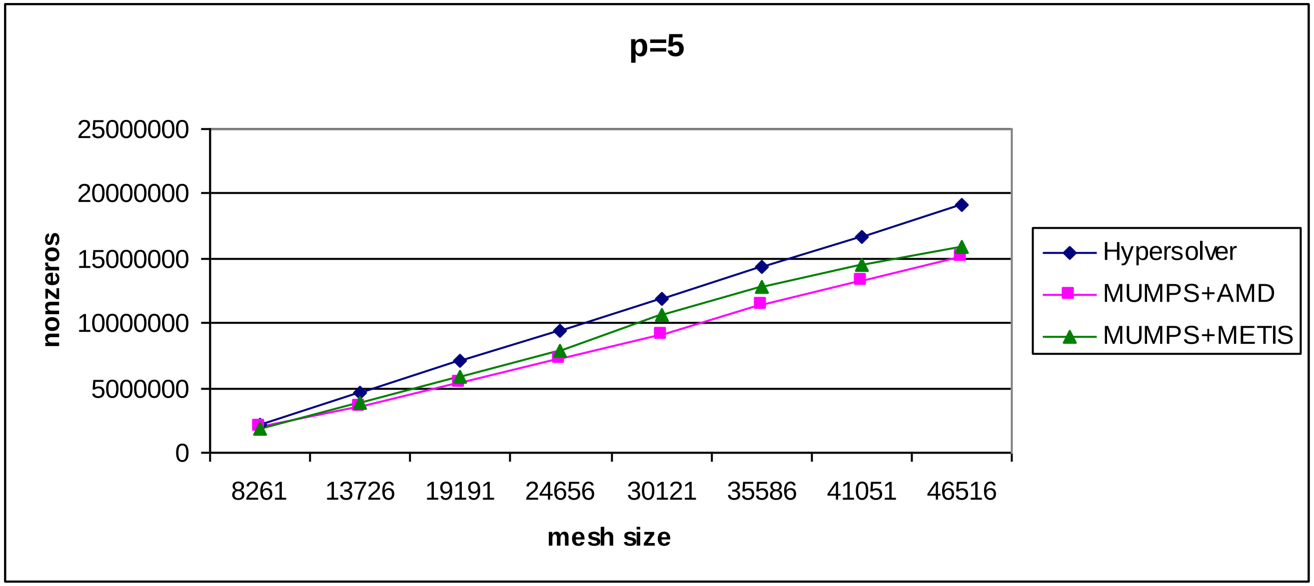


NUMERICAL RESULTS – FICHERA



NON-ZERO ENTRIES





