Frontal and multi-frontal solvers: orderings, elimination trees, refinement trees

Maciej Paszynski

Department of Computer Science

AGH University of Science and Technology, Krakow, Poland

maciej.paszynski@agh.edu.pl

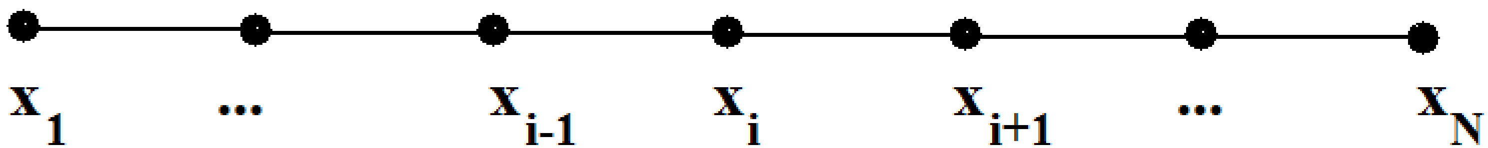
http://www.ki.agh.edu.pl/en/research-groups/a2s

EXEMPLARY SIMPLE 1D NUMERICAL PROBLEM

Find temperature distribution $R \ni x \mapsto u(x) \in R$ such that

$$\frac{d^2u}{dx^2} = 0 \text{ for } x \in [0,1] \qquad u(0) = 0 \qquad u(1) = 20$$

Finite difference disretization



$$u_0 = 0$$

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = 0 \text{ for } i = 1, ..., N-1$$

$$u_N = 20$$

$$h = \frac{1}{N}$$

$$u_i = u(x_i) = u\left(\frac{i}{N}\right)$$

$$[0,1] = \bigcup_{i=1}^{N} [x_{i-1}, x_i]$$

TRIDIAGONAL MATRIX FOR EXEMPLARY 1D PROBLEM

$$u_0 = 0$$

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = 0 \text{ for } i = 1, ..., N-1$$

$$u_N = 20$$

$$\begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & 0 \\ 1/h^2 & -2/h^2 & 1/h^2 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1/h^2 & -2/h^2 & 1/h^2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1/h^2 & -2/h^2 & 1/h^2 \\ 0 & 0 & \dots & 0 & \dots & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \dots \\ x_i \\ \dots \\ x_{N-1} \\ x_N \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ \dots \\ 0 \\ 20 \end{Bmatrix}$$

FRONTAL SOLVER

$$\begin{bmatrix} 1 & 0 & \ldots & 0 & \ldots & 0 & 0 \\ 1/h^2 & -2/h^2 & 1/h^2 & 0 & \ldots & 0 & 0 \\ 0 & 1/h^2 & -2/h^2 & 1/h^2 & 0 & \ldots & 0 \\ 0 & \ldots & 1/h^2 & -2/h^2 & 1/h^2 & \ldots & 0 \\ 0 & \ldots & 0 & 1/h^2 & -2/h^2 & 1/h^2 & 0 \\ 0 & 0 & \ldots & 0 & 1/h^2 & -2/h^2 & 1/h^2 \\ 0 & 0 & \ldots & 0 & \ldots & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \ldots \\ x_i \\ \ldots \\ x_{N-1} \\ x_N \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \ldots \\ 0 \\ \ldots \\ 0 \\ 20 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/h^2 & -2/h^2 & 1/h^2 \end{bmatrix}$$

Front matrix

Frontal matrix focuses on forward elimination of the first row

FRONTAL SOLVER

$$
\begin{bmatrix}
1 & 0 & \dots & 0 & \dots & 0 & 0 \\
1/h^2 & -2/h^2 & 1/h^2 & 0 & \dots & 0 & 0 \\
0 & 1/h^2 & -2/h^2 & 1/h^2 & 0 & \dots & 0 \\
0 & \dots & 1/h^2 & -2/h^2 & 1/h^2 & \dots & 0 \\
0 & \dots & 0 & 1/h^2 & -2/h^2 & 1/h^2 & 0 \\
0 & 0 & \dots & 0 & 1/h^2 & -2/h^2 & 1/h^2 \\
0 & 0 & \dots & 0 & \dots & 0 & 1
\end{bmatrix}
\begin{Bmatrix} x_1 \\ x_2 \\ \dots \\ x_i \\ \dots \\ x_{N-1} \\ x_N \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ \dots \\ 0 \\ 20 \end{Bmatrix}
$$

$$
\begin{matrix}
1 & 0 & 0 \\
1/h^2 & -2/h^2 & 1/h^2
\end{matrix}
$$

Front matrix

2^{nd} row = 2^{nd} row – $1/h^2$ * 1^{st} row

FRONTAL SOLVER

$$\begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & -2/h^2 & 1/h^2 & 0 & \dots & 0 & 0 \\ 0 & 1/h^2 & -2/h^2 & 1/h^2 & 0 & \dots & 0 \\ 0 & \dots & 1/h^2 & -2/h^2 & 1/h^2 & \dots & 0 \\ 0 & \dots & 0 & 1/h^2 & -2/h^2 & 1/h^2 & 0 \\ 0 & 0 & \dots & 0 & 1/h^2 & -2/h^2 & 1/h^2 \\ 0 & 0 & \dots & 0 & \dots & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \dots \\ x_i \\ \dots \\ x_{N-1} \\ x_N \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ \dots \\ 0 \\ 20 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2/h^2 & 1/h^2 \end{bmatrix}$$

Front matrix

2nd row = 2nd row – $1/h^2$ * 1st row

FRONTAL SOLVER

$$\begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & -2/h^2 & 1/h^2 & 0 & \dots & 0 & 0 \\ 0 & 1/h^2 & -2/h^2 & 1/h^2 & 0 & \dots & 0 \\ 0 & \dots & 1/h^2 & -2/h^2 & 1/h^2 & \dots & 0 \\ 0 & \dots & 0 & 1/h^2 & -2/h^2 & 1/h^2 & 0 \\ 0 & 0 & \dots & 0 & 1/h^2 & -2/h^2 & 1/h^2 \\ 0 & 0 & \dots & 0 & \dots & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \dots \\ x_i \\ \dots \\ x_{N-1} \\ x_N \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ \dots \\ 0 \\ 20 \end{Bmatrix}$$

$$\begin{bmatrix} -2/h^2 & 1/h^2 & 0 \\ 1/h^2 & -2/h^2 & 1/h^2 \end{bmatrix}$$

Front matrix

The first row has been eliminated

Frontral matrix focuses on the elimination of the second row

FRONTAL SOLVER

$$\begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & -2/h^2 & 1/h^2 & 0 & \dots & 0 & 0 \\ 0 & 1/h^2 & -2/h^2 & 1/h^2 & 0 & \dots & 0 \\ 0 & \dots & 1/h^2 & -2/h^2 & 1/h^2 & \dots & 0 \\ 0 & \dots & 0 & 1/h^2 & -2/h^2 & 1/h^2 & 0 \\ 0 & 0 & \dots & 0 & 1/h^2 & -2/h^2 & 1/h^2 \\ 0 & 0 & \dots & 0 & \dots & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \dots \\ x_i \\ \dots \\ x_{N-1} \\ x_N \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ \dots \\ 0 \\ 20 \end{Bmatrix}$$

$$\begin{bmatrix} -2/h^2 & 1/h^2 & 0 \\ 1/h^2 & -2/h^2 & 1/h^2 \end{bmatrix}$$

Front matrix

3rd row = 3nd row – [1/h^2] / [-2/h^2] * 2nd row

FRONTAL SOLVER

$$
\begin{bmatrix}
1 & 0 & \dots & 0 & \dots & 0 & 0 \\
0 & -\frac{2}{h^2} & \frac{1}{h^2} & 0 & \dots & 0 & 0 \\
0 & 0 & -\frac{3}{2h^2} & \frac{1}{h^2} & 0 & \dots & 0 \\
0 & \dots & \frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} & \dots & 0 \\
0 & \dots & 0 & \frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} & 0 \\
0 & 0 & \dots & 0 & \frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} \\
0 & 0 & \dots & 0 & \dots & 0 & 1
\end{bmatrix}
\begin{Bmatrix} x_1 \\ x_2 \\ \dots \\ x_i \\ \dots \\ x_{N-1} \\ x_N \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ \dots \\ 0 \\ 20 \end{Bmatrix}
$$

$$
\begin{bmatrix}
-\frac{2}{h^2} & \frac{1}{h^2} & 0 \\
0 & -\frac{3}{2h^2} & \frac{1}{h^2}
\end{bmatrix}
$$

Front matrix

3rd row = 3nd row – [1/h^2] / [-2/h^2] * 2nd row

ELEMENT-WISE DECOMPOSITION

$$\begin{bmatrix} 1 & 0 & \dots & 0 & \dots & 0 & 0 \\ \frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} & 0 & \dots & 0 \\ 0 & \dots & \frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} & \dots & 0 \\ 0 & \dots & 0 & \frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} & 0 \\ 0 & 0 & \dots & 0 & \frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} \\ 0 & 0 & \dots & 0 & \dots & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ \dots \\ x_i \\ \dots \\ x_6 \\ x_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ \dots \\ 0 \\ 20 \end{Bmatrix}$$

Construct multiple frontal matrices in such a way so they sum up to the full matrix

Variables must be split into parts

$$x_i = x_i^{I} + x_i^{II}$$

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2^{II} \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_3^{II} \\ x_4^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_5^{II} \\ x_6^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_6^{II} \\ x_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20 \end{Bmatrix}$$

MULTI-FRONTAL SOLVER ALGORITHM

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2^{II} \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_3^{II} \\ x_4^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_5^{II} \\ x_6^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_6^{II} \\ x_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20 \end{Bmatrix}$$

First all frontal matrices are constructed

MULTI-FRONTAL SOLVER ALGORITHM

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} \\ 0 & \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

+

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2^{II} \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_3^{II} \\ x_4^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_5^{II} \\ x_6^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_6^{II} \\ x_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20 \end{Bmatrix}$$

First and second frontal matrices are sum up into new 3x3 frontal matrix

Its first and second rows are fully assembled

MULTI-FRONTAL SOLVER ALGORITHM

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} \\ 0 & \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

+

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2^{II} \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_3^{II} \\ x_4^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_5^{II} \\ x_6^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_6^{II} \\ x_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20 \end{Bmatrix}$$

First column is eliminated by using first row

2nd row = 2nd row – $[1/h^2]$ * 1st row

MULTI-FRONTAL SOLVER ALGORITHM

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{2}{h^2} & \frac{1}{h^2} \\ 0 & \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

+

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2^{II} \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_3^{II} \\ x_4^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_5^{II} \\ x_6^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_6^{II} \\ x_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20 \end{Bmatrix}$$

2^{nd} row = 2^{nd} row – $[1/h^2]$ * 1^{st} row

MULTI-FRONTAL SOLVER ALGORITHM

$$\begin{bmatrix} -\frac{2}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2 \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} & 0 \\ \frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} \\ 0 & \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_3^{II} \\ x_4 \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

+ +

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2^{II} \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_3^{II} \\ x_4^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_5^{II} \\ x_6^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_6^{II} \\ x_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20 \end{Bmatrix}$$

Third and fourth frontal matrices are sum up into new 3x3 frontal matrix

Now only the second row is fully assembled

MULTI-FRONTAL SOLVER ALGORITHM

$$\begin{bmatrix} -\frac{2}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2 \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{2}{h^2} & \frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} & 0 \\ \frac{1}{h^2} & 0 & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4 \\ x_3^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

+ +

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2^{II} \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_3^{II} \\ x_4^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_5^{II} \\ x_6^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_6^{II} \\ x_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20 \end{Bmatrix}$$

Change of the ordering

MULTI-FRONTAL SOLVER ALGORITHM

$$\begin{bmatrix} -\frac{2}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2 \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \begin{bmatrix} -\frac{2}{h^2} & \frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} & 0 \\ \frac{1}{h^2} & 0 & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4 \\ x_3^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

+ +

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2^{II} \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_3^{II} \\ x_4^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_5^{II} \\ x_6^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_6^{II} \\ x_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20 \end{Bmatrix}$$

Eliminate entries below the diagonal

2nd row = 2nd row – [1/h^2] / [-2/h2] * 1st row

3rd row = 3rd row – [1/h2] / [-2/h2] * 1st row

MULTI-FRONTAL SOLVER ALGORITHM

Why can I substract fully assembled 1st row from not fully assemled rows 2nd and 3rd ?

$$\begin{bmatrix} -\frac{2}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2 \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \begin{bmatrix} -\frac{2}{h^2} & \frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} & 0 \\ \frac{1}{h^2} & 0 & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4 \\ x_3^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2^{II} \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_3^{II} \\ x_4^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_5^{II} \\ x_6^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_6^{II} \\ x_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20 \end{Bmatrix}$$

Eliminate entries below the diagonal

2nd row = 2nd row – $[1/h^2]$ / [-2/h2] * 1st row

3rd row = 3rd row – [1/h2] / [-2/h2] * 1st row

MULTI-FRONTAL SOLVER ALGORITHM

This is because substraction and addition are interchangable (now I am substructing from the 2nd not fully assembled row, and later I will add the remaining part)

Moreover the 1st row which is utilized for the substractions in other columns contains only zeros

$$\begin{bmatrix} -\frac{2}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2 \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \begin{bmatrix} -\frac{2}{h^2} & \frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} & 0 \\ \frac{1}{h^2} & 0 & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4 \\ x_3^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

+ +

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2^{II} \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_3^{II} \\ x_4^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_5^{II} \\ x_6^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_6^{II} \\ x_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20 \end{Bmatrix}$$

Eliminate entries below the diagonal

2nd row = 2nd row – [1/h^2] / [-2/h2] * 1st row

3rd row = 3rd row – [1/h2] / [-2/h2] * 1st row

MULTI-FRONTAL SOLVER ALGORITHM

$$\begin{bmatrix} -\frac{2}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2 \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \begin{bmatrix} -\frac{2}{h^2} & \frac{1}{h^2} & \frac{1}{h^2} \\ 0 & -\frac{1}{h^2} & \frac{1}{2h^2} \\ 0 & \frac{1}{2h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4 \\ x_3^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

+ +

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2^{II} \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_3^{II} \\ x_4^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_5^{II} \\ x_6^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_6^{II} \\ x_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20 \end{Bmatrix}$$

Eliminate entries below the diagonal

2nd row = 2nd row – [1/h^2] / [-2/h2] * 1st row

3rd row = 3rd row – [1/h2] / [-2/h2] * 1st row

MULTI-FRONTAL SOLVER ALGORITHM

$$\begin{bmatrix} -\frac{2}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2 \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{2h^2} \\ \frac{1}{2h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_3^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} & 0 \\ \frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_5^{II} \\ x_6 \\ x_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 20 \end{Bmatrix}$$

+ + +

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2^{II} \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_3^{II} \\ x_4^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_5^{II} \\ x_6^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_6^{II} \\ x_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20 \end{Bmatrix}$$

Fifth and sixth frontal matrices are sum up into new 3x3 frontal matrix
Now the second and third rows are fully assembled

…

MULTI-FRONTAL SOLVER ALGORITHM

root

+

node

+

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{2}{h^2} & \frac{1}{h^2} \\ 0 & \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{2}{h^2} & \frac{1}{h^2} & \frac{1}{h^2} \\ 0 & -\frac{1}{h^2} & \frac{1}{2h^2} \\ 0 & \frac{1}{2h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4 \\ x_3^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{2}{h^2} & \frac{1}{h^2} \\ 0 & -\frac{1}{h^2} & \frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_7 \\ x_6 \\ x_5^{II} \end{Bmatrix} = \begin{Bmatrix} 20 \\ 0 \\ 0 \end{Bmatrix}$$

+ **+** **+**

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2^{II} \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_3^{II} \\ x_4^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_5^{II} \\ x_6^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_6^{II} \\ x_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20 \end{Bmatrix}$$

Continue until root node

PARALLEL MULTI-FRONTAL SOLVER ALGORITHM

$$\begin{bmatrix} 1 & 0 \\ 1/h^2 & -1/h^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Procesor 1

$$\begin{bmatrix} -1/h^2 & 1/h^2 \\ 1/h^2 & -1/h^2 \end{bmatrix} \begin{Bmatrix} x_2^{II} \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Procesor 2

$$\begin{bmatrix} -1/h^2 & 1/h^2 \\ 1/h^2 & -1/h^2 \end{bmatrix} \begin{Bmatrix} x_3^{II} \\ x_4^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Procesor 3

$$\begin{bmatrix} -1/h^2 & 1/h^2 \\ 1/h^2 & -1/h^2 \end{bmatrix} \begin{Bmatrix} x_4^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Procesor 4

$$\begin{bmatrix} -1/h^2 & 1/h^2 \\ 1/h^2 & -1/h^2 \end{bmatrix} \begin{Bmatrix} x_5^{II} \\ x_6^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

