# Frontal and multi-frontal solvers: Graph grammar based model of concurrency

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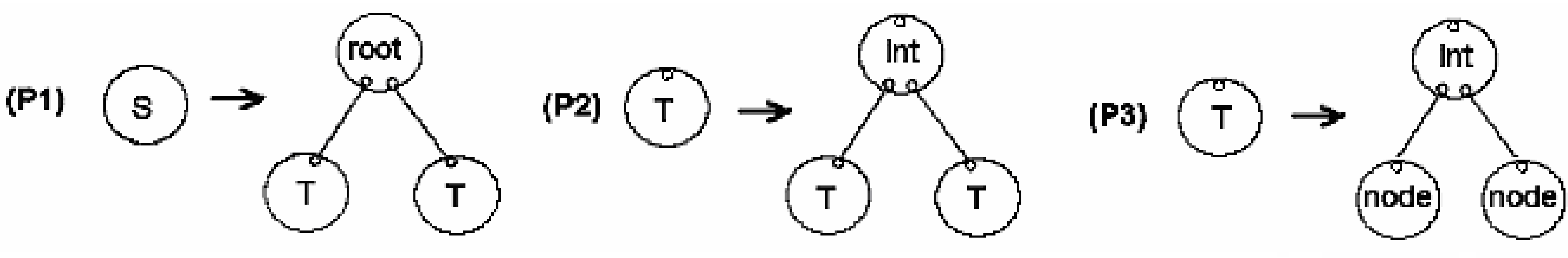
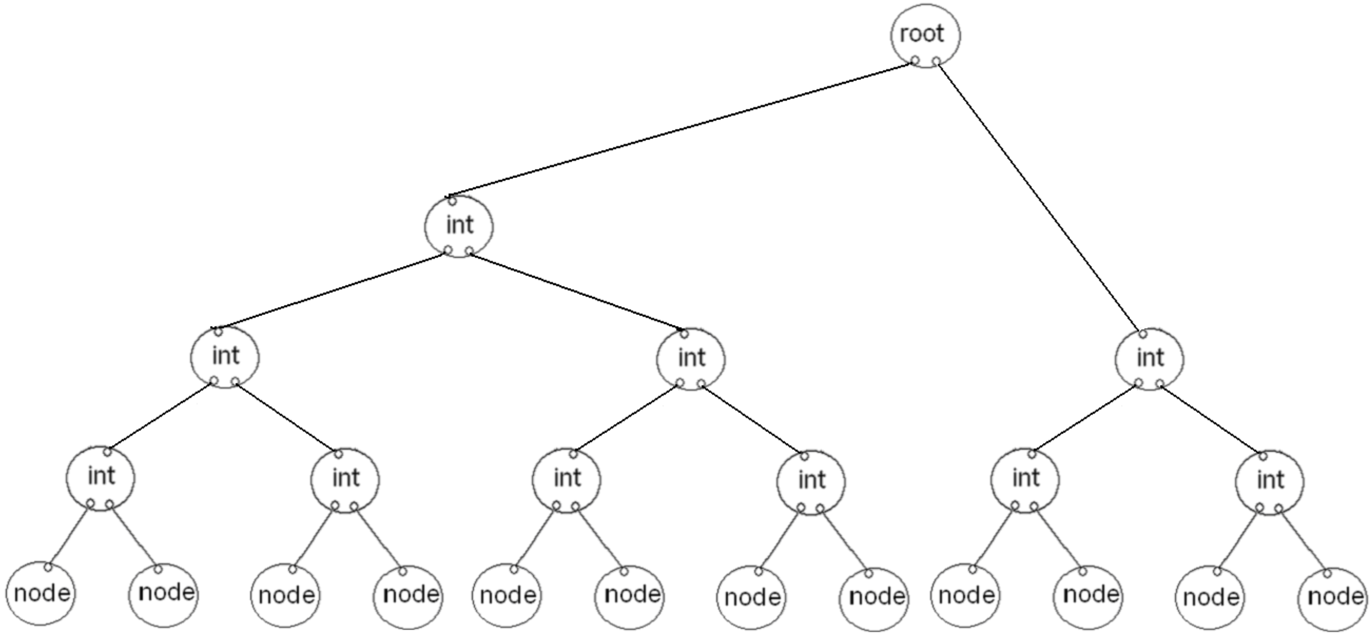
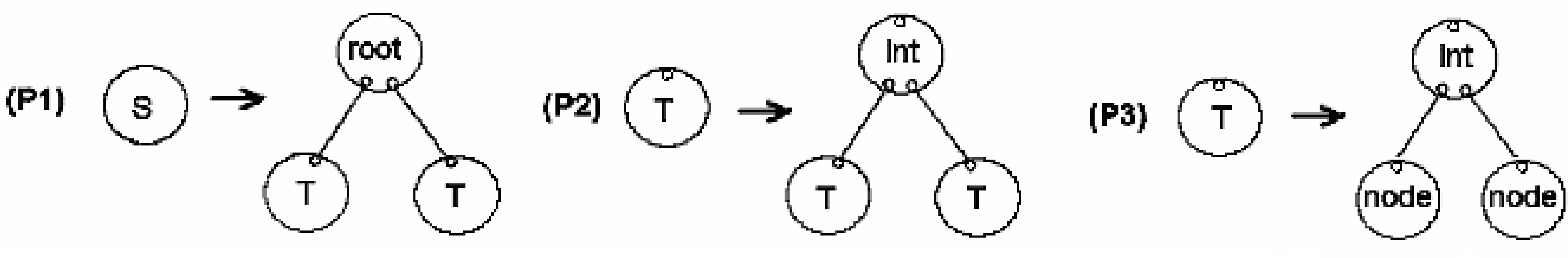
# GENERATION OF 1D ELIMINATION TREE

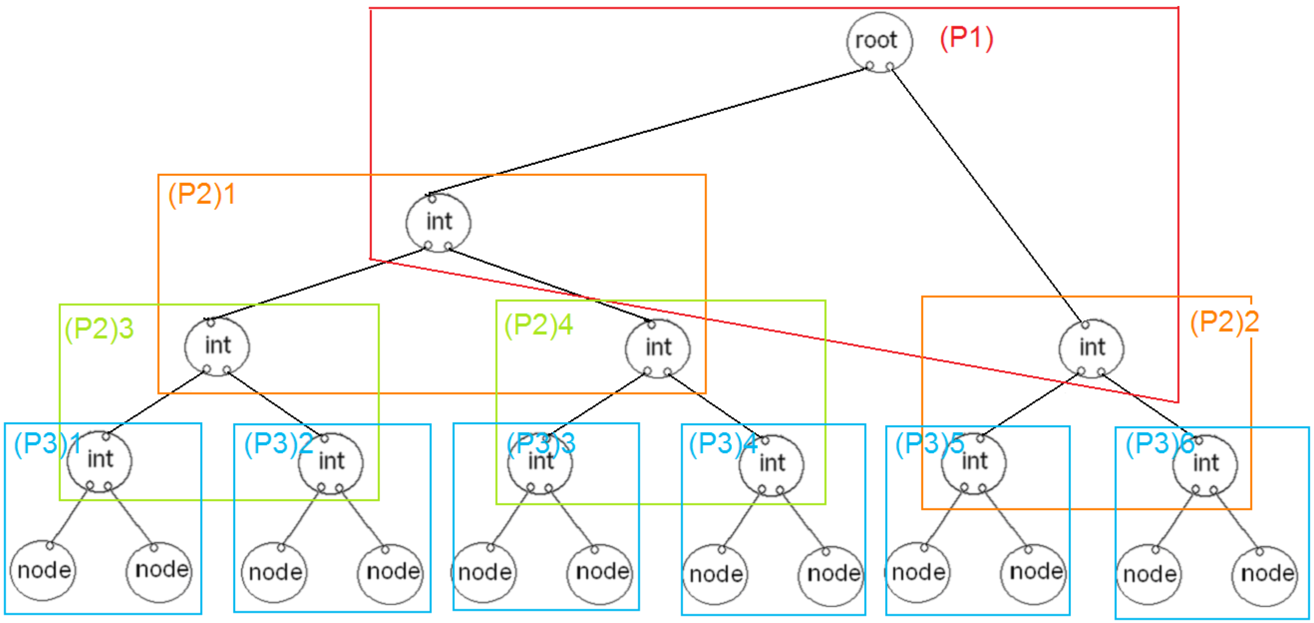



1D elimination tree obtained by executing productions (P1)-(P2)$^2$-(P2)$^2$-(P3)$^6$