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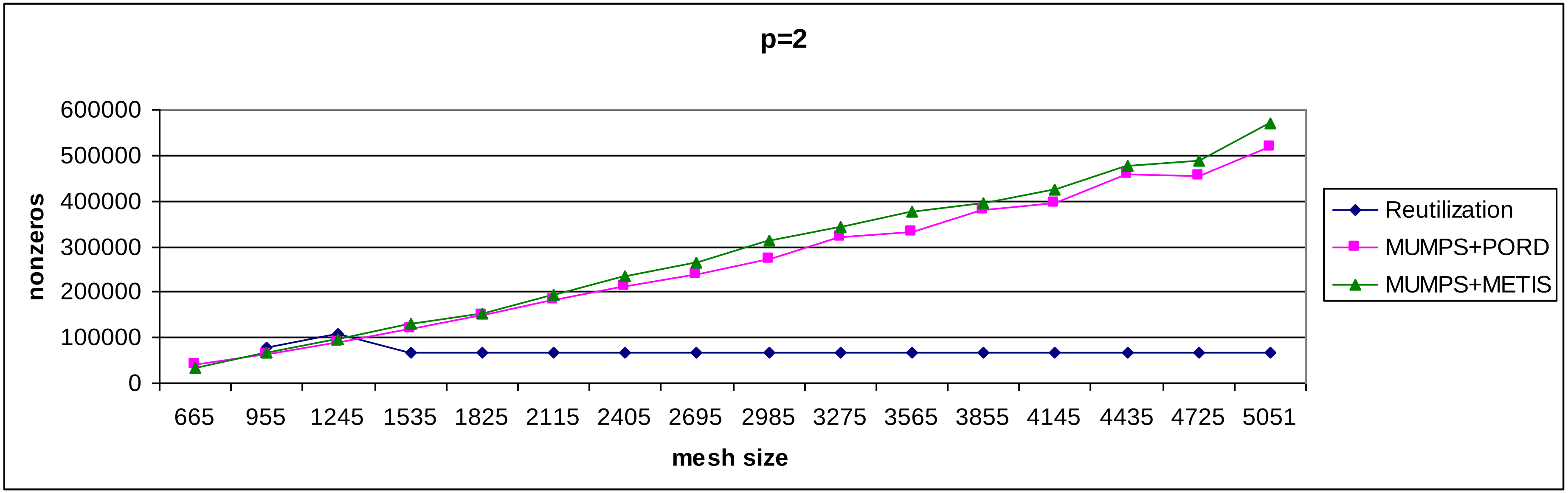