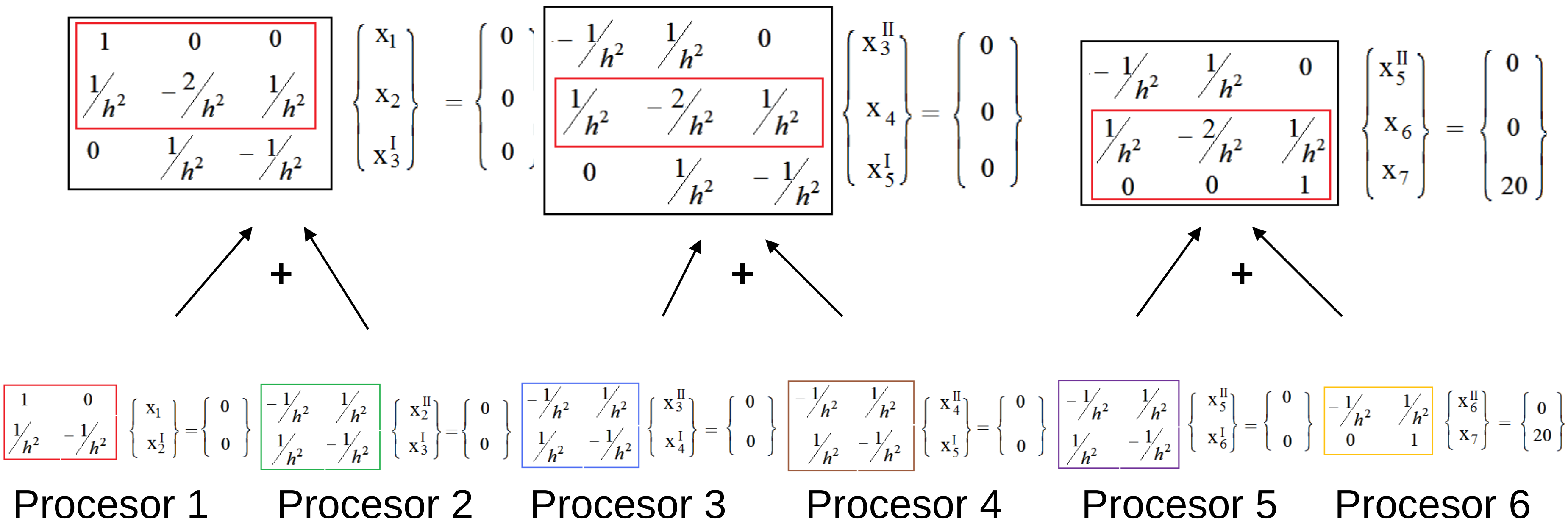
Procesor 5

$$\begin{bmatrix} -1/h^2 & 1/h^2 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_6^{II} \\ x_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20 \end{Bmatrix}$$

Procesor 6

All frontal matrices are generated at the same time

PARALLEL MULTI-FRONTAL SOLVER ALGORITHM



Summing up and elimination are executed at the same time over different pairs of frontal matrices

PARALLEL MULTI-FRONTAL SOLVER ALGORITHM

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{2}{h^2} & \frac{1}{h^2} \\ 0 & \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$+$

$$\begin{bmatrix} -\frac{2}{h^2} & \frac{1}{h^2} & \frac{1}{h^2} \\ 0 & -\frac{1}{h^2} & \frac{1}{2h^2} \\ 0 & \frac{1}{2h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4 \\ x_3^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$+$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{2}{h^2} & \frac{1}{h^2} \\ 0 & -\frac{1}{h^2} & \frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_7 \\ x_6 \\ x_5^{II} \end{Bmatrix} = \begin{Bmatrix} 20 \\ 0 \\ 0 \end{Bmatrix}$$

$+$

Procesor 1: $$\begin{bmatrix} 1 & 0 \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Procesor 2: $$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2^{II} \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Procesor 3: $$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_3^{II} \\ x_4^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

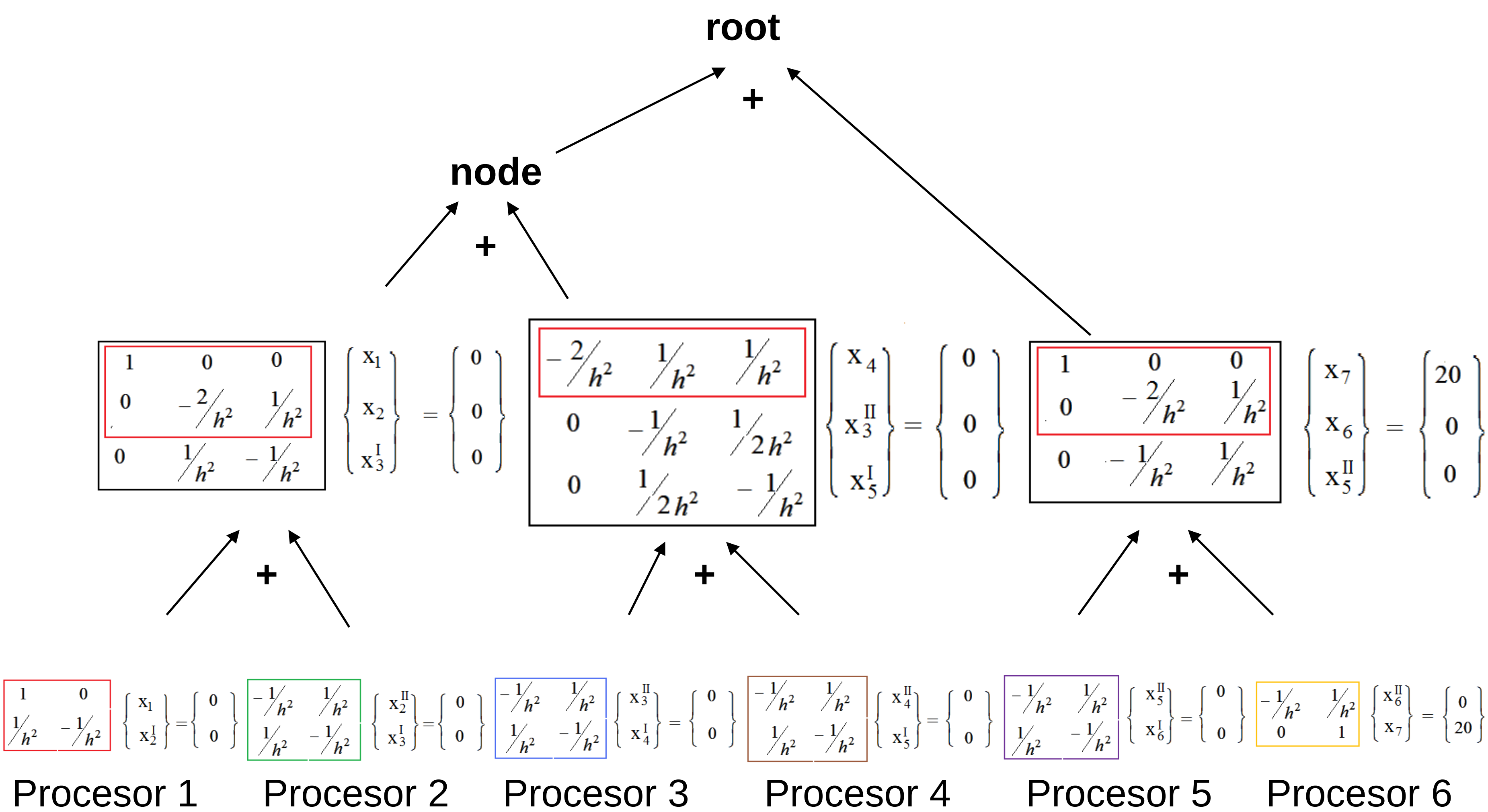
Procesor 4: $$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Procesor 5: $$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_5^{II} \\ x_6^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Procesor 6: $$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_6^{II} \\ x_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20 \end{Bmatrix}$$

Summing up and elimination are executed at the same time over different pairs of frontal matrices

PARALLEL MULTI-FRONTAL SOLVER ALGORITHM



The agorithm is recursively repeated until we reach the root of the tree

The algorithm results in upper trianguler matrix stored in distributed manner

Computational complexity = height of the tree = logN (where N = #unknowns-1)

GENERALIZATION TO 1D FINITE ELEMENT METHOD

Strong formulation

Find $u \in C^2(0,l)$ such that

$$-\left(a(x)u'(x)\right)' + b(x)u'(x) + c(x)u(x) = f(x)$$

$$u(0) = 0$$

$$a(l)u'(l) + \beta u(l) = \gamma$$

Weak formulation

Find $u \in V$ such that

$$\underbrace{\int_0^l \{au'v' + bu'v + cuv\}dx + \beta u(l)v(l)}_{b(u,v)} = \underbrace{\int_0^l f\,v\,dx + \gamma v(l)}_{l(v)}$$

$$\forall v \in V = \left\{v \in H^1(0,l) : v(0) = 0\right\}$$

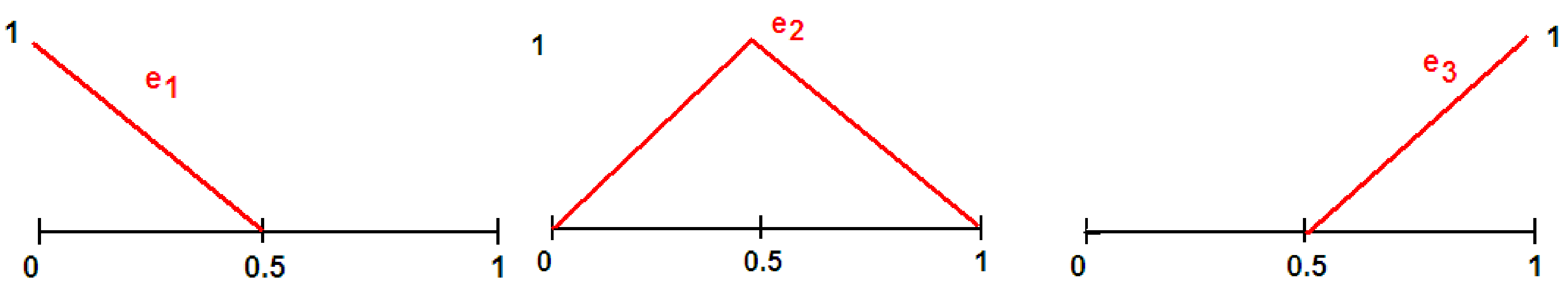
GENERALIZATION TO 1D FINITE ELEMENT METHOD

Finite element method disretization

$$u = \sum_{i=1}^{N} a_i e_i \qquad v = e_j \qquad \sum_{i=1}^{N} a_i b(e_i, e_j) = l(e_j) \quad j = 1,\ldots,N$$

$$b(e_i, e_j) = \int_0^l \{a e_i' e_j' + b e_i' e_j + c e_i e_j\} dx + \beta e_i(l) e_j(l) \qquad l(e_j) = \int_0^l f\, e_j\, dx + \gamma e_j(l)$$

Exemplary basis functions for [0,l] = [0,1], for two finite elements



$$e_1(x) = \begin{cases} 1 - 2x & for\ x \in (0, 0.5) \\ 0 & for\ x \in (0.5, 1) \end{cases} \qquad e_2(x) = \begin{cases} 2x & for\ x \in (0, 0.5) \\ 2 - 2x & for\ x \in (0.5, 1) \end{cases} \qquad e_3(x) = \begin{cases} 0 & for\ x \in (0, 0.5) \\ 2x - 1 & for\ x \in (0.5, 1) \end{cases}$$

$$\frac{de_1(x)}{dx} = \begin{cases} -2 & for\ x \in (0, 0.5) \\ 0 & for\ x \in (0.5, 1) \end{cases} \qquad \frac{de_2(x)}{dx} = \begin{cases} 2 & for\ x \in (0, 0.5) \\ -2 & for\ x \in (0.5, 1) \end{cases} \qquad \frac{de_3(x)}{dx} = \begin{cases} 0 & for\ x \in (0, 0.5) \\ 2 & for\ x \in (0.5, 1) \end{cases}$$

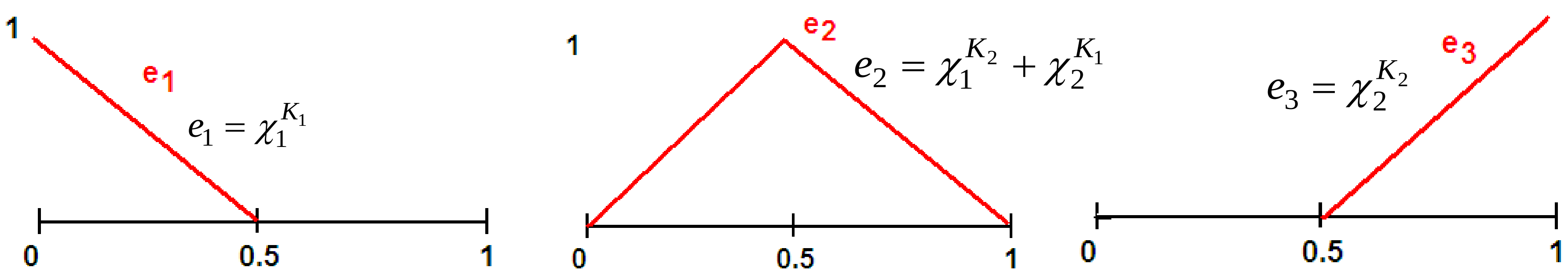
GENERALIZATION TO 1D FINITE ELEMENT METHOD

$$u=\sum_{i=1}^{N} a_i\, e_i \qquad v=e_j$$

$$\begin{bmatrix} b(e_1,e_1) & b(e_2,e_1) & b(e_3,e_1) \\ b(e_1,e_2) & b(e_2,e_2) & b(e_3,e_2) \\ b(e_1,e_3) & b(e_2,e_3) & b(e_3,e_3) \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} l(e_1) \\ l(e_2) \\ l(e_3) \end{Bmatrix}$$

$$b(e_i,e_j)=\int_0^l \{a e_i' e_j' + b e_i' e_j + c e_i e_j\}dx + \beta\, e_i(l) e_j(l) \qquad l(e_j)=\int_0^l f\, e_j\, dx + \gamma\, e_j(l)$$

Exemplary basis functions for [0,l] = [0,1], for two finite elements



$$b(e_1,e_3)=\int_0^1 \{a e_1' e_3' + b e_1' e_3 + c e_1 e_3\}dx + \beta\, e_1(l) e_3(l) = 0$$

$$b(e_1,e_2)=\int_0^1 \{a e_1' e_2' + b e_1' e_2 + c e_1 e_2\}dx + \beta\, e_1(l) e_1(l) = \int_0^{0.5} \left\{ a \frac{d\chi_1^{K_1}}{dx} \frac{d\chi_2^{K_1}}{dx} + b \frac{d\chi_1^{K_1}}{dx} \chi_2^{K_1} + c \chi_1^{K_1} \chi_2^{K_1} \right\} dx$$

$$b(e_2,e_3)=\int_0^1 \{a e_2' e_3' + b e_2' e_3 + c e_2 e_3\}dx + \beta\, e_2(l) e_3(l) = \int_{0.5}^{1} \left\{ a \frac{d\chi_1^{K_2}}{dx} \frac{d\chi_2^{K_2}}{dx} + b \frac{d\chi_1^{K_2}}{dx} \chi_2^{K_2} + c \chi_1^{K_2} \chi_2^{K_2} \right\} dx$$

GENERALIZATION TO 1D FINITE ELEMENT METHOD

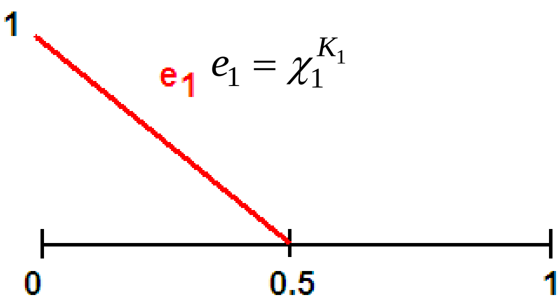
$$u = \sum_{i=1}^{N} a_i e_i \qquad v = e_j$$

$$\begin{bmatrix} b(e_1,e_1) & b(e_2,e_1) & b(e_3,e_1) \\ b(e_1,e_2) & b(e_2,e_2) & b(e_3,e_2) \\ b(e_1,e_3) & b(e_2,e_3) & b(e_3,e_3) \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} l(e_1) \\ l(e_2) \\ l(e_3) \end{Bmatrix}$$

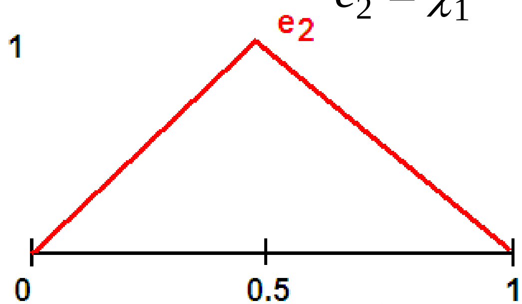
$$b(e_i, e_j) = \int_0^l \{a e_i' e_j' + b e_i' e_j + c e_i e_j\} dx + \beta e_i(l) e_j(l) \qquad l(e_j) = \int_0^l f e_j \, dx + \gamma e_j(l)$$