# GRAPH GRAMMAR PRODUCTIONS AS ATOMIC TASKS




We assign indices to grammar productions in order to localize the places where the graph grammar productions were fired

$(P1)\text{-}(P2)_1\text{-}(P2)_2\text{-}(P2)_3\text{-}(P2)_4\text{-}(P3)_1\text{-}(P3)_2\text{-}(P3)_3\text{-}(P3)_4\text{-}(P3)_5\text{-}(P3)_6$

# TRACE THEORY BASED SCHEDULER

Alphabet:

$A = \{(P1), (P2)_1, (P2)_2, (P2)_3, (P2)_4, (P3)_1, (P3)_2, (P3)_3, (P3)_4, (P3)_5, (P3)_6\}$

Dependency relation for construction of the elimination tree

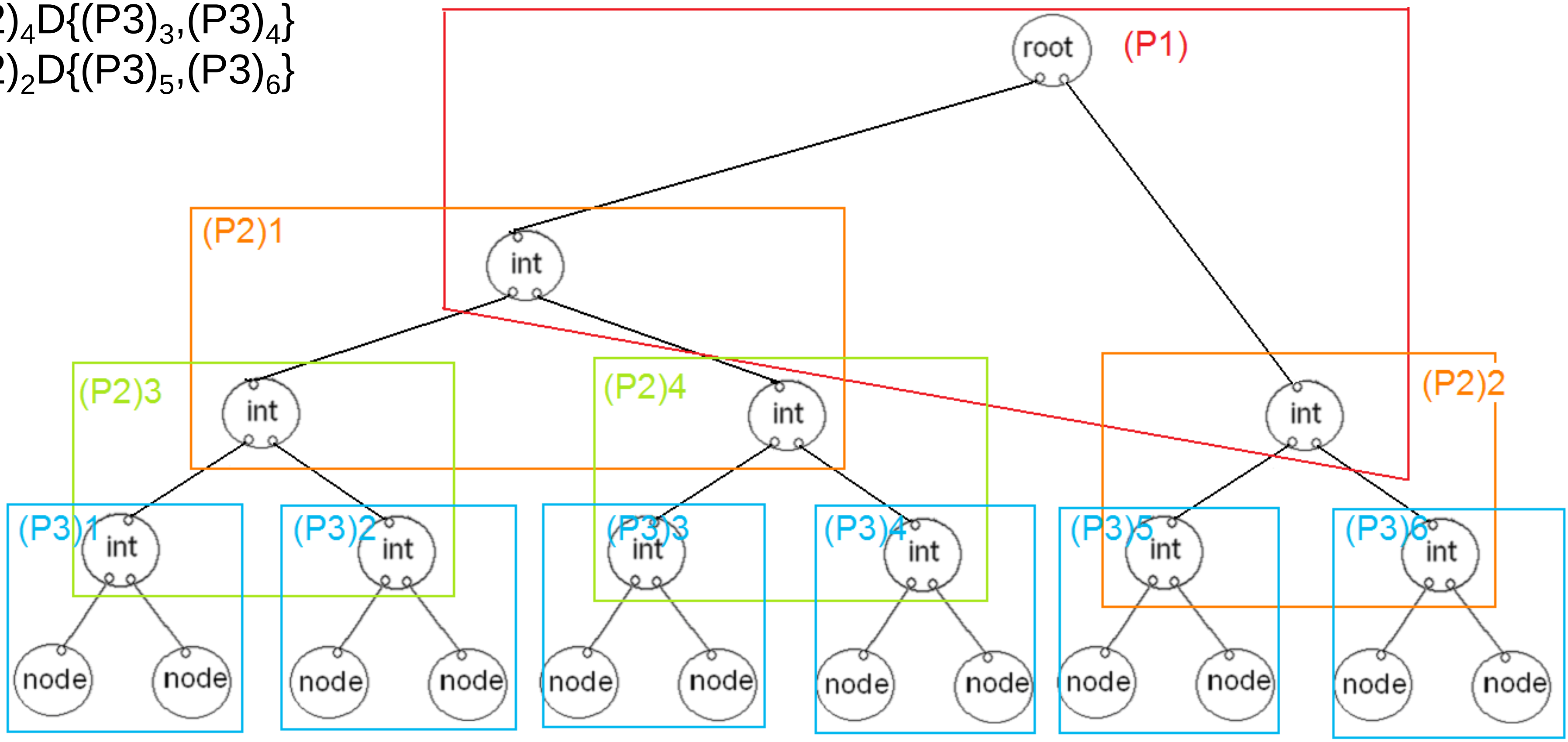
$(P1)D\{(P2)_1,(P2)_2\}$

$(P2)_1D\{(P2)_3,(P2)_4\}$

$(P2)_3D\{(P3)_1,(P3)_2\}$

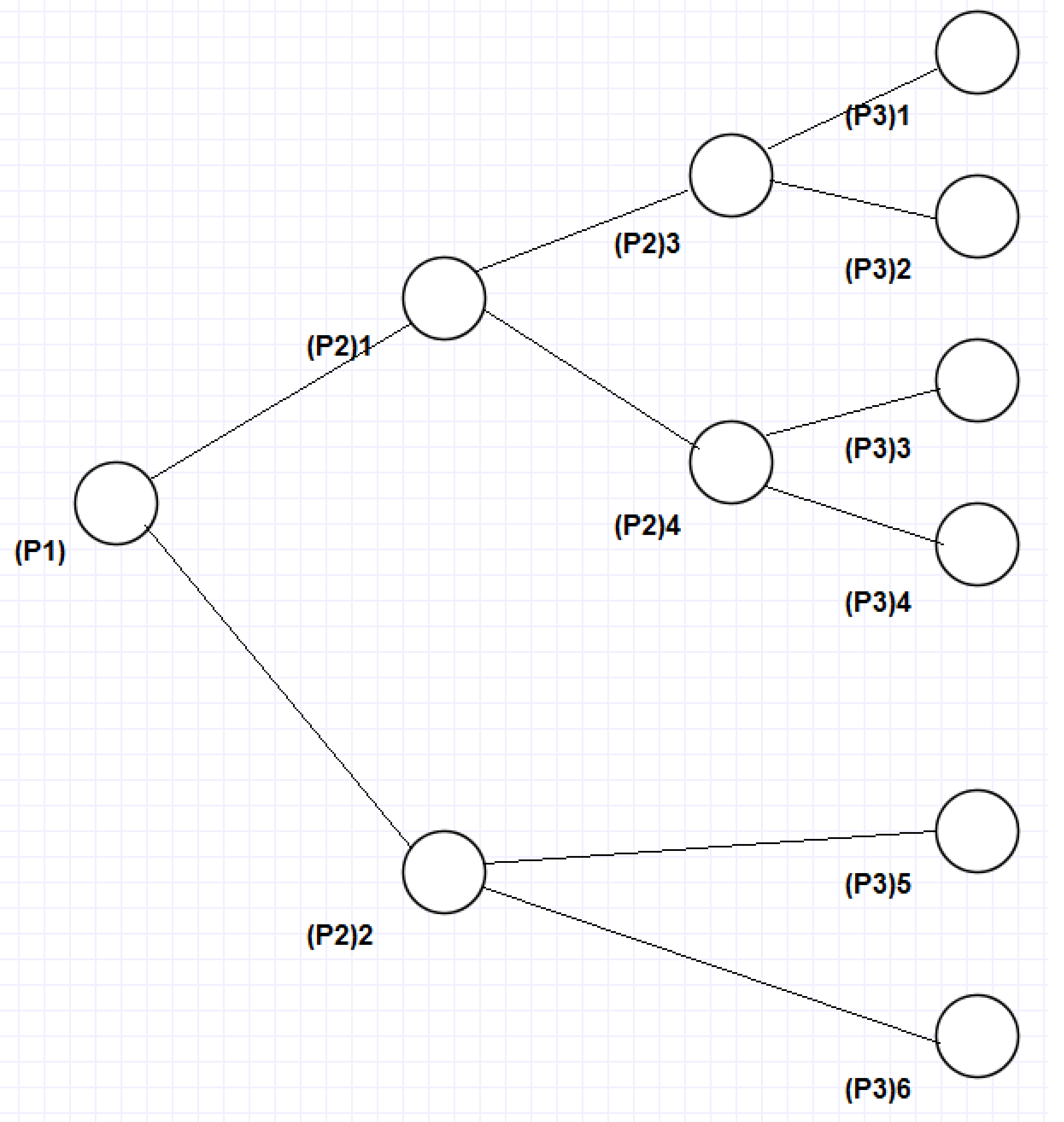
$(P2)_4D\{(P3)_3,(P3)_4\}$

$(P2)_2D\{(P3)_5,(P3)_6\}$



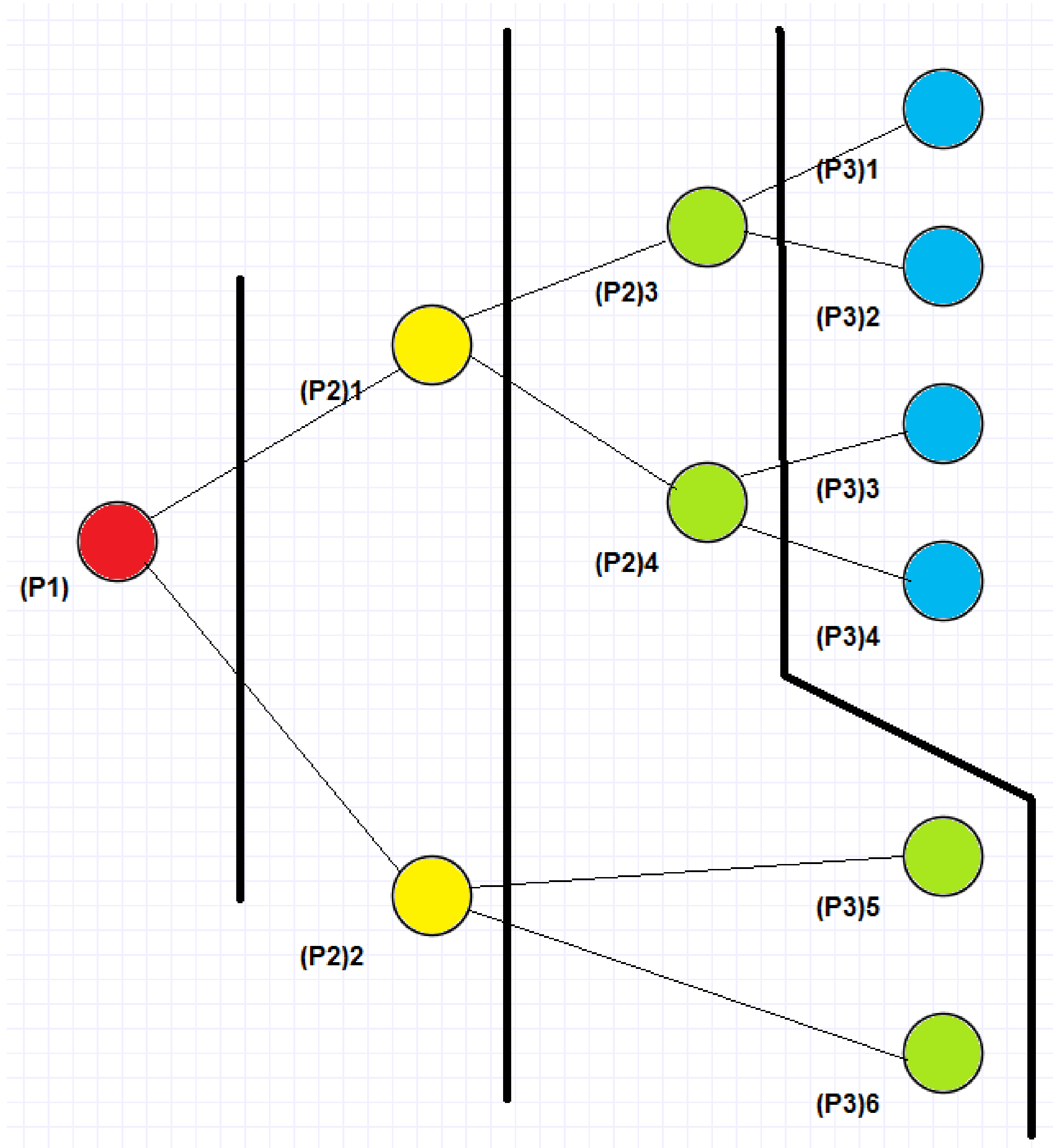
# TRACE THEORY BASED SCHEDULER

Dependency graph



# TRACE THEORY BASED SCHEDULER

Dependency graph