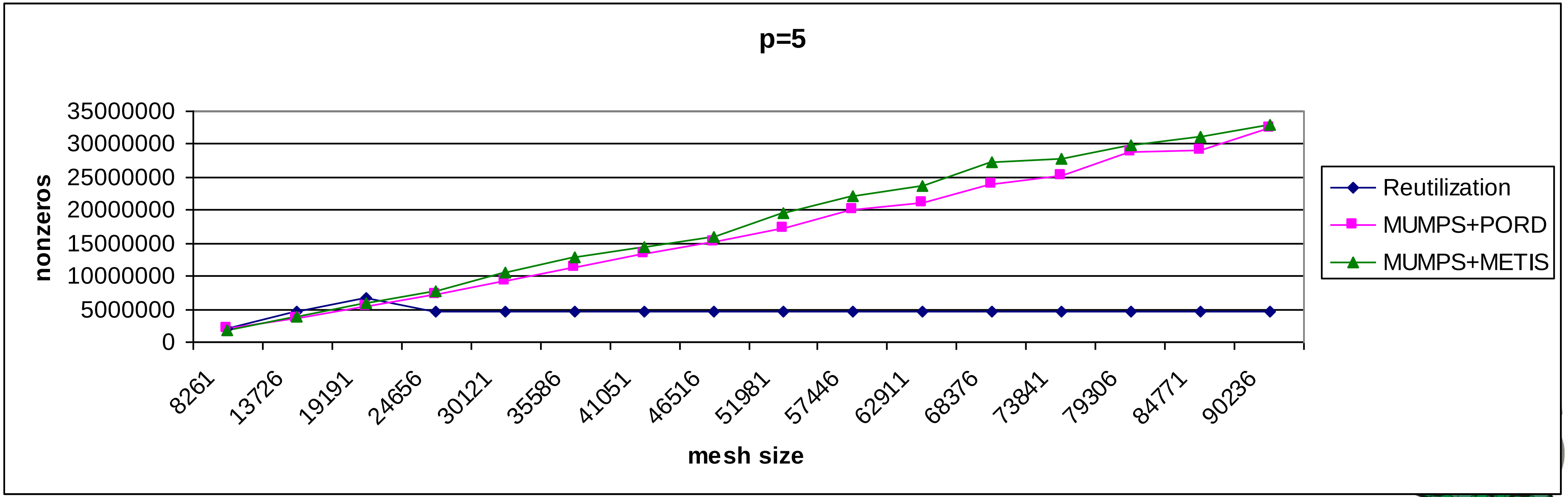
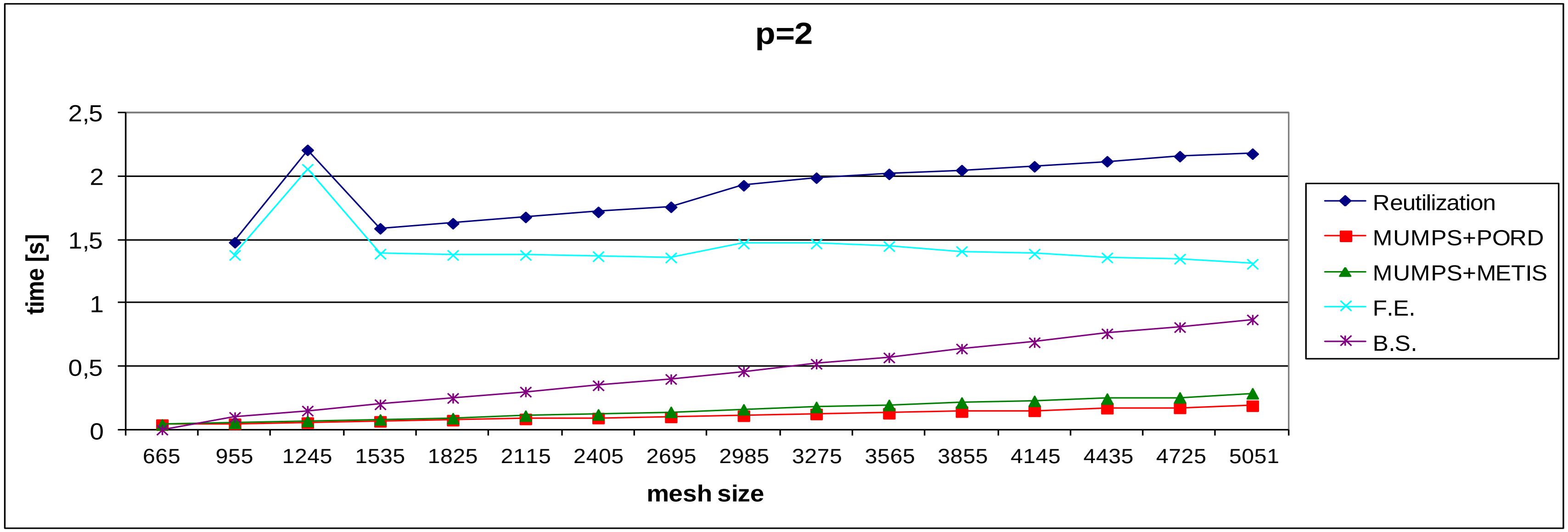
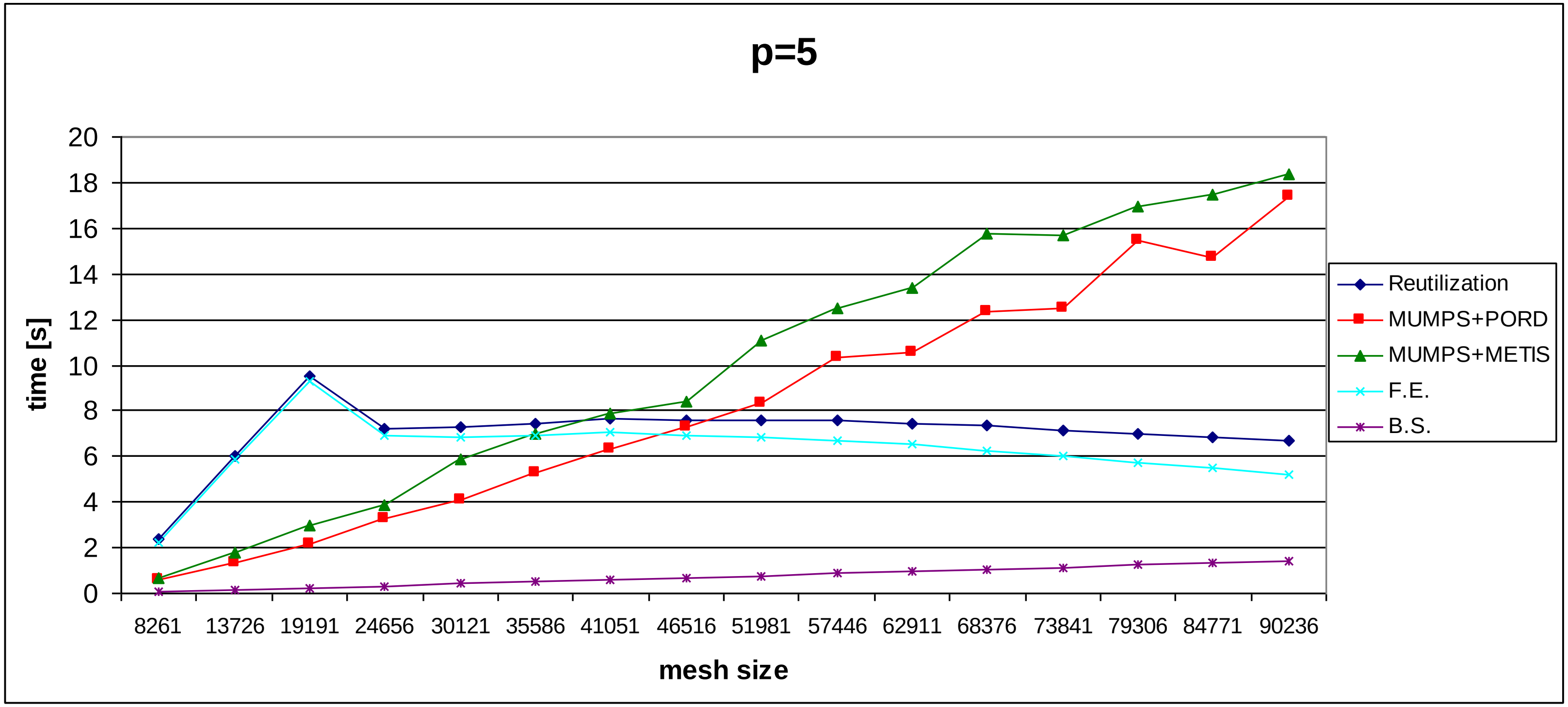