Exemplary basis functions for [0,l] = [0,1], for two finite elements

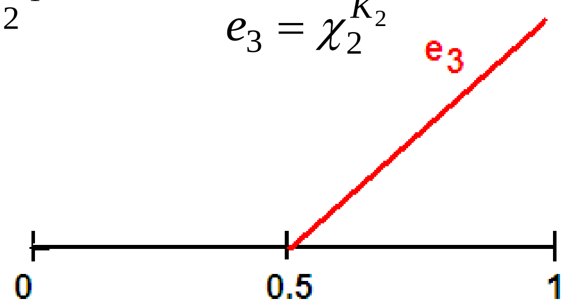
$$e_1 = \chi_1^{K_1}$$



$$e_2 = \chi_1^{K_2} + \chi_2^{K_1}$$



$$e_3 = \chi_2^{K_2}$$



$$b(e_1, e_1) = \int_0^1 \{a e_1' e_1' + b e_1' e_1 + c e_1 e_1\} dx + \beta e_1(l) e_1(l) = \int_0^{0.5} \left\{ a \frac{d\chi_1^{K_1}}{dx} \frac{d\chi_1^{K_1}}{dx} + b \frac{d\chi_1^{K_1}}{dx} \chi_1^{K_1} + c\left(\chi_1^{K_1}\right)^2 \right\} dx$$

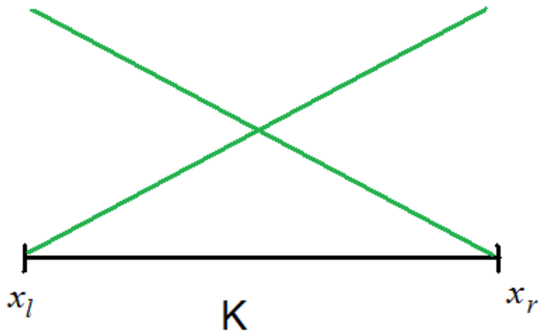
$$b(e_2, e_2) = \int_0^1 \{a e_2' e_2' + b e_2' e_2 + c e_2 e_2\} dx + \beta e_2(l) e_2(l) =$$

$$\int_0^{0.5} \left\{ a \frac{d\chi_2^{K_1}}{dx} \frac{d\chi_2^{K_1}}{dx} + b \frac{d\chi_2^{K_1}}{dx} \chi_2^{K_1} + c\left(\chi_2^{K_1}\right)^2 \right\} dx + \int_{0.5}^1 \left\{ a \frac{d\chi_1^{K_2}}{dx} \frac{d\chi_1^{K_2}}{dx} + b \frac{d\chi_1^{K_2}}{dx} \chi_1^{K_2} + c\left(\chi_1^{K_2}\right)^2 \right\} dx$$

$$b(e_3, e_3) = \int_0^1 \{a e_3' e_3' + b e_3' e_3 + c e_3 e_3\} dx + \beta e_3(l) e_3(l) = \int_{0.5}^1 \left\{ a \frac{d\chi_2^{K_2}}{dx} \frac{d\chi_2^{K_2}}{dx} + b \frac{d\chi_2^{K_2}}{dx} \chi_2^{K_2} + c\left(\chi_2^{K_2}\right)^2 \right\} dx + \beta$$

GENERALIZATION TO 1D FINITE ELEMENT METHOD

$$\chi_1^K = 1 - \frac{x_r - x}{x_r - x_l} \qquad \chi_2^K = \frac{x_r - x}{x_r - x_l}$$



$$\begin{bmatrix} b(e_1,e_1) & b(e_2,e_1) & b(e_3,e_1) \\ b(e_1,e_2) & b(e_2,e_2) & b(e_3,e_2) \\ b(e_1,e_3) & b(e_2,e_3) & b(e_3,e_3) \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} l(e_1) \\ l(e_2) \\ l(e_3) \end{Bmatrix}$$

↓

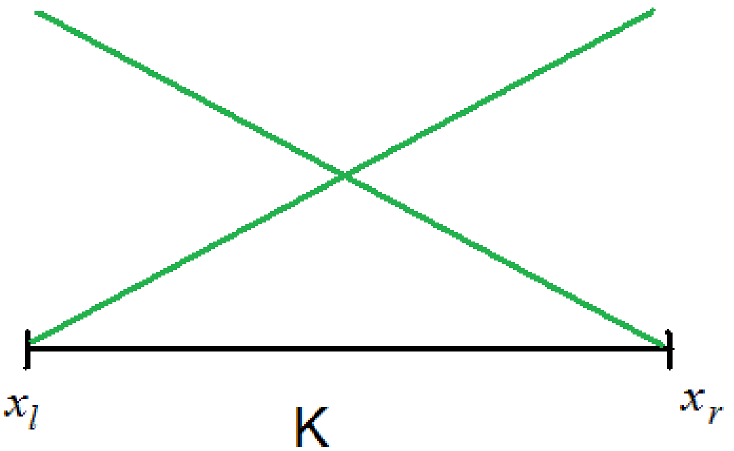
$$\begin{bmatrix} b\left(\chi_1^{K_1},\chi_1^{K_1}\right) & b\left(\chi_2^{K_1},\chi_1^{K_1}\right) & 0 \\ b\left(\chi_1^{K_1},\chi_2^{K_1}\right) & b\left(\chi_2^{K_1},\chi_2^{K_1}\right)+b\left(\chi_1^{K_2},\chi_1^{K_2}\right) & b\left(\chi_2^{K_2},\chi_1^{K_2}\right) \\ 0 & b\left(\chi_1^{K_2},\chi_2^{K_2}\right) & b\left(\chi_2^{K_2},\chi_2^{K_2}\right) \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} l\left(\chi_1^{K_1}\right) \\ l\left(\chi_2^{K_1}\right)+l\left(\chi_1^{K_2}\right) \\ l\left(\chi_2^{K_2}\right) \end{Bmatrix}$$

Notice that when we switch from finite difference to finite elements, it only changes the local systems of equations at tree nodes

$$\begin{bmatrix} b\left(\chi_1^K,\chi_1^K\right) & b\left(\chi_2^K,\chi_1^K\right) \\ b\left(\chi_1^K,\chi_2^K\right) & b\left(\chi_2^K,\chi_2^K\right) \end{bmatrix} \begin{Bmatrix} x_1^K \\ x_2^K \end{Bmatrix} = \begin{Bmatrix} l\left(\chi_1^K\right) \\ l\left(\chi_2^K\right) \end{Bmatrix}$$

GENERALIZATION TO 1D FINITE ELEMENT METHOD

$$\chi_1^K = 1 - \frac{x_r - x}{x_r - x_l} \qquad \chi_2^K = \frac{x_r - x}{x_r - x_l}$$



Global basis functions are composed with local shape functions, e.g.

$$e_2 = \chi_1^{K_2} + \chi_2^{K_1}$$

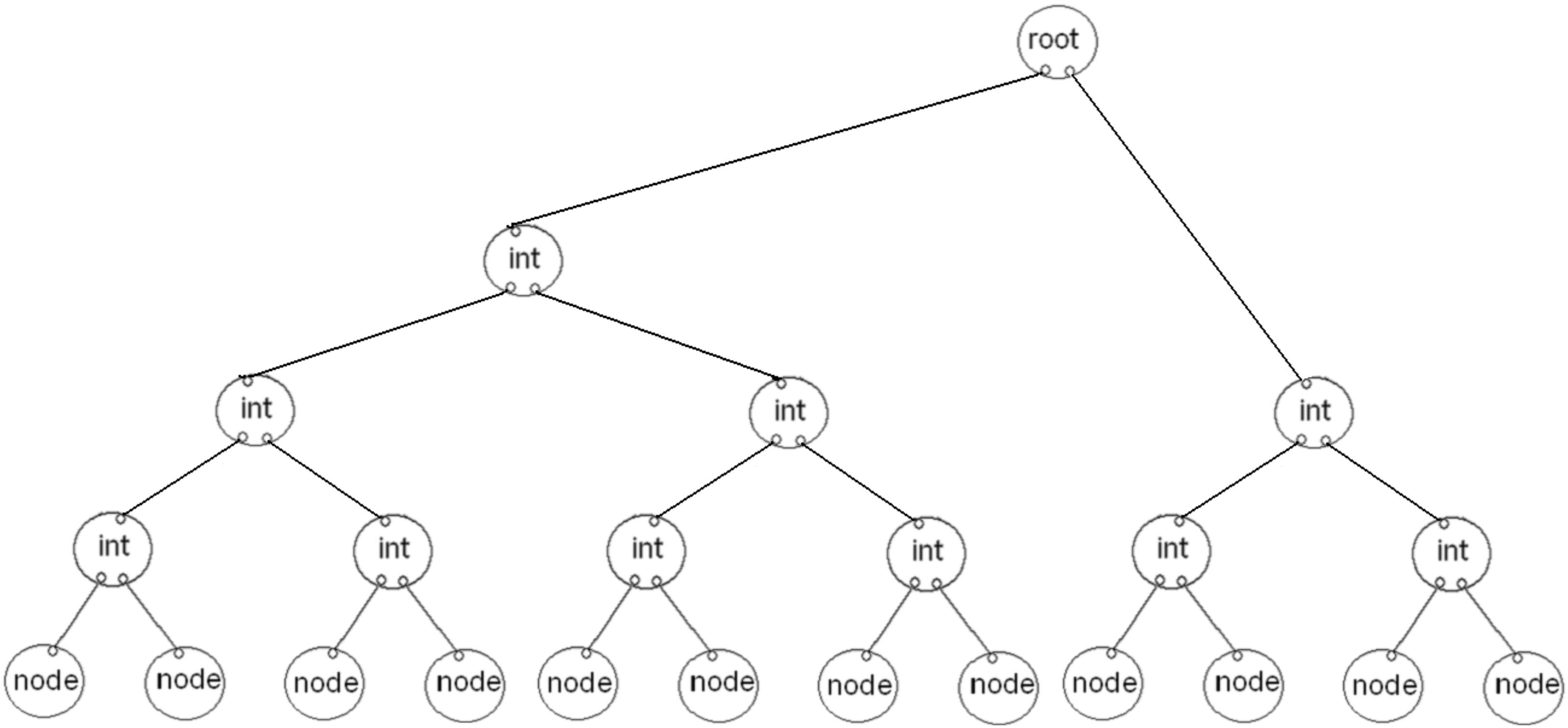
$$b\left(\chi_i^K, \chi_j^K\right) = \int_K \left\{a \chi_i^K{}' \chi_j^K{}' + b \chi_i^K{}' \chi_j^K + c \chi_i^K \chi_j^K\right\} dx + \beta \chi_i^K(l) \chi_i^K(l)$$

$$l\left(e_j\right) = \int_K f \chi_j^K \, dx + \gamma \chi_j^K(l)$$

Local system of equations generated over the element K

$$\begin{bmatrix} b\left(\chi_1^K, \chi_1^K\right) & b\left(\chi_2^K, \chi_1^K\right) \\ b\left(\chi_1^K, \chi_2^K\right) & b\left(\chi_2^K, \chi_2^K\right) \end{bmatrix} \begin{Bmatrix} x_1^K \\ x_2^K \end{Bmatrix} = \begin{Bmatrix} l\left(\chi_1^K\right) \\ l\left(\chi_2^K\right) \end{Bmatrix}$$

GENERALIZATION TO 1D FINITE ELEMENT METHOD



$$\begin{bmatrix} b(\chi_1^K,\chi_1^K) & b(\chi_2^K,\chi_1^K) \\ b(\chi_1^K,\chi_2^K) & b(\chi_2^K,\chi_2^K) \end{bmatrix} \begin{Bmatrix} x_1^K \\ x_2^K \end{Bmatrix} = \begin{Bmatrix} l(\chi_1^K) \\ l(\chi_2^K) \end{Bmatrix}$$

$$\begin{bmatrix} b(\chi_1^K,\chi_1^K) & b(\chi_2^K,\chi_1^K) \\ b(\chi_1^K,\chi_2^K) & b(\chi_2^K,\chi_2^K) \end{bmatrix} \begin{Bmatrix} x_1^K \\ x_2^K \end{Bmatrix} = \begin{Bmatrix} l(\chi_1^K) \\ l(\chi_2^K) \end{Bmatrix}$$

$$\begin{bmatrix} b(\chi_1^K,\chi_1^K) & b(\chi_2^K,\chi_1^K) \\ b(\chi_1^K,\chi_2^K) & b(\chi_2^K,\chi_2^K) \end{bmatrix} \begin{Bmatrix} x_1^K \\ x_2^K \end{Bmatrix} = \begin{Bmatrix} l(\chi_1^K) \\ l(\chi_2^K) \end{Bmatrix}$$

$$\begin{bmatrix} b(\chi_1^K,\chi_1^K) & b(\chi_2^K,\chi_1^K) \\ b(\chi_1^K,\chi_2^K) & b(\chi_2^K,\chi_2^K) \end{bmatrix} \begin{Bmatrix} x_1^K \\ x_2^K \end{Bmatrix} = \begin{Bmatrix} l(\chi_1^K) \\ l(\chi_2^K) \end{Bmatrix}$$

$$\begin{bmatrix} b(\chi_1^K,\chi_1^K) & b(\chi_2^K,\chi_1^K) \\ b(\chi_1^K,\chi_2^K) & b(\chi_2^K,\chi_2^K) \end{bmatrix} \begin{Bmatrix} x_1^K \\ x_2^K \end{Bmatrix} = \begin{Bmatrix} l(\chi_1^K) \\ l(\chi_2^K) \end{Bmatrix}$$

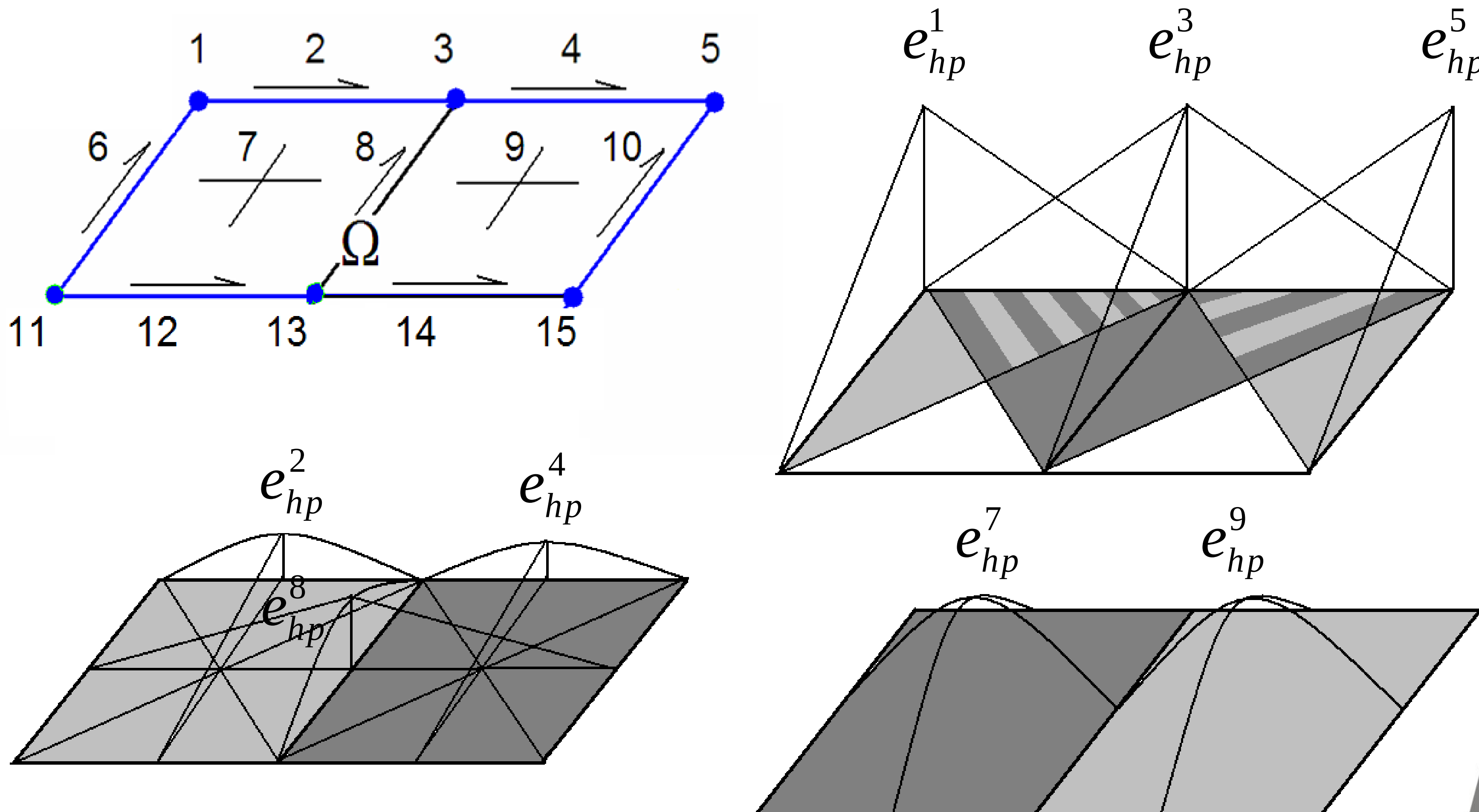
$$\begin{bmatrix} b(\chi_1^K,\chi_1^K) & b(\chi_2^K,\chi_1^K) \\ b(\chi_1^K,\chi_2^K) & b(\chi_2^K,\chi_2^K) \end{bmatrix} \begin{Bmatrix} x_1^K \\ x_2^K \end{Bmatrix} = \begin{Bmatrix} l(\chi_1^K) \\ l(\chi_2^K) \end{Bmatrix}$$