# TRACE THEORY BASED SCHEDULER

$(P1)$-$(P2)_1$-$(P2)_2$-$(P2)_3$-$(P2)_4$- $(P3)_1$-$(P3)_2$-$(P3)_3$-$(P3)_4$-$(P3)_5$-$(P3)_6$

Foata Normal Form

$$\left[a_1^1 a_2^1 \ldots a_{l_1}^1\right]\left[a_1^2 a_2^2 \ldots a_{l_1}^2\right]\ldots\left[a_1^n a_2^n \ldots a_{l_n}^n\right]$$

$a_i^k \in A$ (alphabet)

$\forall k \ \forall i, j \in \{1, \ldots, l_k\} \quad a_i^k I a_j^k$ i<>j where $I = AxA \backslash D$

$\forall k \ \forall i \in \{1, \ldots, l_k\} \ \exists j \in \{1, \ldots, l_{k-1}\} \quad a_i^{k-1} D a_j^k$

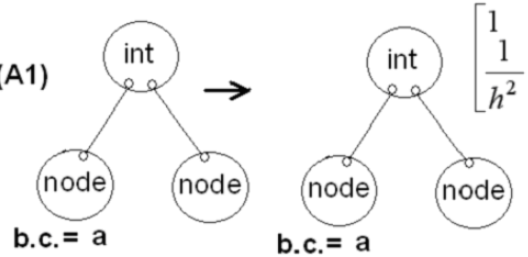
Scheduling according to Foata Normal Form:

$[(P1)][(P2)_1(P2)_2][(P2)_3(P2)_4(P3)_5(P3)_6][(P3)_1(P3)_2(P3)_3(P3)_4]$

Thus, the execution of the solver consists of several steps, where independent tasks are executed in concurrent, interchanged with the synchronization barriers.