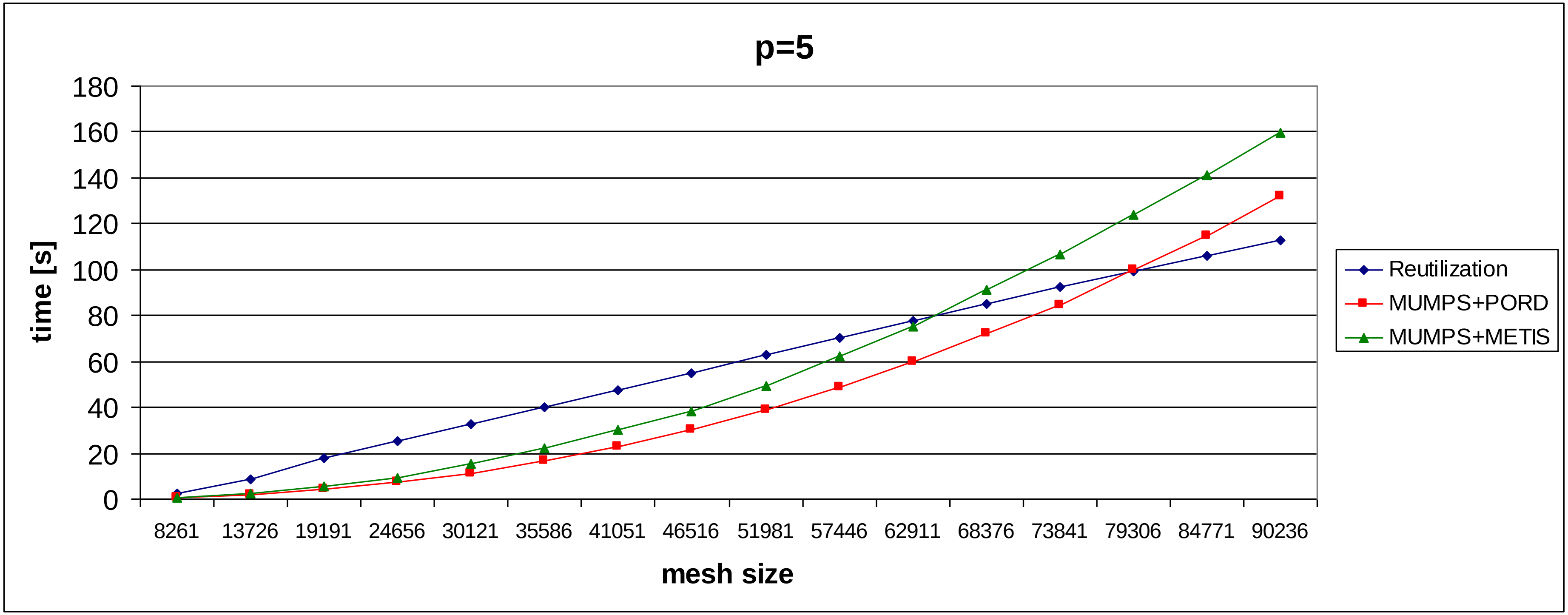


EXECUTION TIME

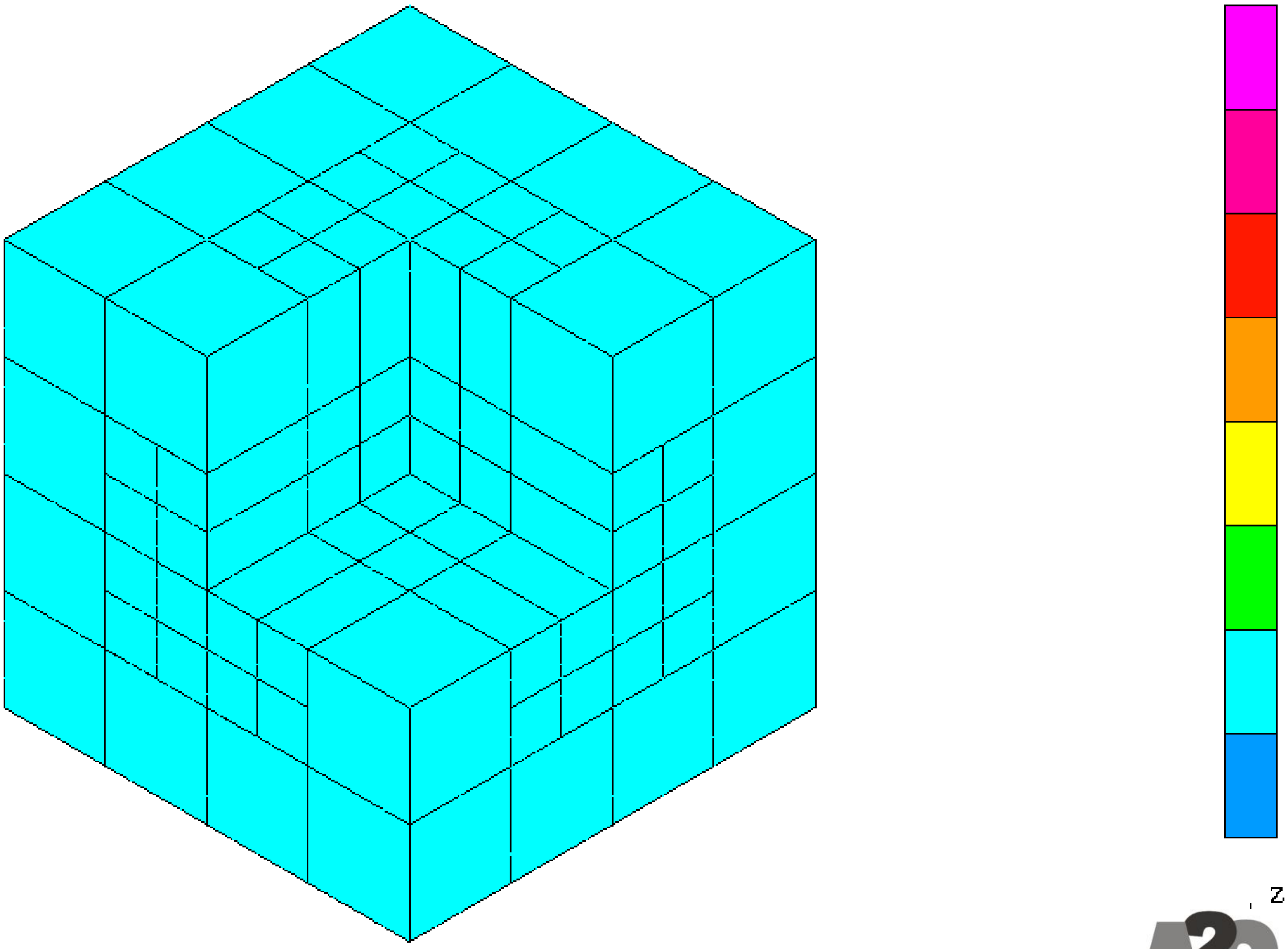




REUTILIZATION vs CLASSICAL APPROACH