Notice that when we switch from finite difference to finite elements, it only changes the local systems of equations at tree nodes

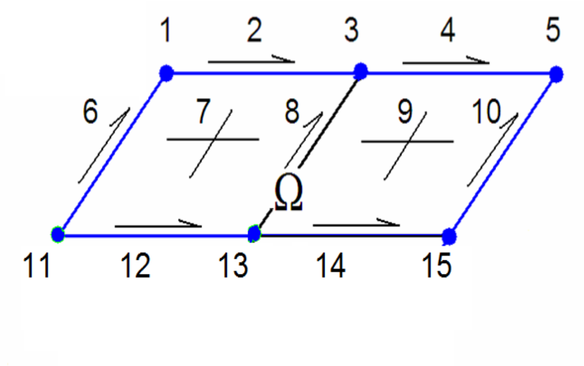
2D *hp* FINITE ELEMENT METHOD

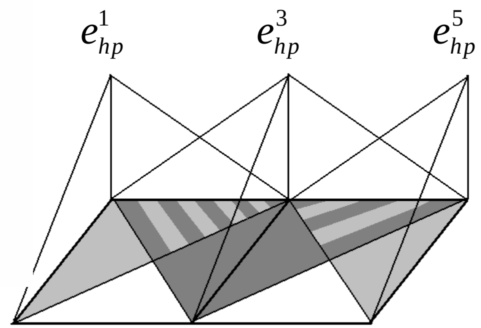
$$u = \sum_{i=1}^{15} u_{hp}^i e_{hp}^i$$

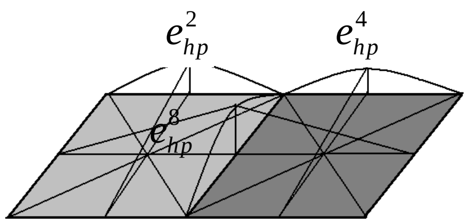
We seek the solution ***u*** of some weak form of PDE as a linear combination of shape functions e_{hp}^i spread over finite element mesh



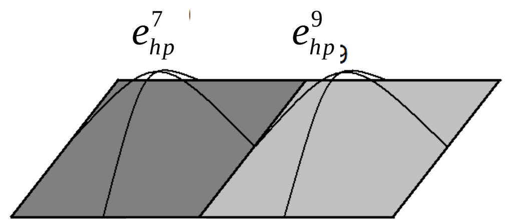
2D *hp* FINITE ELEMENT METHOD



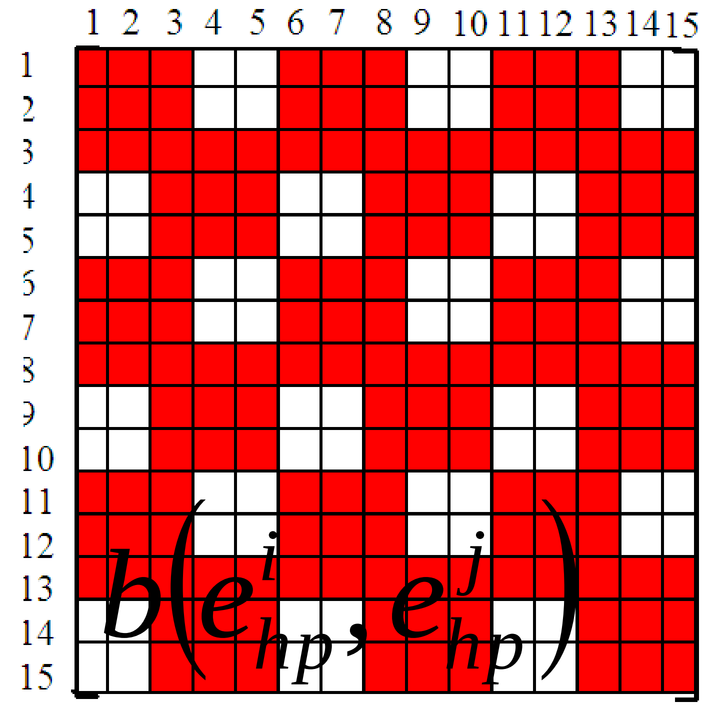




c)



l)



The coefficients u^i_{hp}

(called „degrees of freedom” **d.o.f.**)

are obtained by solving

system of linear equations –

finite element disecretization

of a weak (variational) form of PDE

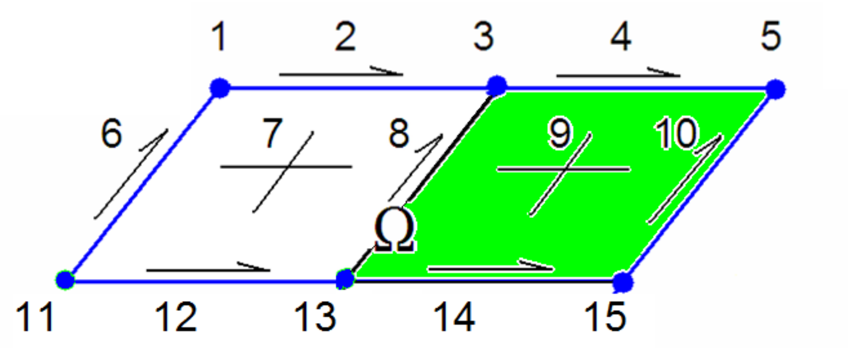
$$\sum_{i=1}^{15} u^i_{hp} b\left(e^i_{hp}, e^j_{hp}\right) = l\left(e^j_{hp}\right) \quad j = 1,\ldots,15$$

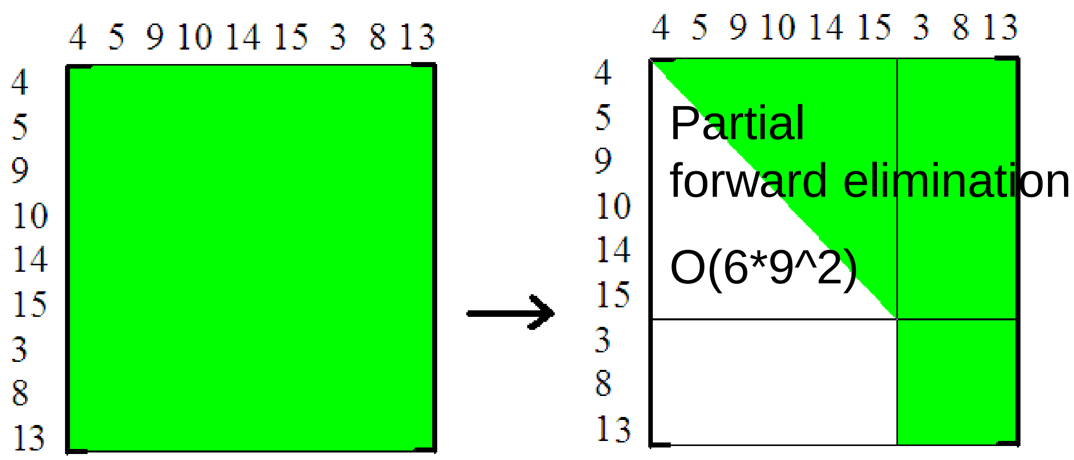
where $b\left(e^i_{hp}, e^j_{hp}\right)$ and $l\left(e^j_{hp}\right)$

are some integrals of shape functions

e^i_{hp}, e^j_{hp}

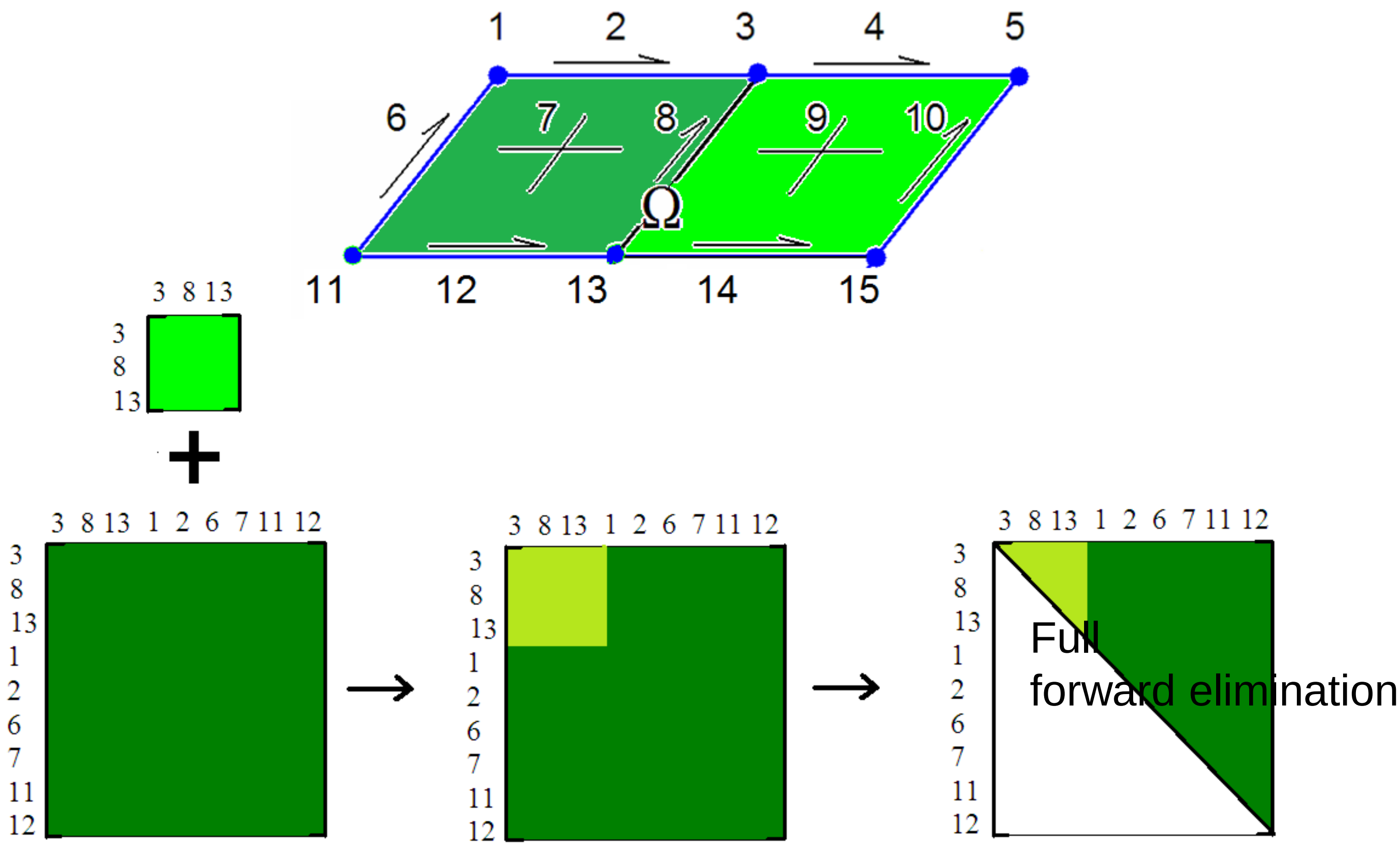
FRONTAL SOLVER
SOLUTION BASED ON LINEAR ORDER OF ELEMENTS





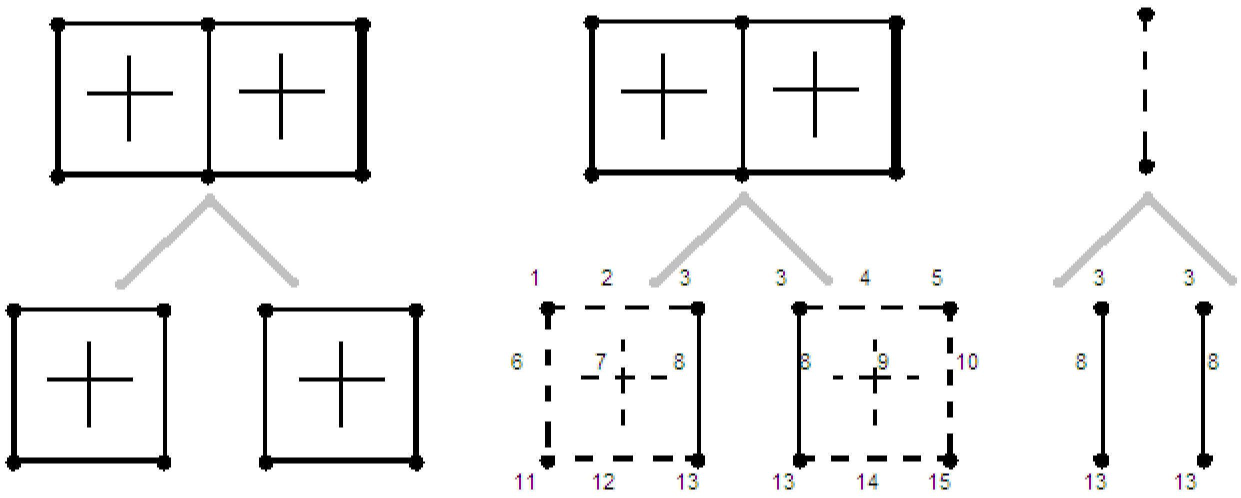
Generates frontal matrix for the first element,
eliminates fully assembled degrees of freedom

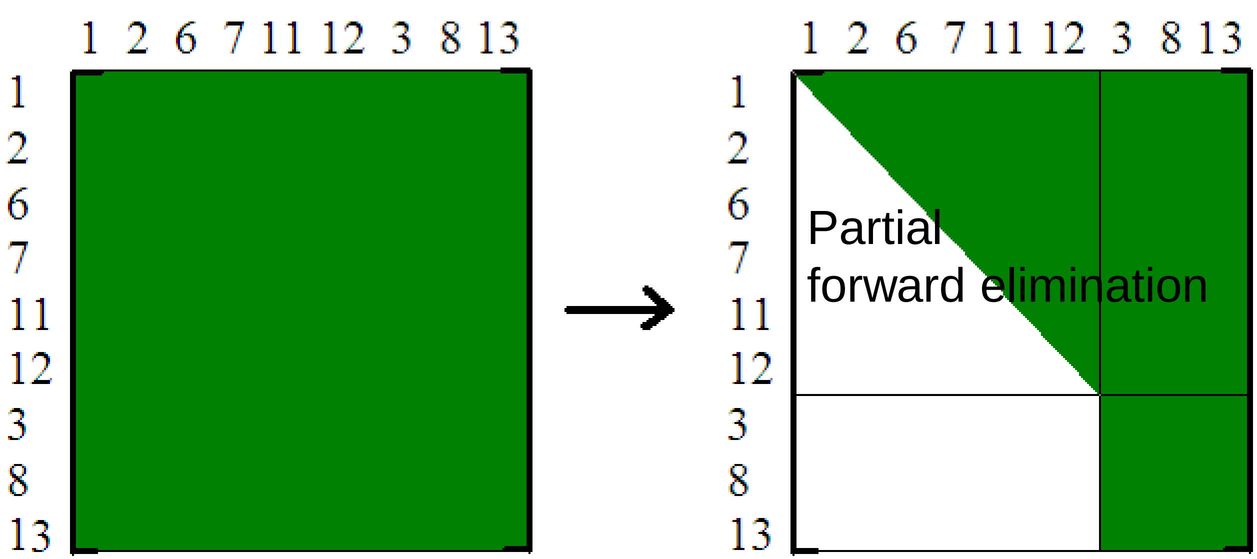
FRONTAL SOLVER
SOLUTION BASED ON LINEAR ORDER OF ELEMENTS



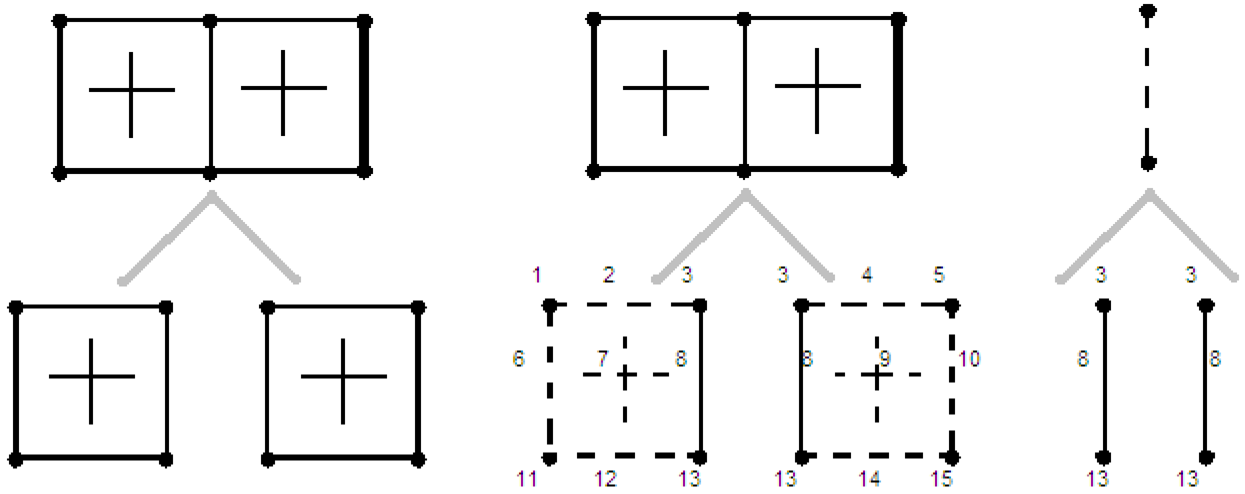
Generates frontal matrix for the second element,
merges with the current frontal matrix
eliminates fully assembled degrees of freedom