# PROCESS OF THE ELIMINATION
# EXPRESSED BY GRAPH GRAMMAR PRODUCTIONS

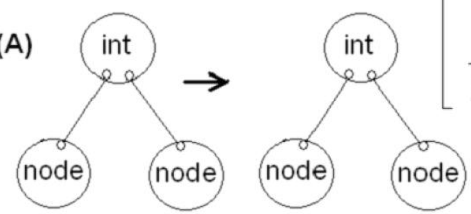
(A1)




$$\begin{bmatrix} 1 & 0 \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} a \\ 0 \end{Bmatrix}$$

Graph grammar production construction local matrix for the first sub-interval

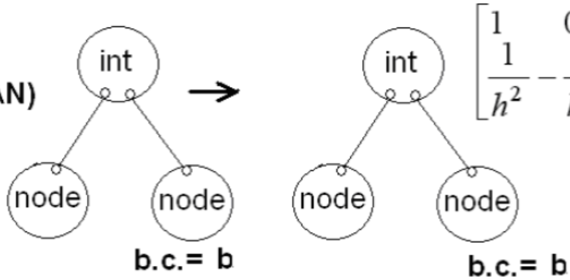
(A)




$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Graph grammar production construction local matrix for the i-th sub-interval

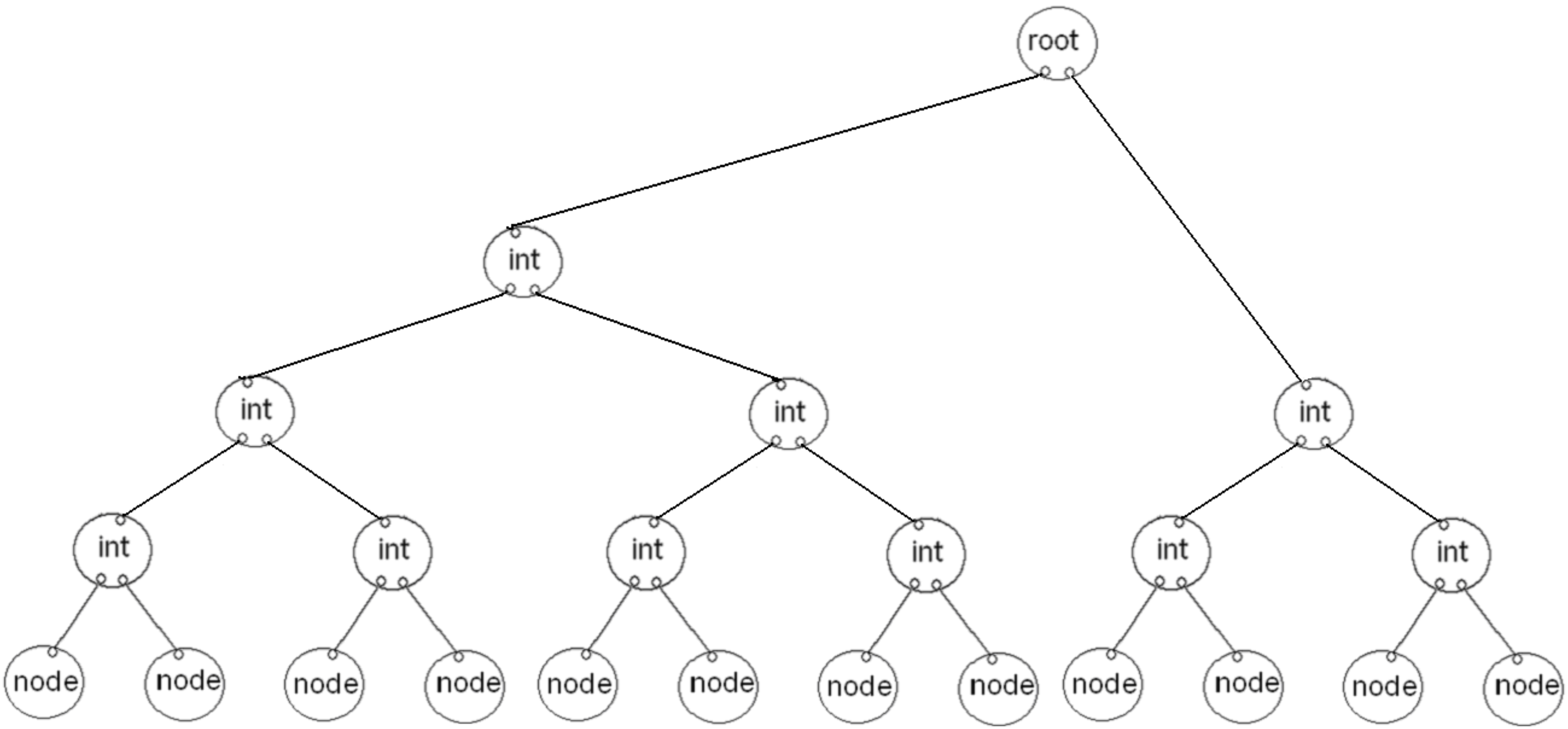
(AN)




$$\begin{bmatrix} 1 & 0 \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} b \\ 0 \end{Bmatrix}$$

Graph grammar production construction local matrix for the last sub-interval

# PROCESS OF THE ELIMINATION
# EXPRESSED BY GRAPH GRAMMAR PRODUCTIONS




$$\begin{bmatrix} 1 & 0 \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2^{II} \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_3^{II} \\ x_4^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_5^{II} \\ x_6^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_6^{II} \\ x_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20 \end{Bmatrix}$$

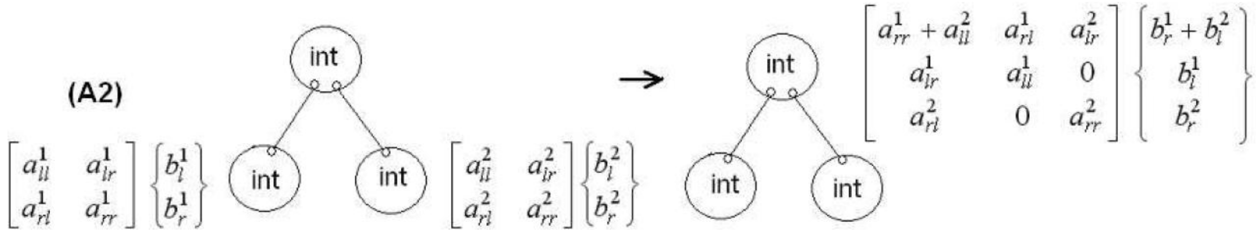
Generation of frontal matrices at leaves of the eliminaton tree expressed as the execution of graph grammar productions (A1)-$(A)^4$-(AN)

# PROCESS OF THE ELIMINATION EXPRESSED BY GRAPH GRAMMAR PRODUCTIONS

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{bmatrix} x_{i-1} \\ x_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \bigoplus \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{bmatrix} x_i \\ x_{i+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} =$$