LIMITATION – EDGE SINGULARITY



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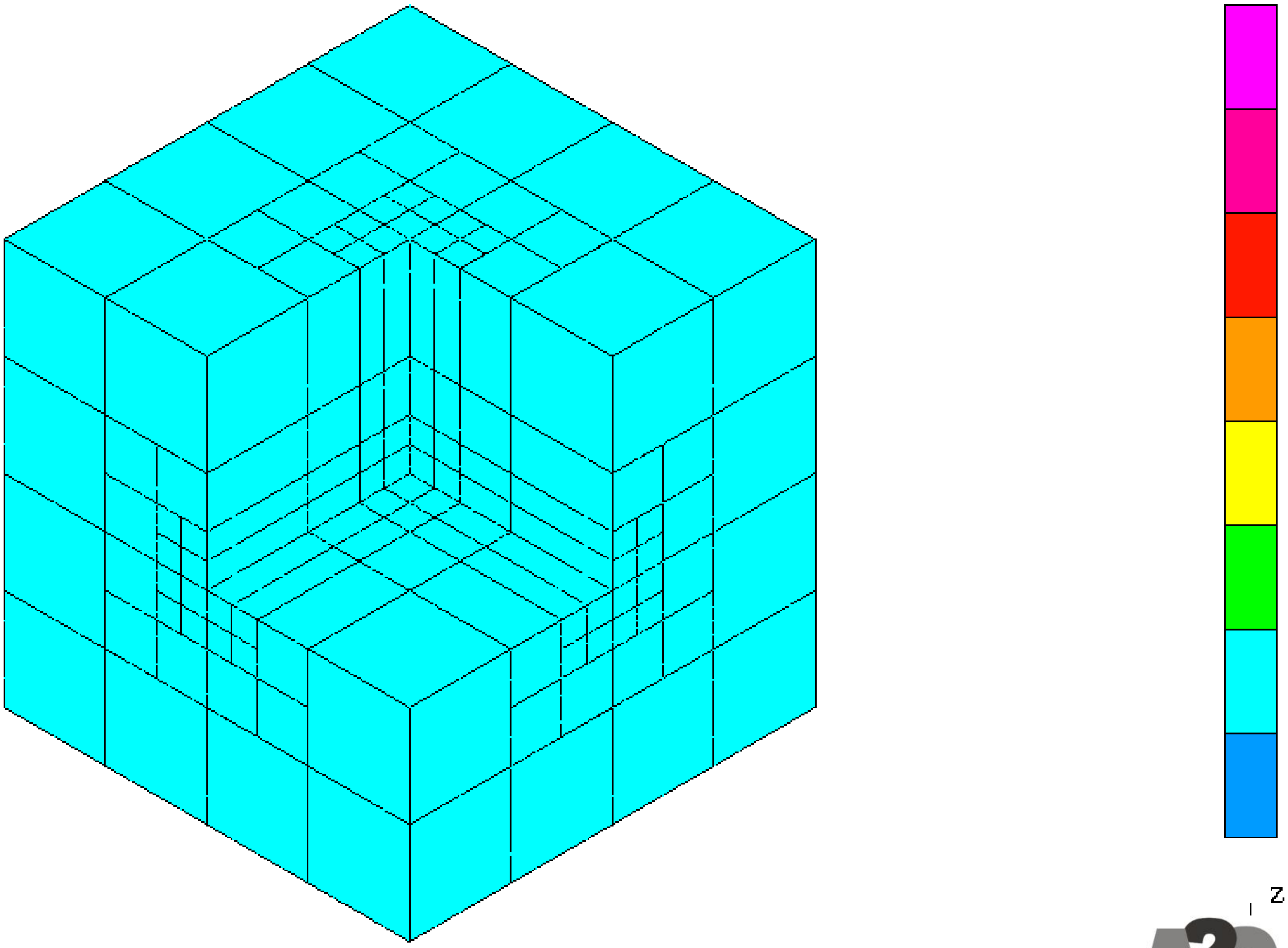