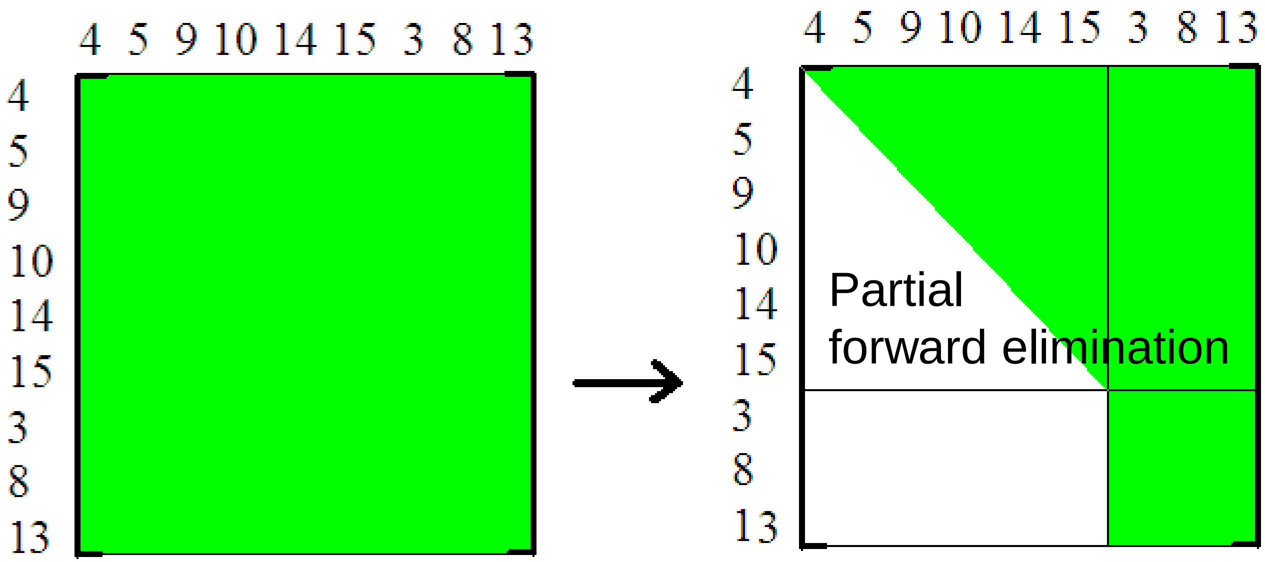
MULTI-FRONTAL SOLVER
SOLUTION BASED ON THE ELIMINATION TREE



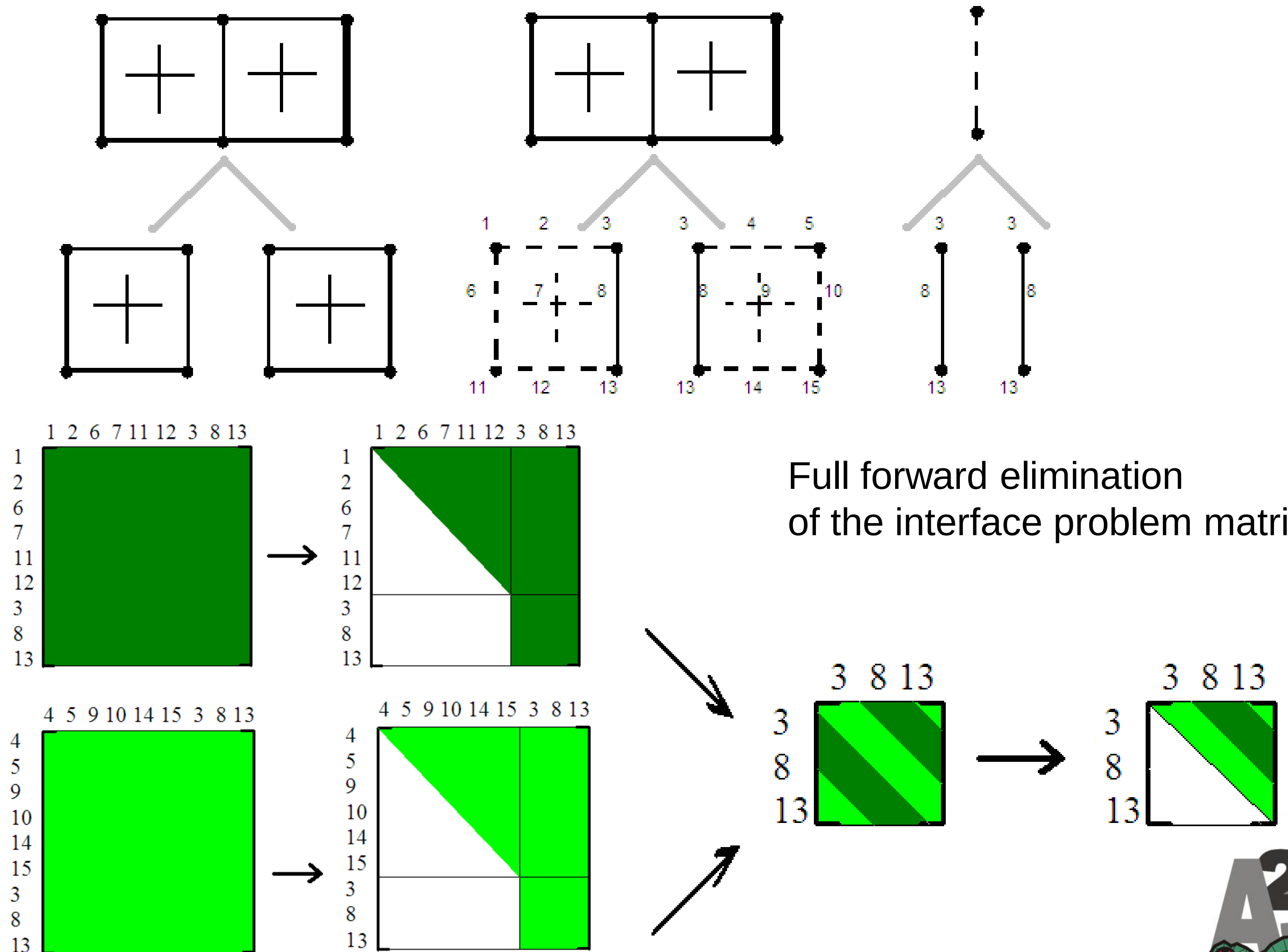


MULTI-FRONTAL SOLVER
SOLUTION BASED ON THE ELIMINATION TREE



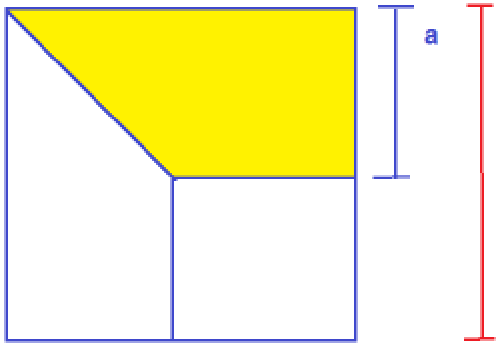


MULTI-FRONTAL SOLVER
SOLUTION BASED ON THE ELIMINATION TREE



Full forward elimination
of the interface problem matrix

COMPARISON OF COSTS



Number of operations for partial forward elimination

$$\sum_{m=1}^{b} m^2 - \sum_{m=1}^{(b-a)} m^2 = \frac{a(6b^2 - 6ab + 6b + 2a^2 - 3a + 1)}{6}$$

Frontal solver

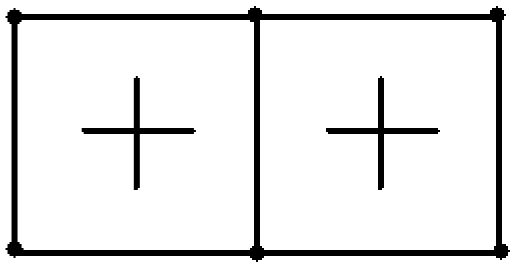
Step	a	b	computational cost $= \frac{a\left(6b^2 - 6ab + 6b + 2a^2 - 3a + 1\right)}{6}$
1	6	9	271
2	9	9	729
Total			1000

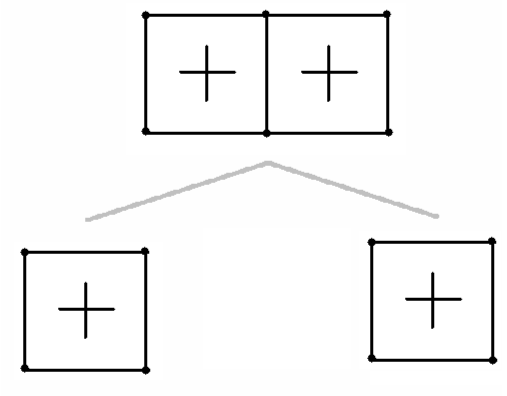
Multi-frontal solver

Step	a	b	computational cost $= \frac{a\left(6b^2 - 6ab + 6b + 2a^2 - 3a + 1\right)}{6}$
1	6	9	271
2	6	9	271
3	3	3	27
Total			569

CONSTRUCTION OF THE ELIMINATION TREE BASED ON THE HISTORY OF MESH REFINEMENTS

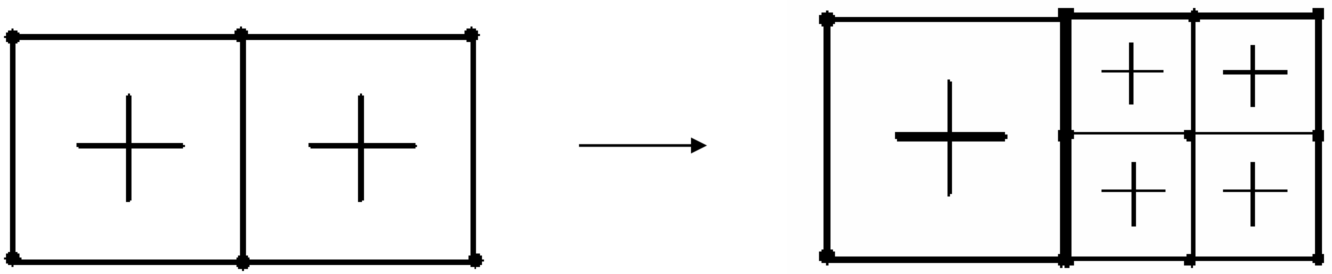
For any initial mesh, the elimination tree can be created based on nested dissection algorithm.

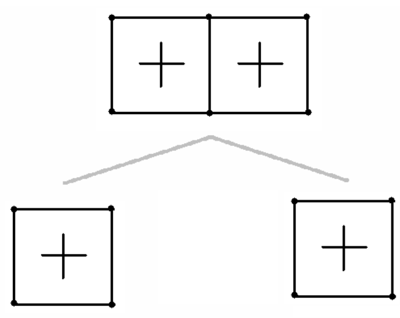


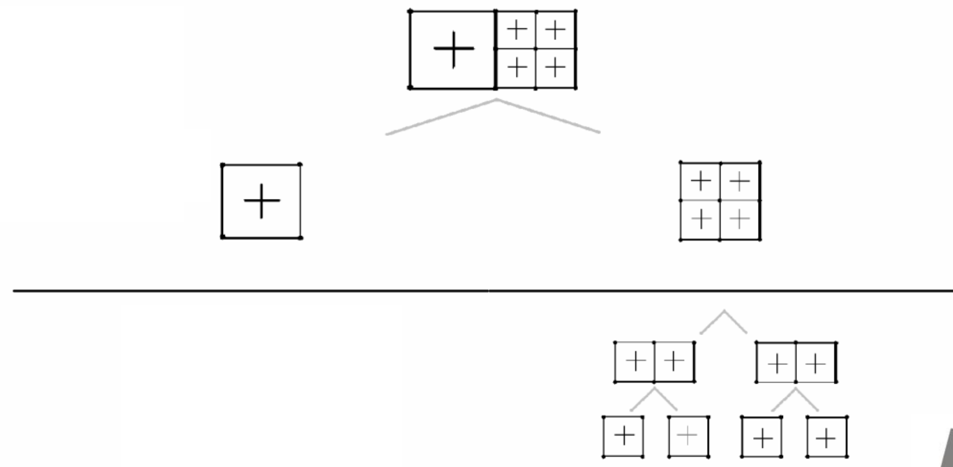


CONSTRUCTION OF THE ELIMINATION TREE BASED ON THE HISTORY OF MESH REFINEMENTS

The elimination tree created for the initial mesh is updated when the mesh is refined **(elimination tree is constructed dynamically, during mesh refinements)**