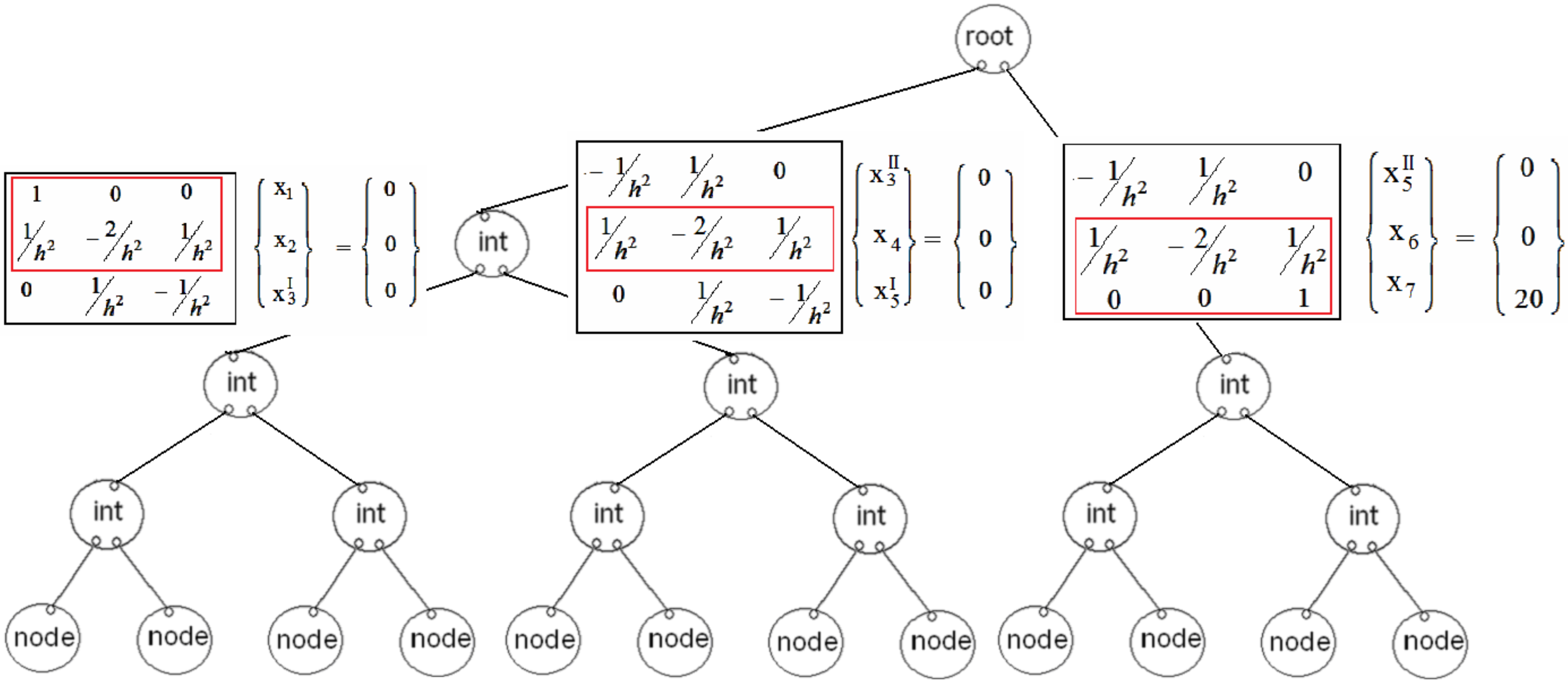
$$\begin{bmatrix} -\frac{2}{h^2} & \frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} & 0 \\ \frac{1}{h^2} & 0 & -\frac{1}{h^2} \end{bmatrix} \begin{bmatrix} x_i \\ x_{i-1} \\ x_{i+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Exemplary merging of two internal contributions




Graph grammar production expressing the merging process

# ASSEMBLING AT PARENT LEVEL




Expression of the solver execution by graph grammar productions

(A1)-$(A)^4$-(AN) (generation of frontal matrices at leaves of the elimination trees)

$(A2)^3$ (merging contributions at father nodes)

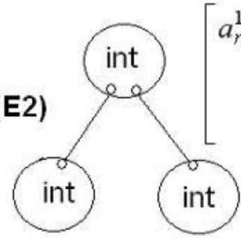
# PROCESS OF THE ELIMINATION
# EXPRESSED BY GRAPH GRAMMAR PRODUCTIONS

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{bmatrix} x_{i-1} \\ x_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \bigoplus \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{bmatrix} x_i \\ x_{i+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} =$$

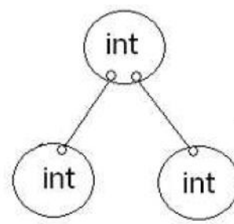
$$\begin{bmatrix} -\frac{2}{h^2} & \frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} & 0 \\ \frac{1}{h^2} & 0 & -\frac{1}{h^2} \end{bmatrix} \begin{bmatrix} x_i \\ x_{i-1} \\ x_{i+1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

After merging of the two internal contributions, $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ the i-th equation is fully assembled, and can be eliminated

**(E2)**




$$\begin{bmatrix} a^1_{rr} + a^2_{ll} & a^1_{rl} & a^2_{lr} \\ a^1_{lr} & a^1_{ll} & 0 \\ a^2_{rl} & 0 & a^2_{rr} \end{bmatrix} \begin{Bmatrix} b^1_r + b^2_l \\ b^1_l \\ b^2_r \end{Bmatrix} \rightarrow$$




$$\begin{bmatrix} 1 & \hat{a}_{12} & \hat{a}_{13} \\ 0 & \hat{a}_{ll} & \hat{a}_{lr} \\ 0 & \hat{a}_{rl} & \hat{a}_{rr} \end{bmatrix} \begin{Bmatrix} \hat{b} \\ \hat{b}_l \\ \hat{b}_r \end{Bmatrix}$$

Graph grammar production expressing the elimination process

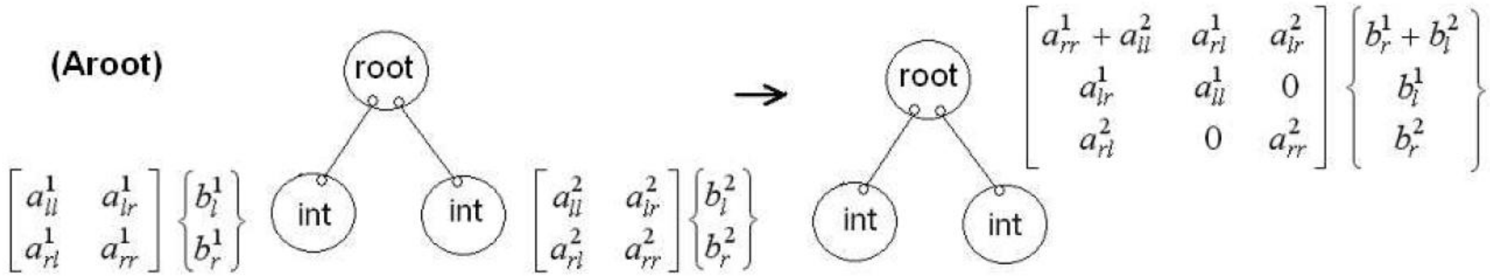
Expression of the solver execution by graph grammar productions

(A1)-(A)$^4$-(AN) (generation of frontal matrices at leaves of the elimination trees)

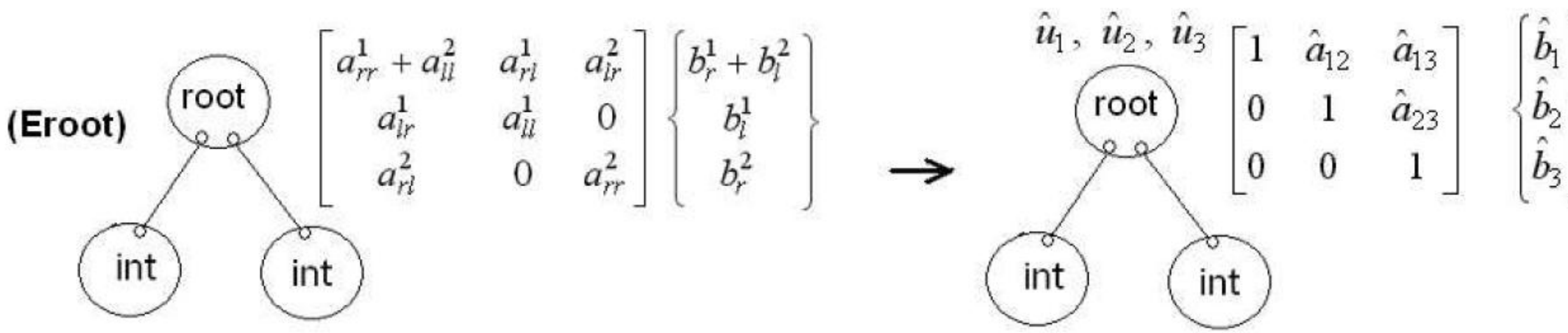
(A2)$^3$ (merging contributions at father nodes)