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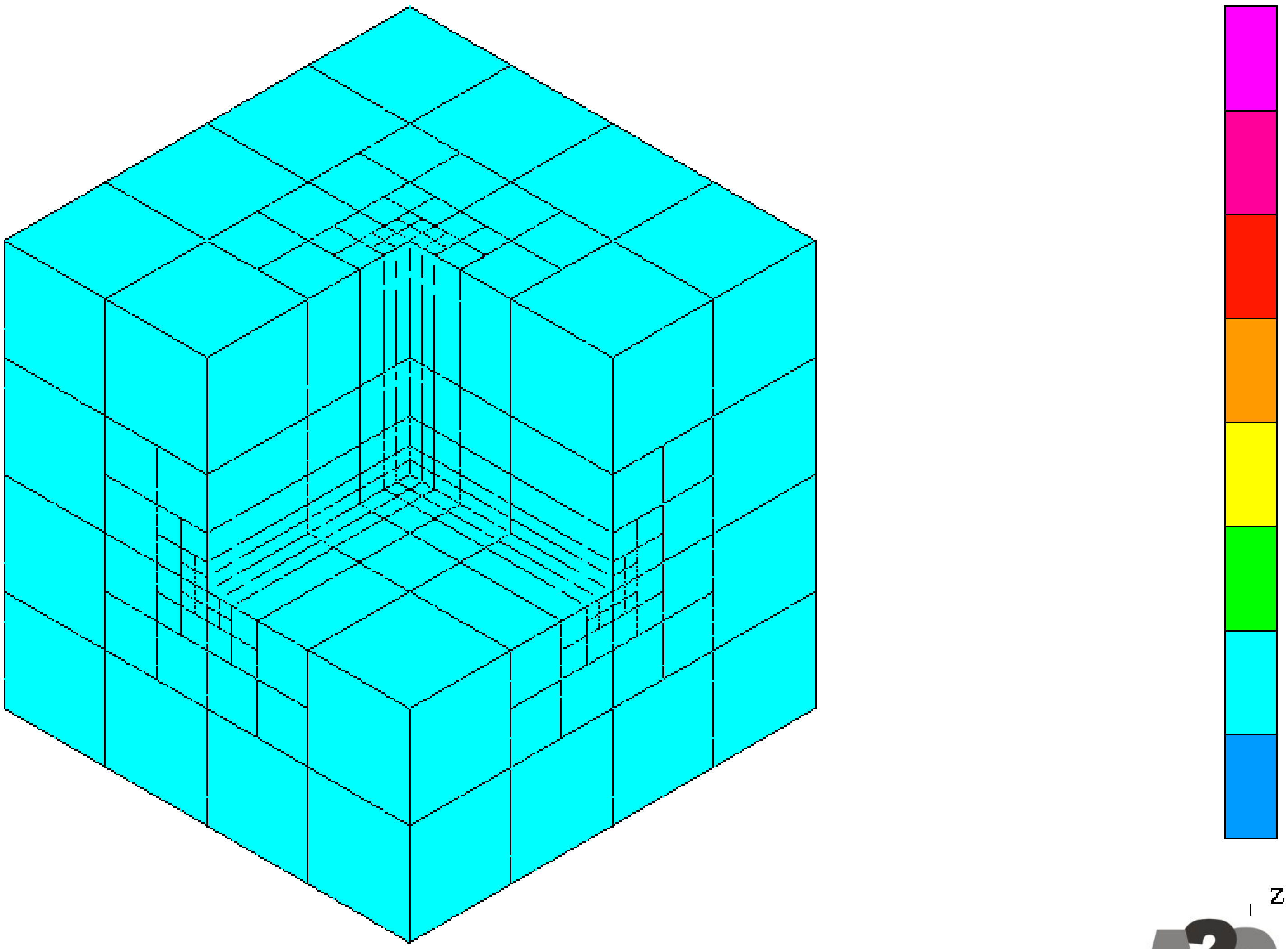