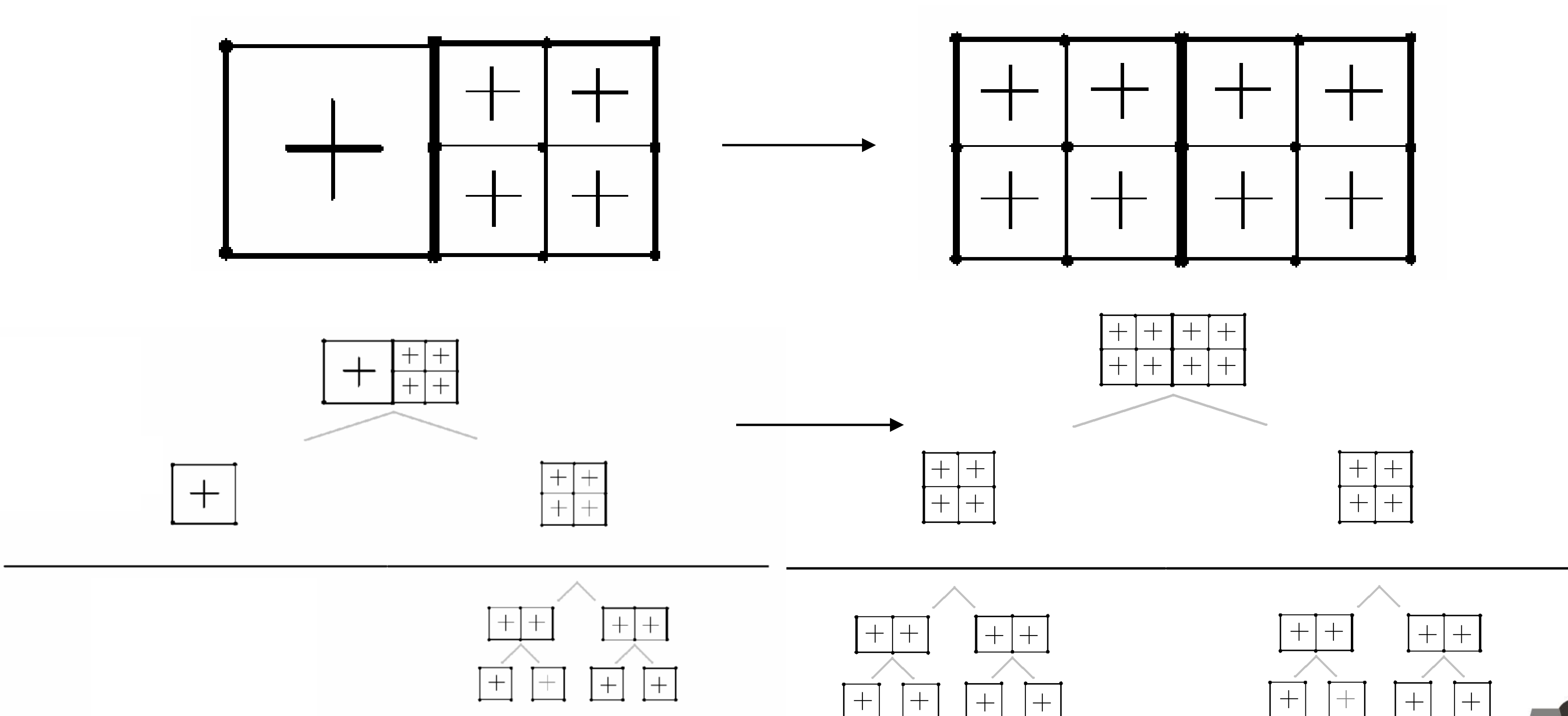




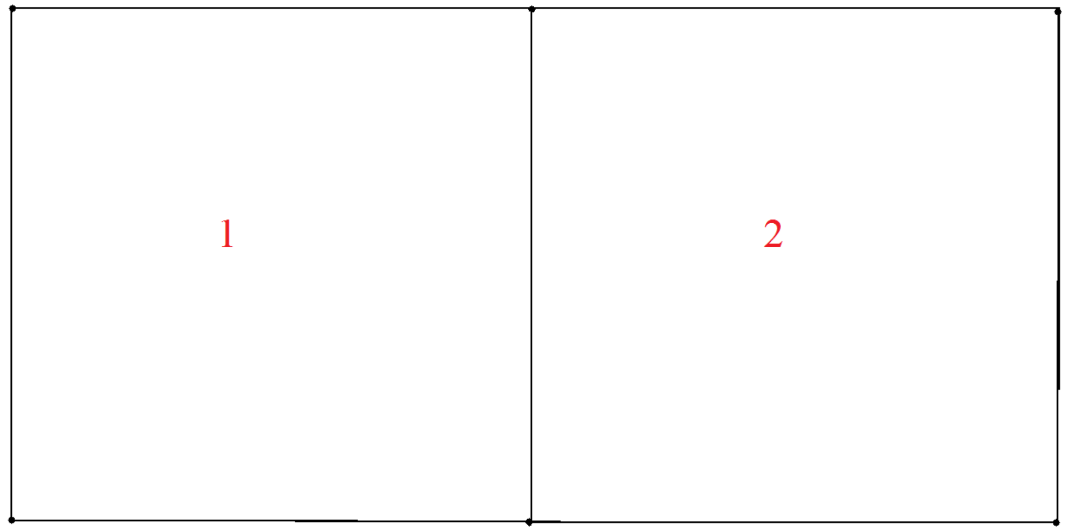


CONSTRUCTION OF THE ELIMINATION TREE BASED ON THE HISTORY OF MESH REFINEMENTS

The elimination tree created for the initial mesh is updated when the mesh is refined **(elimination tree is constructed dynamically, during mesh refinements)**