(E2)$^3$ (elimination of fully assembled nodes)

# PROCESS OF THE ELIMINATION
# EXPRESSED BY GRAPH GRAMMAR PRODUCTIONS




Finally, we reach the root of the elimination tree




At the root node, all three equations are fully assembled, and the local system can be solved now

# ELIMINATION OF FULLY ASSEMBLED NODES

Expression of the solver execution by graph grammar productions

(A1)-$(A)^4$-(AN) (generation of frontal matrices at leaves of the elimination trees)

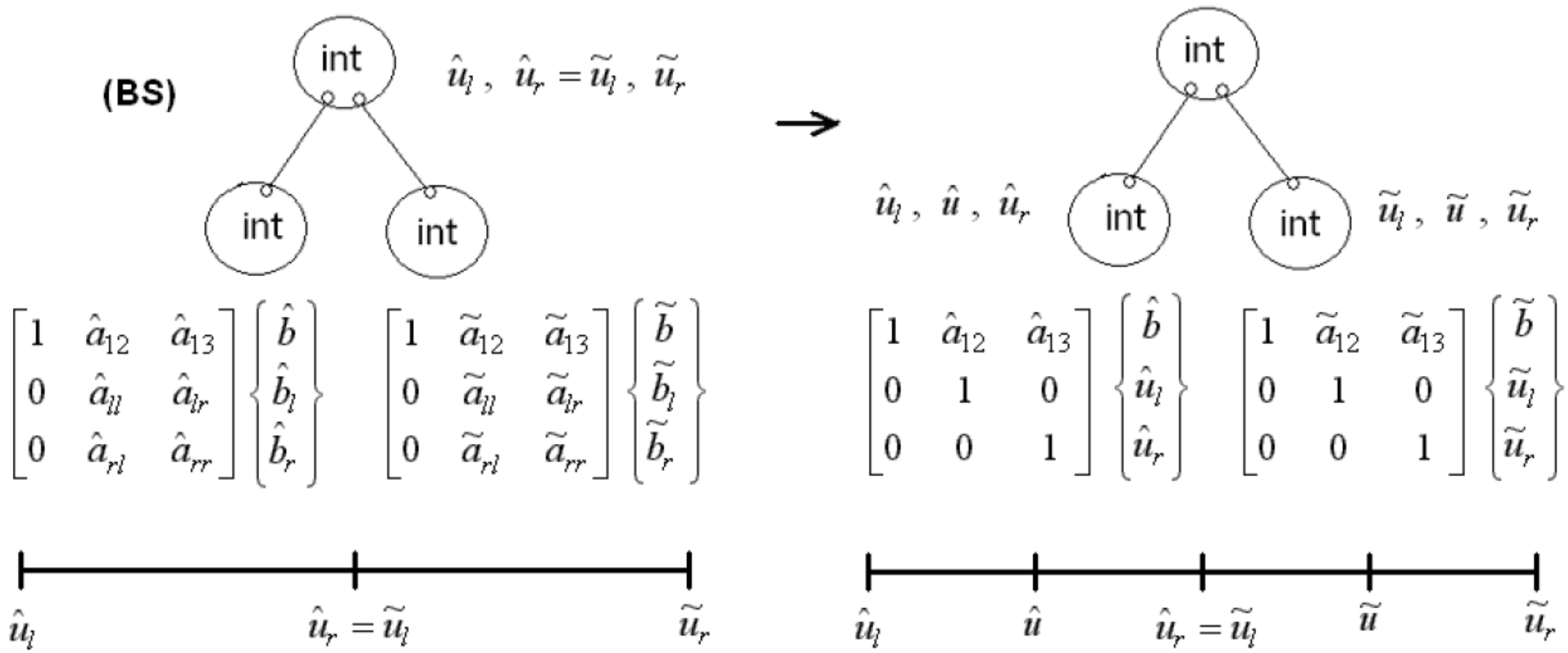
$(A2)^3$ (merging contributions at father nodes)

$(E2)^3$ (elimination of fully assembled nodes)

(A2) – (E2) (merging at parent node followed by elimination)

(Aroot) – (Eroot) (merging at root node followed by full forward elimination)

# PROCESS OF THE BACKWARD SUBSTITUTIONS EXPRESSED BY GRAPH GRAMMAR PRODUCTIONS




At the last stage of the solver execution, we execute partial backward substitutions

# TRACE THEORY BASED SCHEDULER

Alphabet:

A={(A1), $(A)_1$ , $(A)_2$ , $(A)_3$ , $(A)_4$ , (AN), $(A2)_1$ , $(A2)_2$ , $(A2)_3$ , $(E2)_1$ , $(E2)_2$ , $(E2)_3$ , $(A2)_4$ , $(E2)_4$ , (Aroot) , (Eroot) , $(BS)_1$ , $(BS)_2$ , $(BS)_3$ , $(BS)_4$ }