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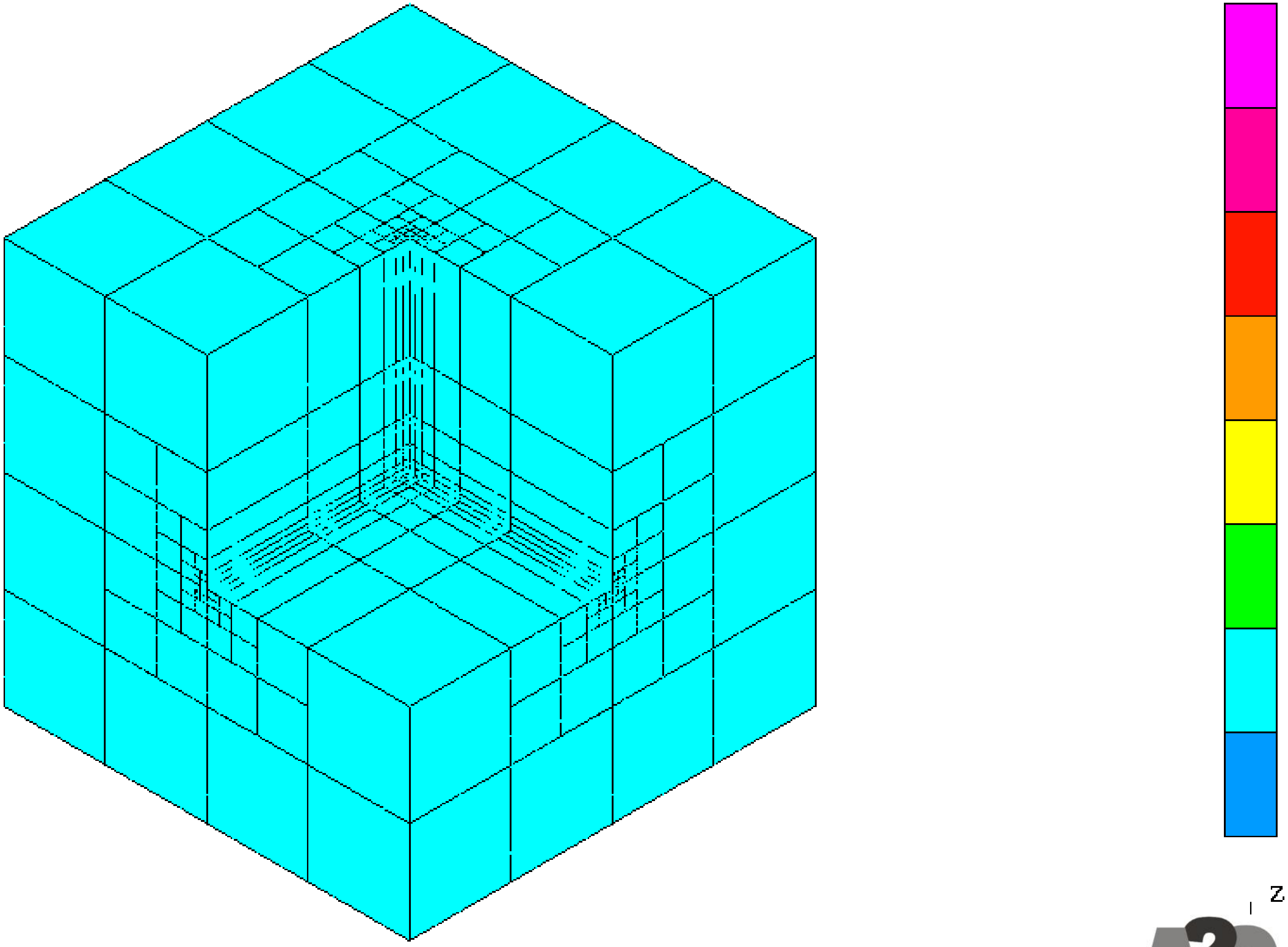

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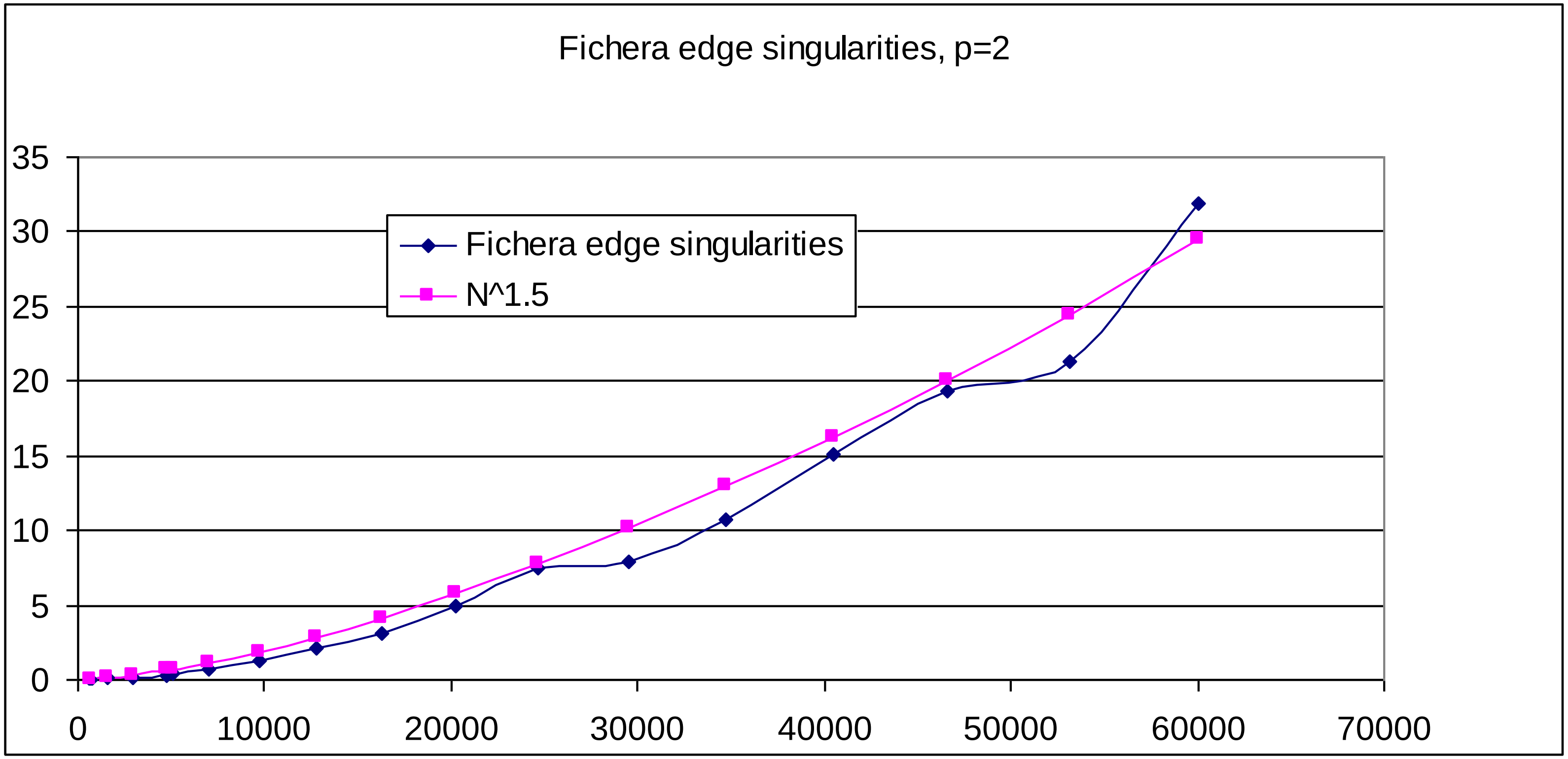