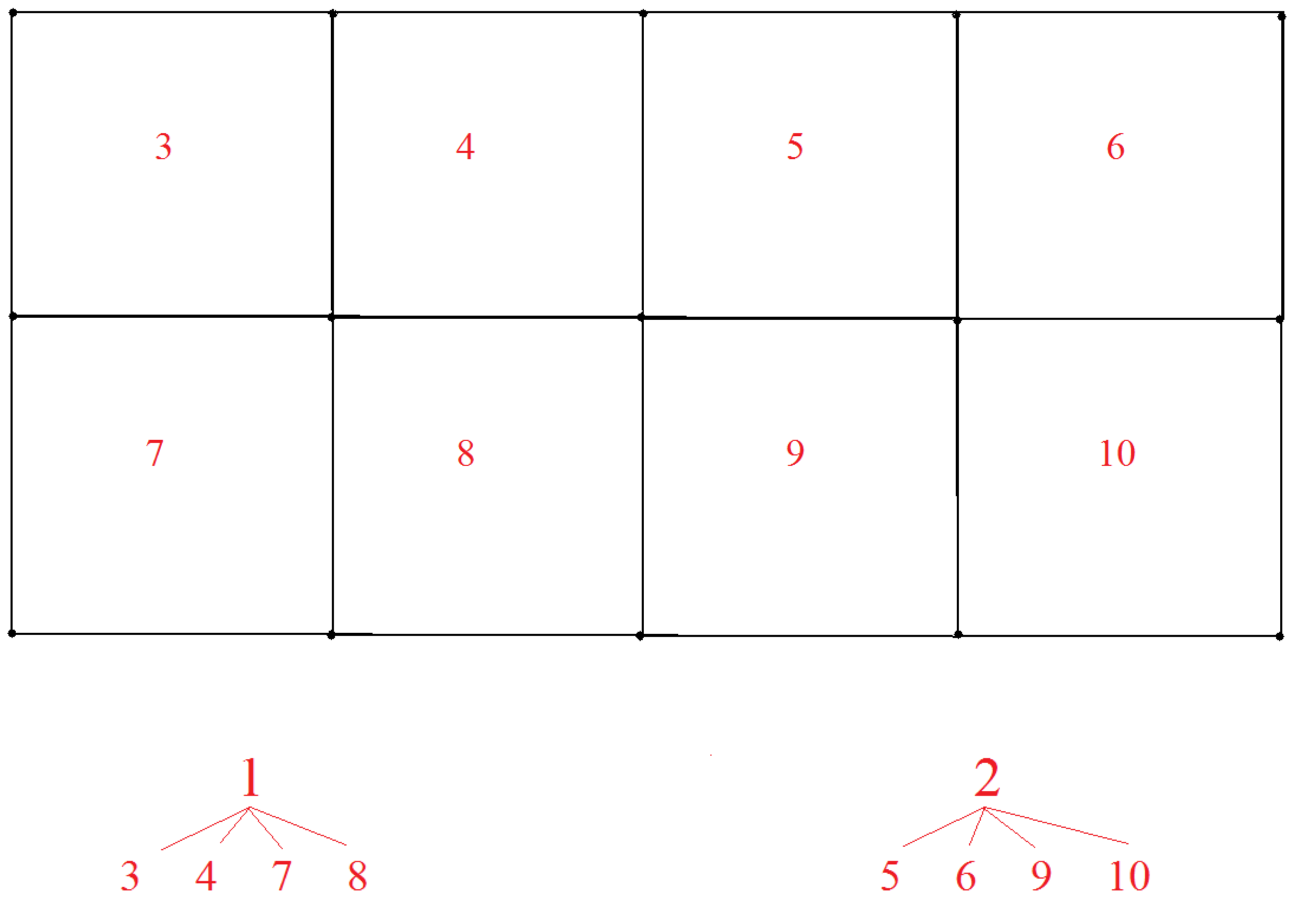


CONSTRUCTION OF THE REFINEMENT TREES

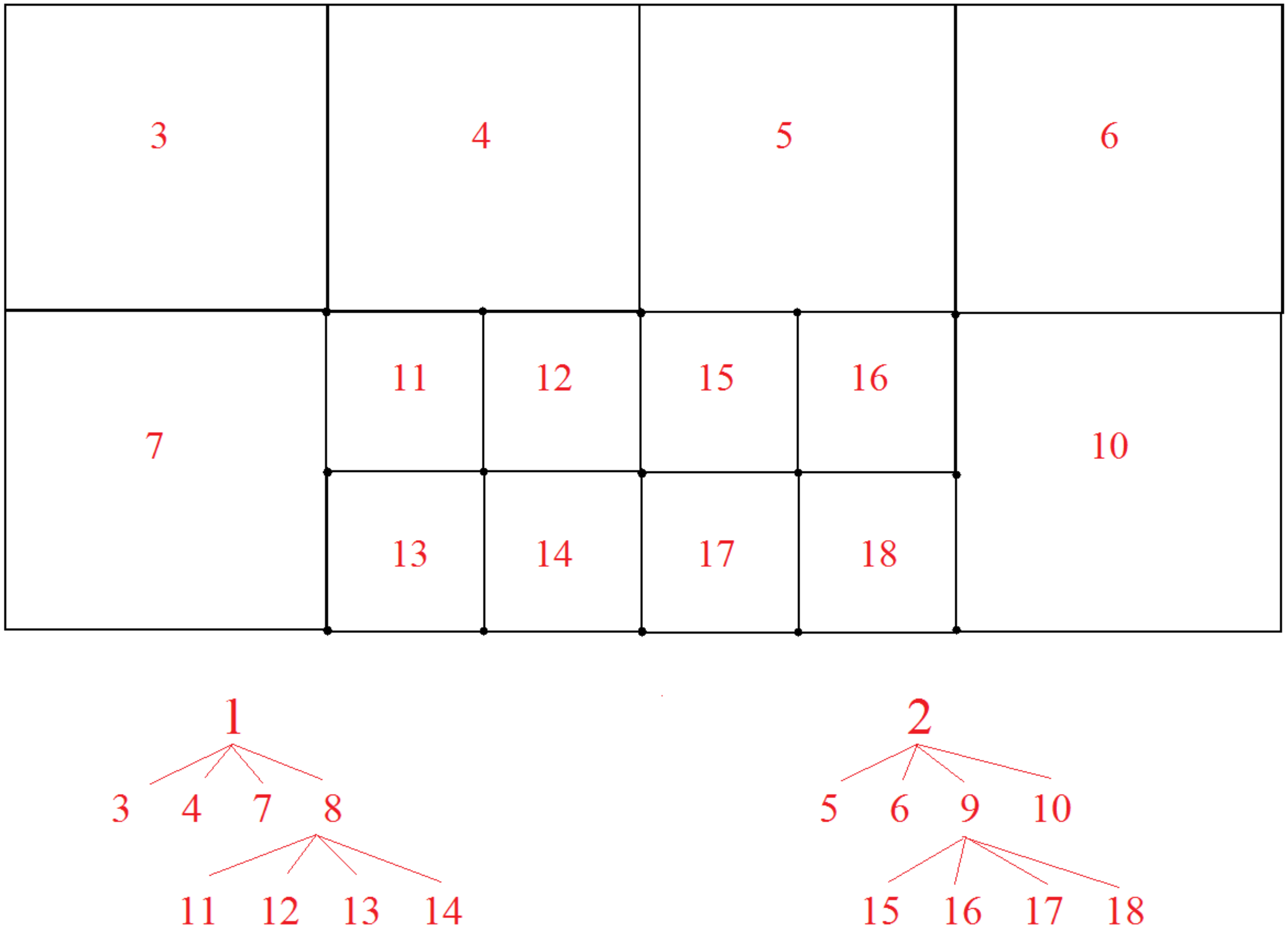


1 2

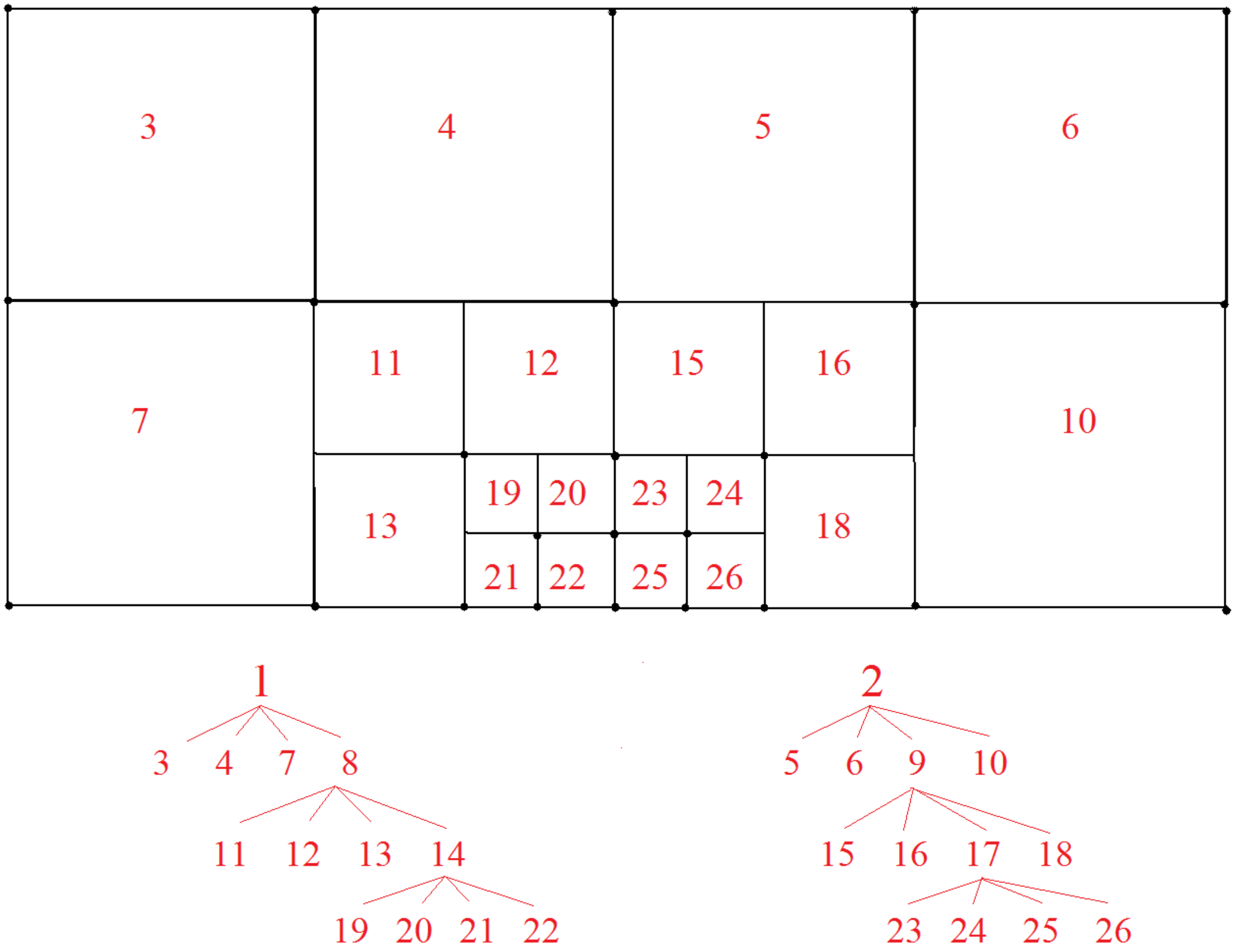
CONSTRUCTION OF THE REFINEMENT TREES



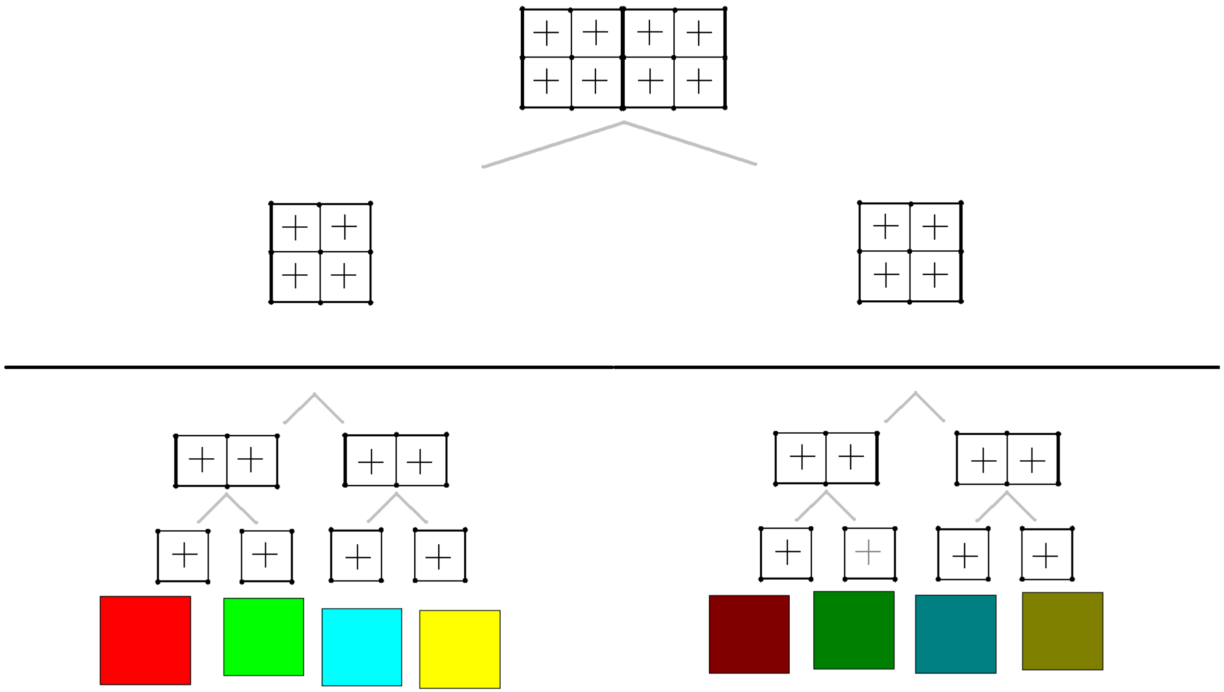
CONSTRUCTION OF THE REFINEMENT TREES



CONSTRUCTION OF THE REFINEMENT TREES



ELIMINATIONS BASED ON REFINEMENT TREES

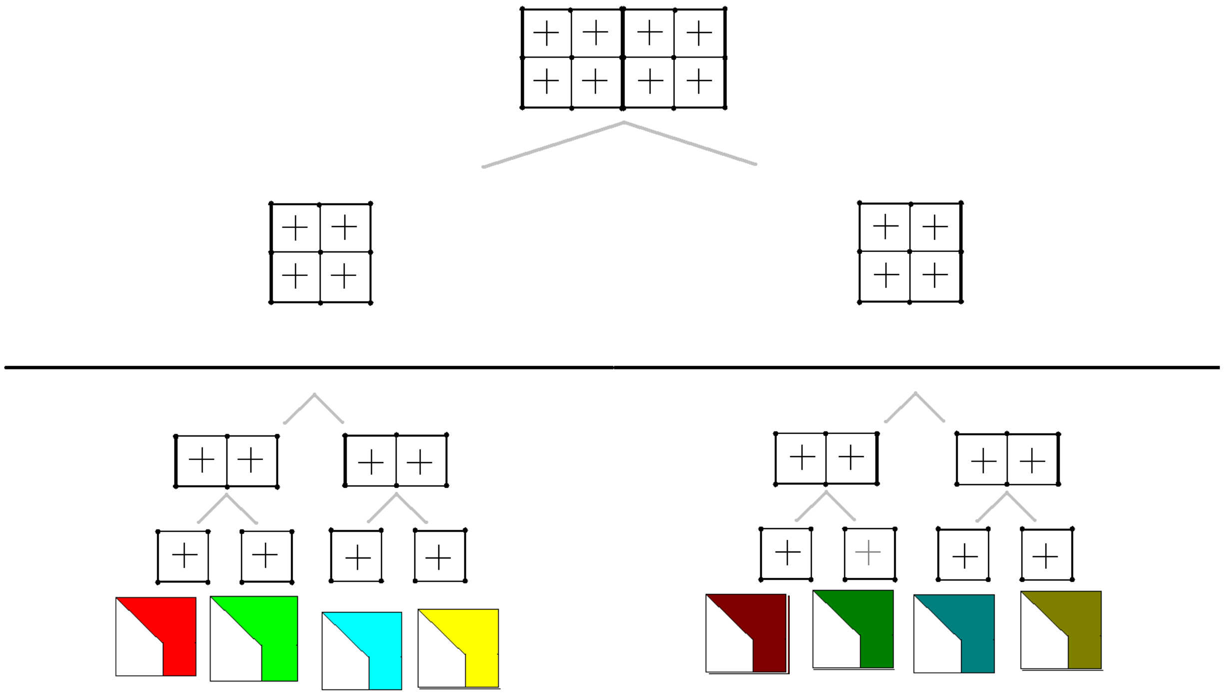


Level of
initial mesh elements

Level of
refinement trees

- Local matrices for active elements – leaves of the elimination tree – are created

ELIMINATIONS BASED ON REFINEMENT TREES

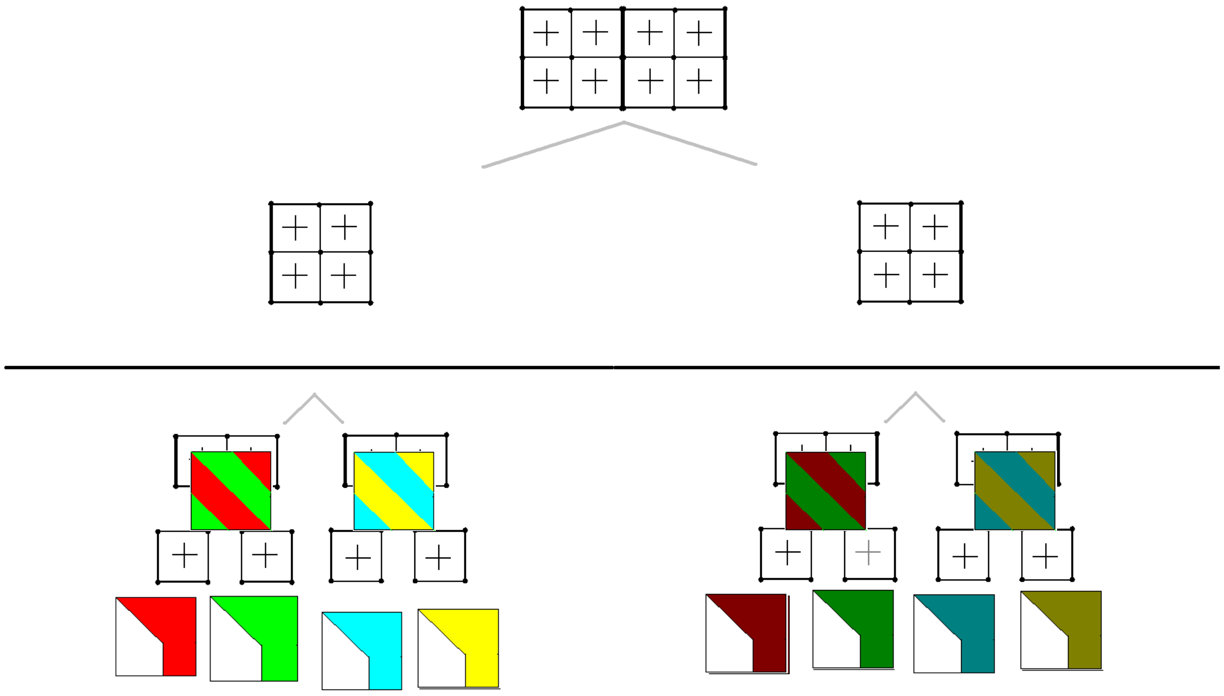


Level of
initial mesh elements

Level of
refinement trees

- Interior degrees of freedom are eliminated at every leaf

ELIMINATIONS BASED ON REFINEMENT TREES

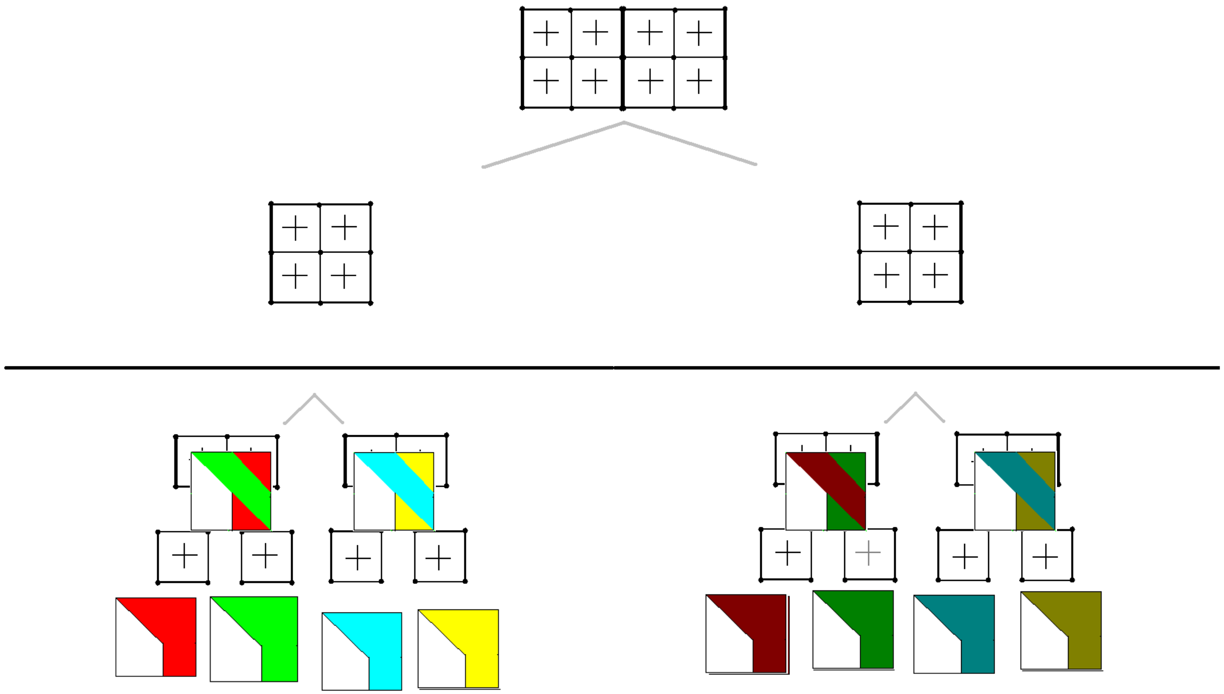


Level of
initial mesh elements

Level of
refinement trees

- Schur complement contributions are merged at parent level

ELIMINATIONS BASED ON REFINEMENT TREES

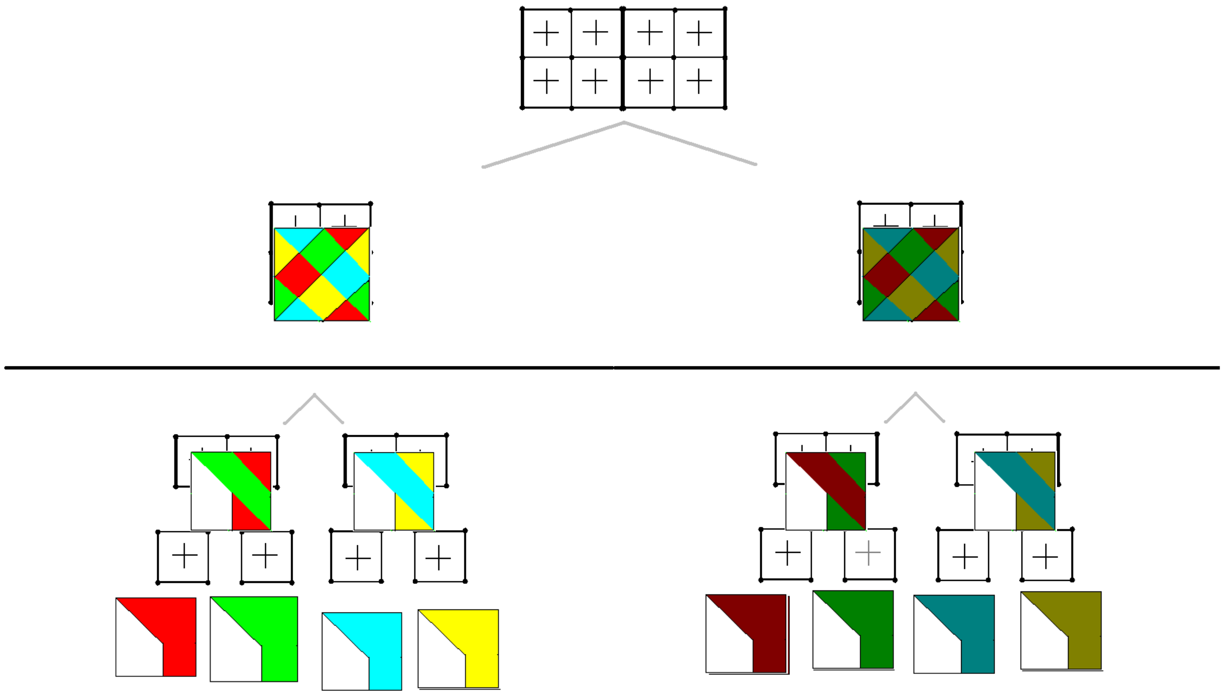


Level of
initial mesh elements

Level of
refinement trees

- Fully assembled degrees of freedom are eliminated at parent nodes level (degrees of freedom related to common edges shared by both son elements can be eliminated now)