Dependency relation for the solver algorithm

{(A1),$(A)_1$}D$(A2)_1$

{$(A)_2$,$(A)_3$}D$(A2)_2$

{$(A)_4$,(AN)}D$(A2)_3$

$(A2)_1$D$(E2)_1$

$(A2)_2$D$(E2)_2$

$(A2)_3$D$(E2)_3$

{$(E2)_1$,$(E2)_2$}D$(A2)_4$

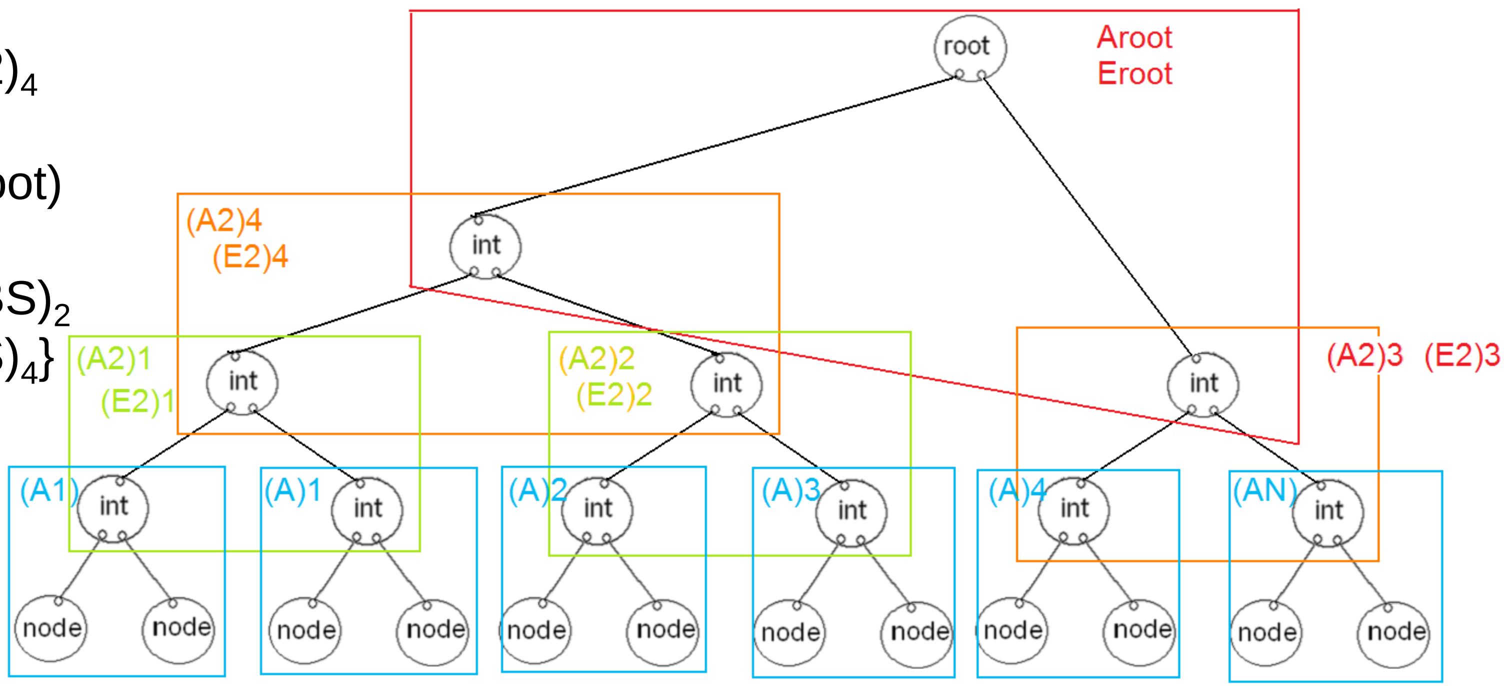
$(A2)_4$D$(E2)_4$

{$(E2)_3$$(E2)_4$}D(Aroot)

(Aroot)D(Eroot)