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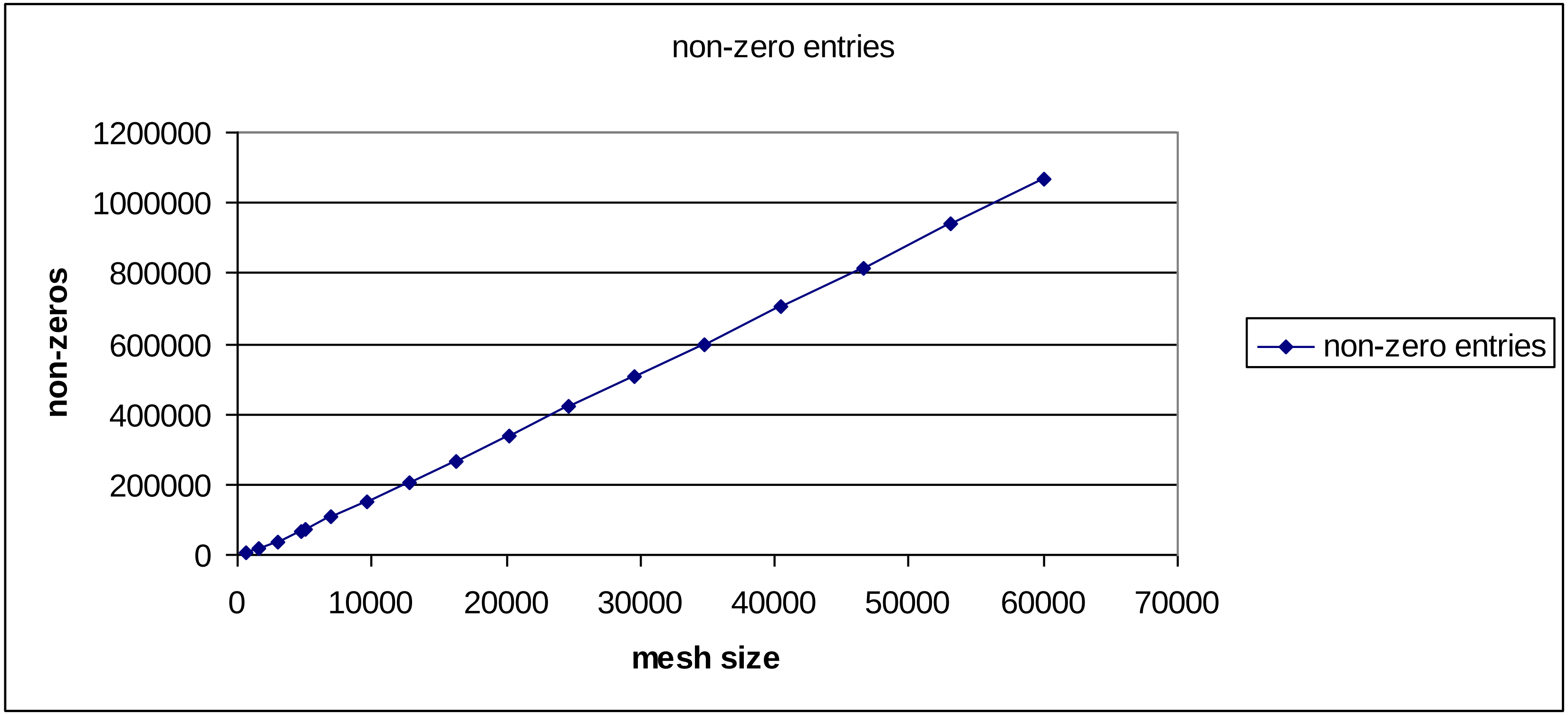

PAPERS

Maciej Paszyński, David Pardo, Victor Calo
A DIRECT SOLVER WITH REUTILIZATION OF PREVIOUSLY-COMPUTED LU FACTORIZATIONS FOR H-ADAPTIVE FINITE ELEMENT GRIDS WITH POINT SINGULARITIES
Computers and Mathematics with Applications, 65, 8 (2013) 1140-1151

Hassan AbouEisha, Mikhail Moshkhov, Maciej Paszynski, Piotr Gurgul, Victor Calo
ALGORITHM TO FIND AN OPTIMAL ELIMINATION TREE FOR A CLASS OF 2D FINITE ELEMENT GRIDS
submitted to ***SIAM Journal of Applied Mathematics***, 2013