ELIMINATIONS BASED ON REFINEMENT TREES

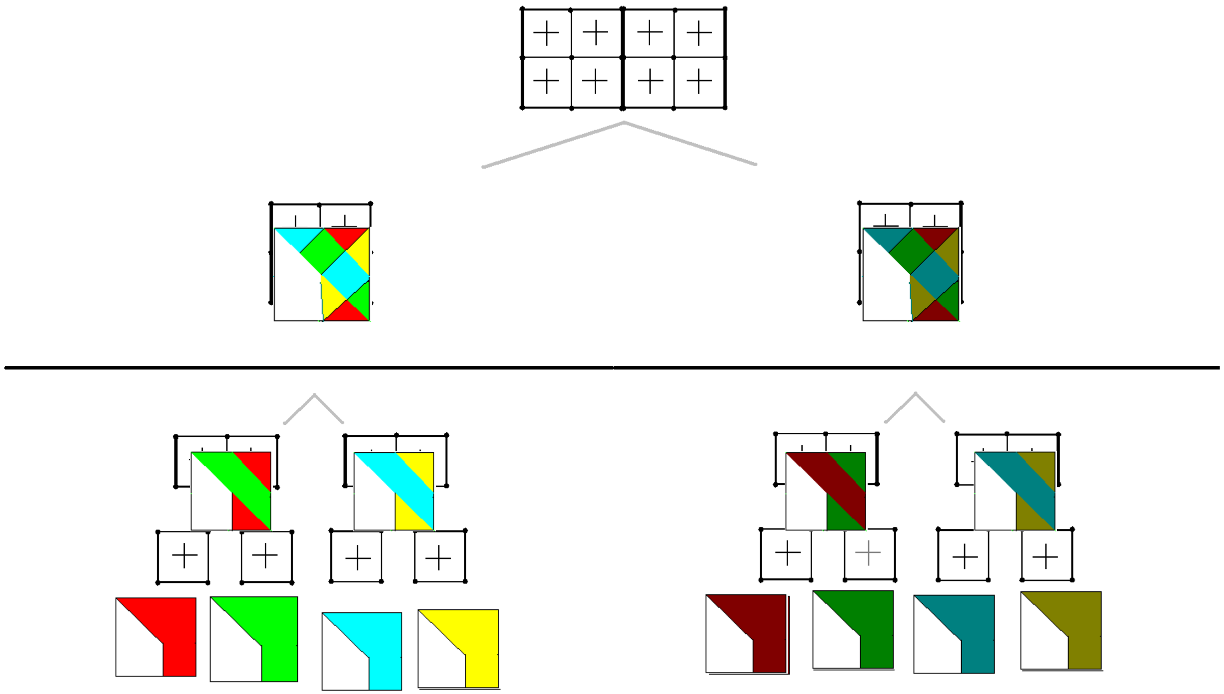


Level of
initial mesh elements

Level of
refinement trees

- Contributions to Schur complement are merged again at the next level

ELIMINATIONS BASED ON REFINEMENT TREES

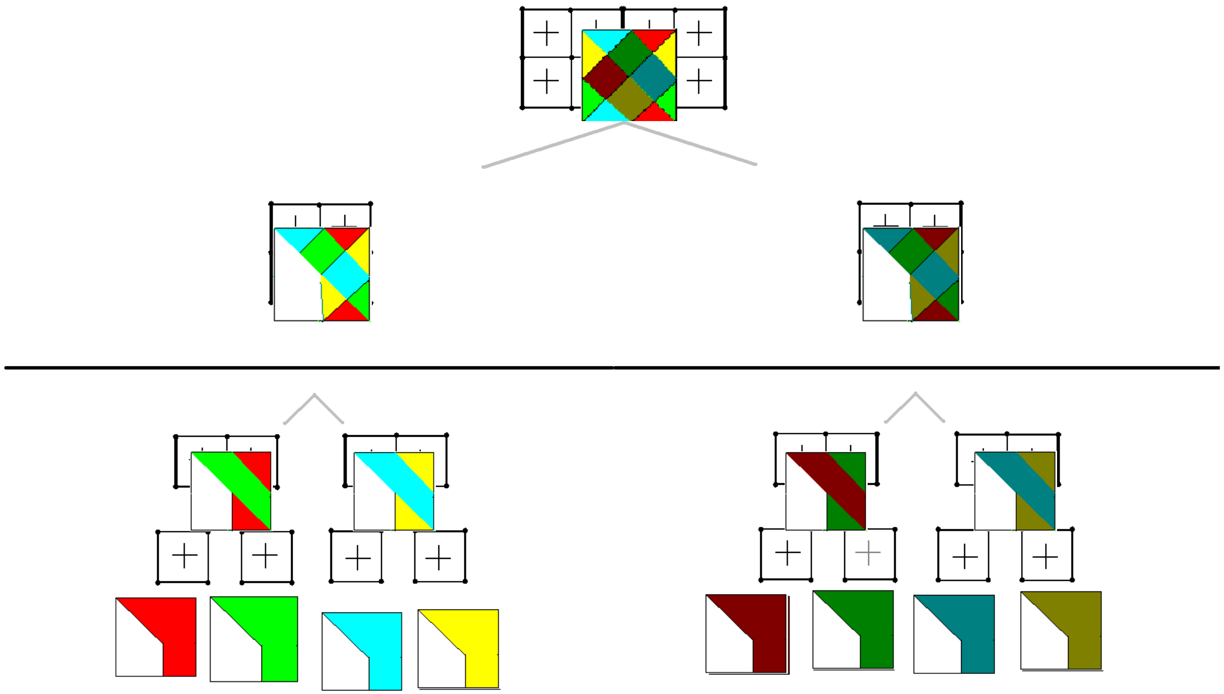


Level of
initial mesh elements

Level of
refinement trees

- Fully assembled degrees of freedom are eliminated

ELIMINATIONS BASED ON REFINEMENT TREES

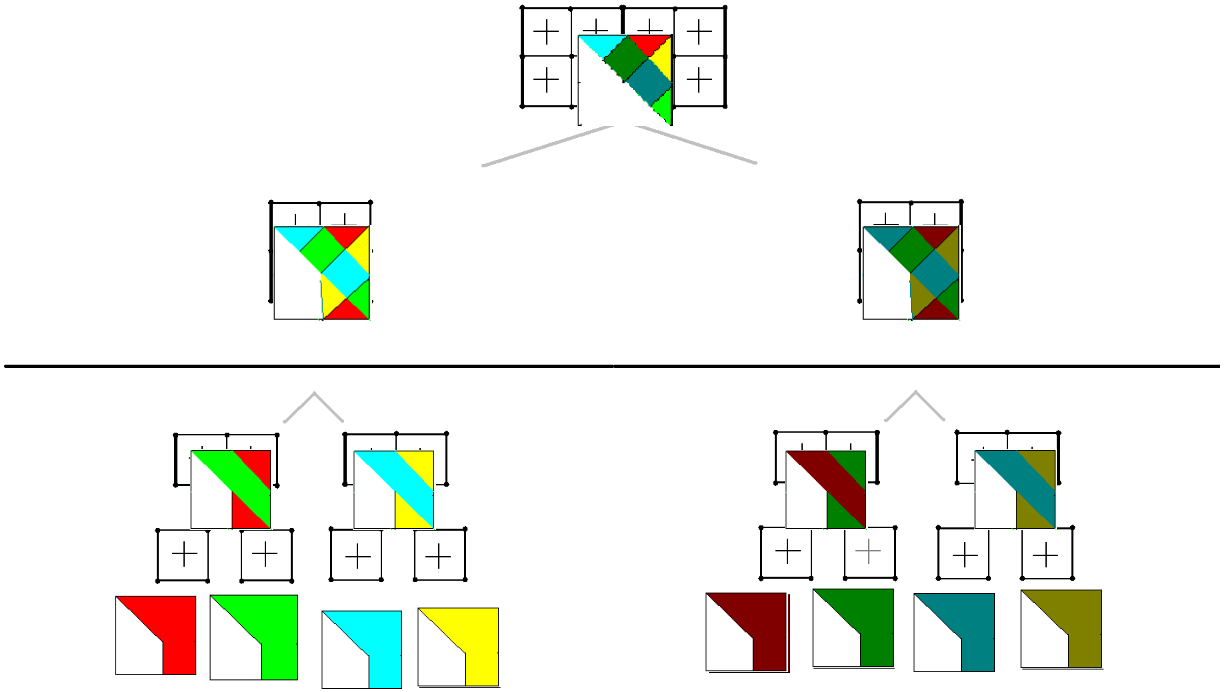


Level of
initial mesh elements

Level of
refinement trees

- Finally, Schur complement contributions are merged at the tree root node

ELIMINATIONS BASED ON REFINEMENT TREES



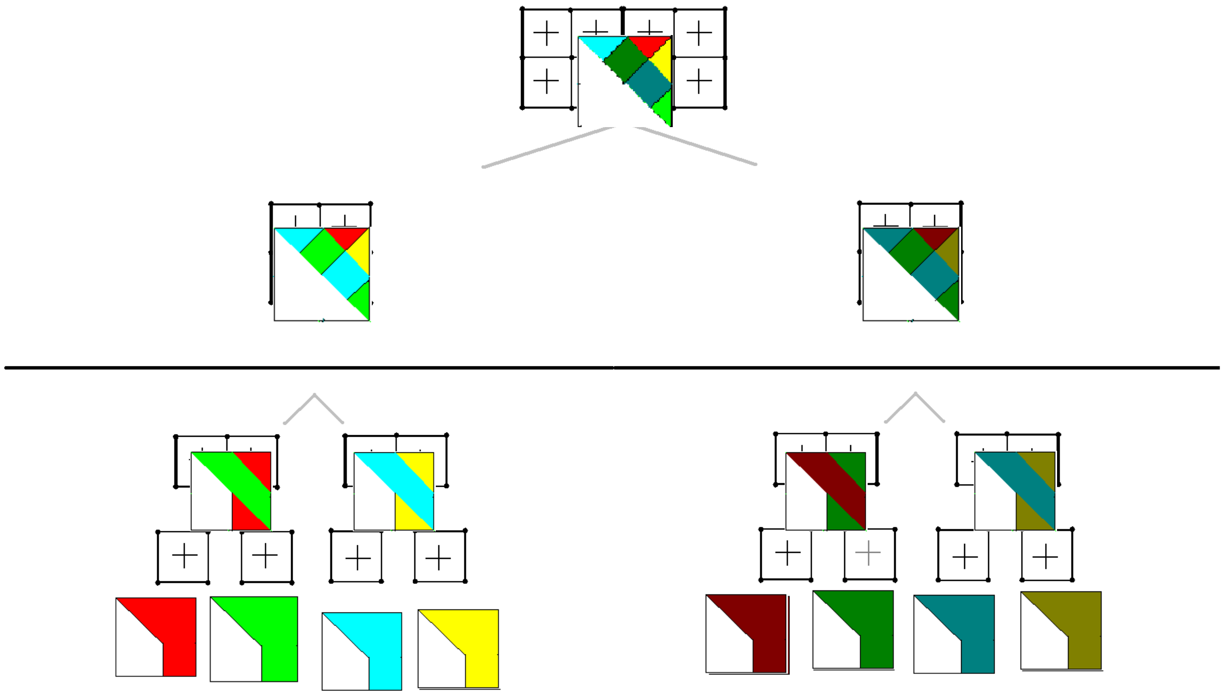
Level of
initial mesh elements

Level of
refinement trees

- The common interface problem is solved at the tree root node

(The size of the common interface problem corresponds to the size of crossection of the entire domain)

ELIMINATIONS BASED ON REFINEMENT TREES

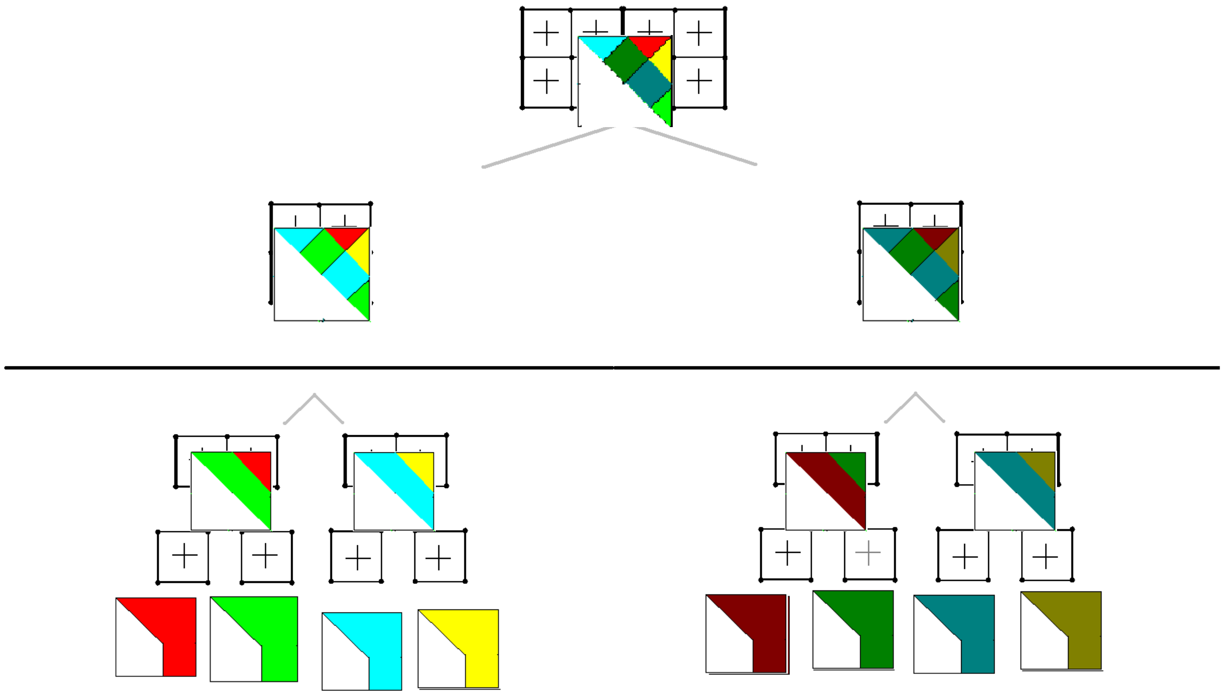


Level of
initial mesh elements

Level of
refinement trees

- The solution obtained at root node is utilized at son nodes.
- The backward substitution is executed

ELIMINATIONS BASED ON REFINEMENT TREES

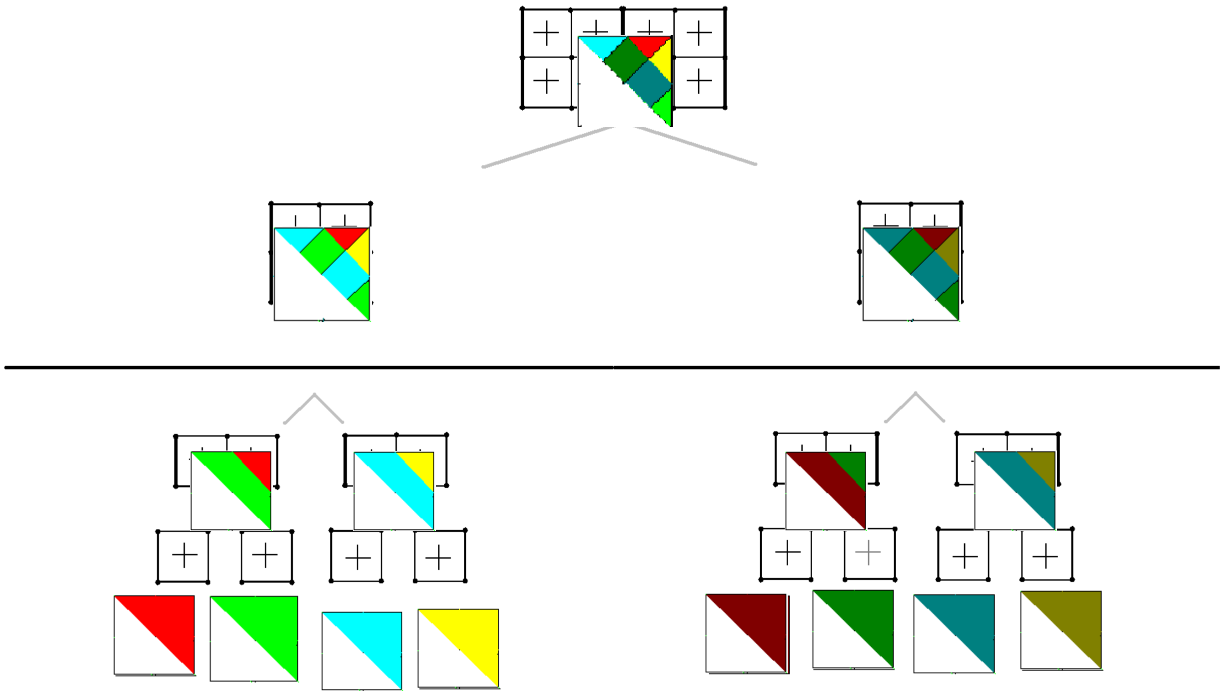


Level of
initial mesh elements

Level of
refinement trees

- The solution utilized at the second level nodes is utilizes at their sone nodes.
- The backward substitution is executed

ELIMINATIONS BASED ON REFINEMENT TREES



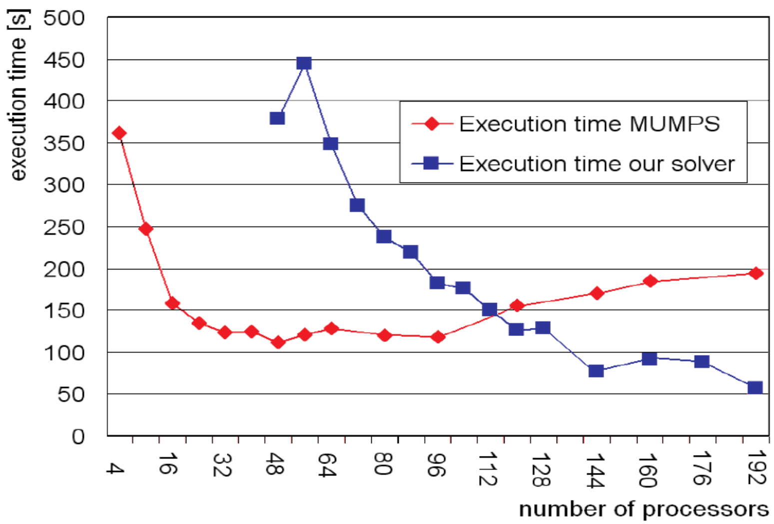
Level of
initial mesh elements

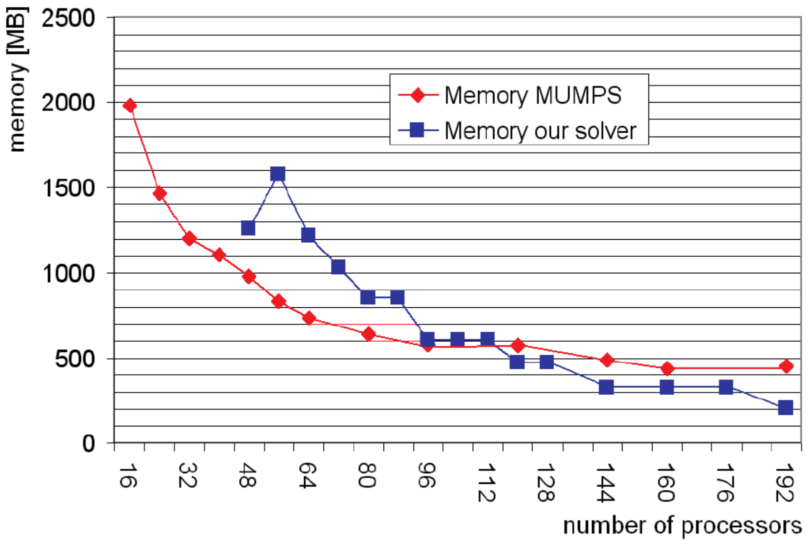
Level of
refinement trees

- The solution obtained at the third level nodes is utilized at leaf nodes.
- The backward substitution is executed on the level of leaf nodes

PERFORMANCE OF THE 1st VERSION OF THE DISTRIBUTED MEMORY MULTI-FRONTAL PARALLEL SOLVER

1,482,570 degrees of freedom (68,826,475 non-zeros)





- Embeded into hp code

+ Outperforms MUMPS for large number of processors

- Slower than MUMPS for low number of processors

Profiling showed that the time consuming part for low number of processors was actually process of merging of matrices –
performed on the level of unknowns, with moving of matrix entries.

Switch to the level of nodes, do not touch matrix entries –work with pointers

PAPERS
http://home.agh.edu.pl/~paszynsk/Publications.html

Maciej Paszyński, David Pardo, Carlos Torres-Verdin, Leszek Demkowicz, Victor Calo
A PARALLEL DIRECT SOLVER FOR SELF-ADAPTIVE HP FINITE ELEMENT METHOD
Journal of Parallel and Distributed Computing, 70, 3 (2013) 270-281

Anna Paszynska, Maciej Paszynski, Konrad Jopek, Maciej Wozniak, Damian Goik, Piotr Gurgul, Hassan AbouEisha, Mikhail Moshkov, Victor Calo, Andrew Lenharth, Donald Nguyen, Keshav Pingali
QUASI-OPTIMAL ELIMINATION TREES FOR 2D GRIDS WITH SINGULARITIES
Scientiffic Programming, (2015) Article ID 303024, 1-18

Maciej Paszynski, David Pardo, Anna Paszynska, Leszek Demkowicz
OUT-OF-CORE MULTI-FRONTAL SOLVER FOR MULTI-PHYSICS HP ADAPTIVE PROBLEMS
Procedia Computer Science, 4 (2011) 1788-1797