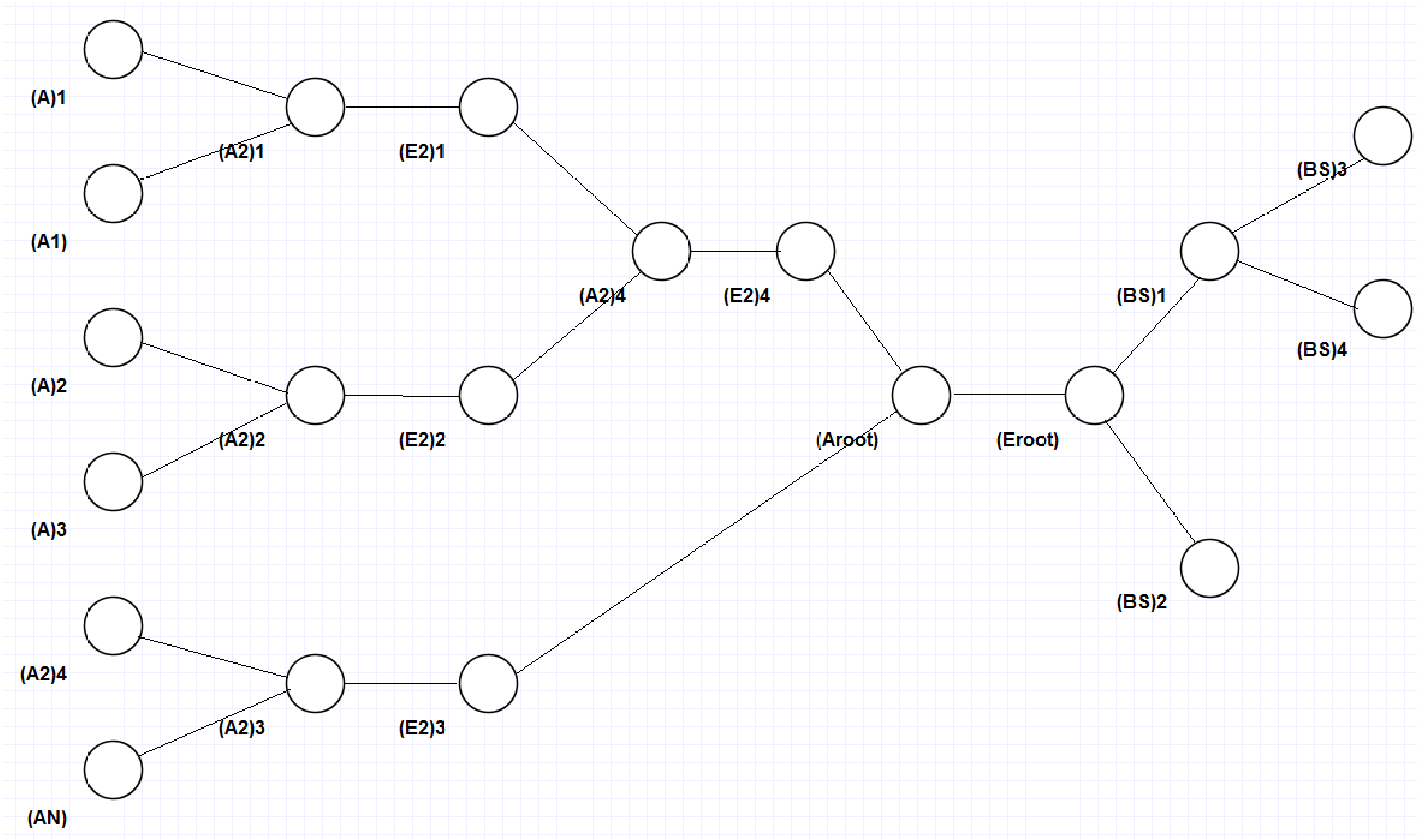
(Eroot)D{$(BS)_1$,$(BS)_2$

$(BS)_1$D{$(BS)_3$,$(BS)_4$}



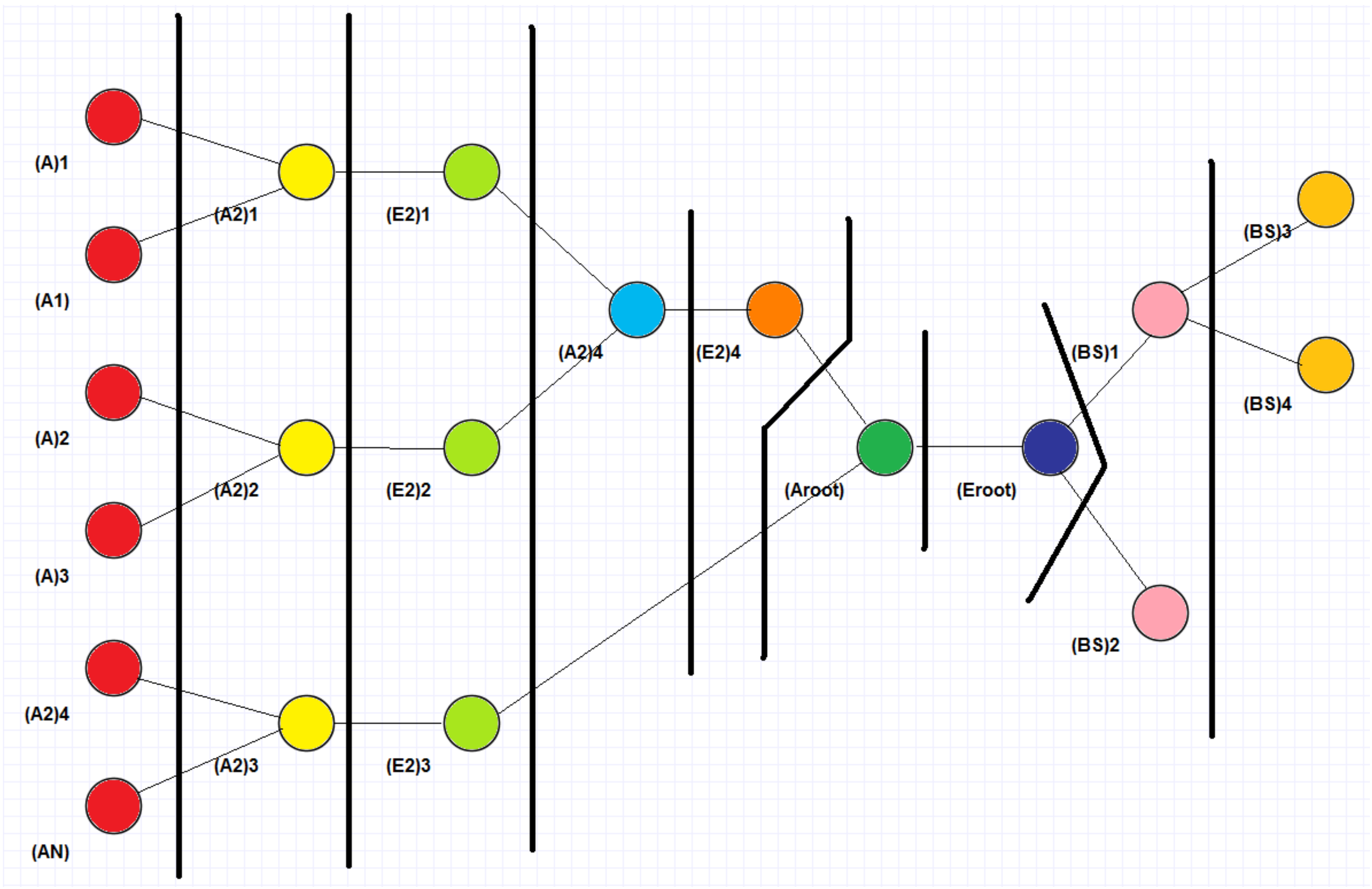
# TRACE THEORY BASED SCHEDULER

## Dependency graph



# TRACE THEORY BASED SCHEDULER

Dependency graph



# TRACE THEORY BASED SCHEDULER

(A1)-$(A)_1$-$(A)_2$-$(A)_3$-$(A)_4$- (AN)-$(A2)_1$-$(A2)_2$- $(A2)_3$-$(E2)_1$-$(E2)_2$-$(E2)_3$- $(A2)_4$- $(E2)_4$-
(Aroot)-(Eroot)-$(BS)_1$-$(BS)_2$-$(BS)_3$-$(BS)_4$

Foata Normal Form

$$\left[a_1^1 a_2^1 \ldots a_{l_1}^1\right]\left[a_1^2 a_2^2 \ldots a_{l_1}^2\right] \ldots \left[a_1^n a_2^n \ldots a_{l_n}^n\right]$$

$a_i^k \in A$ (alphabet)

$$\forall k\ \forall i, j \in \{1, \ldots, l_k\} \quad a_i^k I a_j^k$$

$$\forall k\ \forall i \in \{1, \ldots, l_k\}\ \exists j \in \{1, \ldots, l_{k-1}\} \quad a_i^{k-1} D a_j^k$$

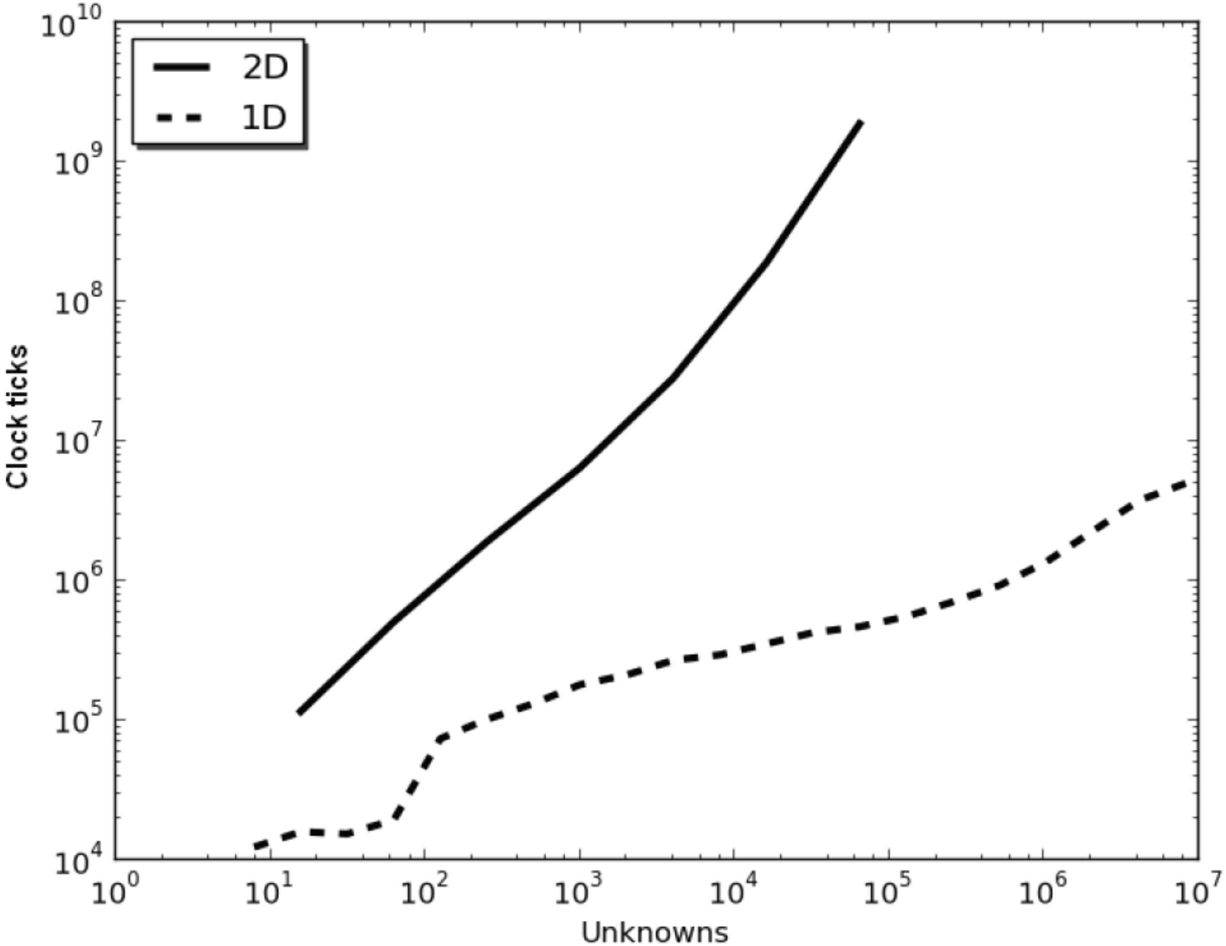
Scheduling according to Foata Normal Form:

[(A1)$(A)_1$$(A)_2$$(A)_3$$(A)_4$(AN)][$(A2)_1$$(A2)_2$$(A2)_3$][$(E2)_1$$(E2)_2$$(E2)_3$] [$(A2)_4$][$(E2)_4$]
[(Eroot)][(Aroot)][(Eroot)][$(BS)_1$$(BS)_2$][$(BS)_3$$(BS)_4$]

Thus, the execution of the solver consists of several steps, where independent tasks are executed in concurrent, interchanged with the synchronization barriers.

# NUMERICAL EXPERIMENTS

NVIDIA GeForce 8800 gt with 16 multiprocessors, each having 8 cores (128 cores total)




1D solver $O(\log N)$ 2D solver $O(N\log N)$

When the number of leaves $n$ is larger than number of processors, the execution time must be multiplied by $n/p$

# PAPERS

Paweł Obrok, Paweł Pierzchała, Arkadiusz Szymczak, Maciej Paszyński
**GRAPH GRAMMAR BASED MULTI-THREAD MULTI-FRONTAL PARALLEL SOLVER WITH THRACE THEORY BASED SCHEDULER**
***Procedia Computer Science,*** 1, 1 (2010) 1993-2001