THE MOTION PLANNING PROBLEM AND EXPONENTIAL STABILIZATION OF A HEAVY CHAIN. PART I

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Dedicated to Frank M. Callier on the occasion of his 65th birthday

Abstract. A model of a heavy chain system with a punctual load (tip mass) in the form of a system of partial differential equations is interpreted as an abstract semigroup system on a Hilbert state space. Our aim is to solve the output motion planning problem of the same nature as in the case of an unloaded heavy chain [13]. In order to solve this problem we first analyse its well–posedness and some basic properties. Next we solve the output motion planning problem using a substitute of the inverse of the input–output operator represented in terms of the Laplace transforms.

A problem of exponential stabilization is also formulated and solved using a stabilizer of the colocated–type. The exponential stabilization is proved using the method of Lyapunov functionals combined with some frequency–domain tools. The method of Lyapunov functionals can be replaced by the spectral or exact controllability approach as shown in the second part [15] of the present paper.

A laboratory setup which allow to verify the results in practice is described in details. Its dynamical model is used as an example to illustrate the theoretical results.

1. Heavy chain with a punctual load

1.1. System equations and formulation of the problem. Following [26] and [27] consider the motion planning problem for heavy chain control system with a lumped mass \( m > 0 \) depicted in Figure 1.1. Let \( \rho \) be the density of a material of which the chain of length \( L \) is made, and let \( g \) be the acceleration due to gravity. For any fixed time \( t \) two forces are acting on the small piece of the chain located between points \( P = (\xi, \phi(\xi, t)) \) and \( Q = (\xi + \Delta \xi, \phi(\xi + \Delta \xi, t)) \). The gravity force corresponding to the lower part of the chain acts at \( P \). It is directed down, tangentially to the chain. Therefore this force vector creates an angle \( \alpha(\xi, t) \) with the spatial axis \( 0\xi \),

\[
\cos \alpha(\xi, t) = \frac{1}{\sqrt{1 + \phi^2(\xi, t)}} \approx 1, \quad \sin \alpha(\xi, t) = \frac{\phi(\xi, t)}{\sqrt{1 + \phi^2(\xi, t)}} \approx \phi(\xi, t)
\]

because \( \tan[\alpha(\xi, t)] = \phi(\xi, t) \). The tension force of the chain acts at \( Q \) and is directed up, tangentially to the chain, creating an angle \( \alpha(\xi + \Delta \xi, t) \) with the vertical axis. The sum of the vertical components of these two forces tends to 0 as \( \Delta \xi \) tends to 0, while the second principle of dynamics yields the following equation for the horizontal components:

\[
S\rho\Delta \xi \phi_{tt}(\xi, t) = g [m + S\rho(\xi + \Delta \xi)] \phi(\xi + \Delta \xi, t) - g [m + S\rho \xi] \phi(\xi, t).
\]
\[ \phi(L, t) = u(t) \]

\( \phi(\xi, t) \)

\( \Phi(\xi, t) := \phi(\xi, t) - 1(\xi)u(t) \)

\( \Psi(\xi, t) := \phi_t(\xi, t) - 1(\xi)\dot{u}(t) \)

\[ \begin{align*}
\dot{\nu}(t) &= g\Phi_\xi(0, t) - \ddot{u}(t) \\
\Phi_t(\xi, t) &= \Psi(\xi, t) \\
\Psi_t(\xi, t) &= [g(\xi + \mu)\Phi_\xi(\xi, t)]_\xi - 1(\xi)\ddot{u}(t) \\
\nu(0) &= -\dot{u}(0) \\
\Phi(\xi, 0) &= 0 \\
\Psi(\xi, 0) &= -1(\xi)\dot{u}(0) \\
\nu(t) &= \Psi(0, t) \\
\Phi(L, t) &= 0 \\
y(t) &= \Phi(0, t) + u(t)
\end{align*} \]

Problem 1.1. Find a control \( u \) which gives rise to a given, sufficiently smooth, output trajectory.
In [25] and [13] this problem has been solved for the unloaded chain, i.e., when \( m = 0 \). A solution of Problem 1.1 is given in Section 3, where we follow the results of [19] with several corrections and improvements. The main difference in comparison with the result of [27] is our main representations (3.3) and (3.4) of Section 3 are numerically more convenient than those of [27].

If a final position of the chain is reached then the next task is to maintain the final position. The following problem is important in this context.

**Problem 1.2.** Find a feedback controller \( u \) which exponentially stabilizes the system.

It will be proved in Section 4, using the Lyapunov method (the method of Riesz bases or the exact controllability approach can alternatively be applied as shown in [15]) combined with the frequency–domain tools, that the chain position can be exponentially stabilized by means of a boundary negative proportional–integral linear feedback controller of the colocated–type.

A laboratory test bed has been built and described in Section 5. For this particular system the representations (3.3) and (3.4) can be effectively applied to find the control solving the output motion planning problem. It is also verified that the proposed exponential stabilizer of the colocated–type works well in practice.

Finally, a discussion of results is presented in Section 6.

2. Abstract semigroup model

2.1. **The abstract state space.** Let \( H := \mathbb{R} \oplus H^1_0(0, L) \oplus L^2(0, L) \) where

\[
H^1_0(0, L) := \{ \Phi \in H^1(0, L) : \Phi(L) = 0 \}
\]

is a closed subspace of the Sobolev space \( H^1(0, L) \). We endow \( H \) with the energetic scalar product, which is equivalent to the natural scalar product of \( H \),

\[
\left\langle \begin{bmatrix} v \\ \phi \\ \psi \end{bmatrix}, \begin{bmatrix} V \\ \Phi \\ \Psi \end{bmatrix} \right\rangle = \mu vv + \int_0^L g(\xi + \mu)\phi'(\xi)\phi'(\xi)d\xi + \int_0^L \psi(\xi)\psi(\xi)d\xi.
\]

Observe that both the state space and the energetic scalar product substantially differ from those introduced for the unloaded case [13].

2.2. **Semigroup model.** Treating, for any fixed \( t \geq 0 \), the vector \( x(t) \),

\[
x(t)(\xi) = \begin{bmatrix} v(t) \\ \Phi(\xi, t) \\ \psi(\xi, t) \end{bmatrix}, \quad \xi \in [0, L]
\]

as an element of \( H \) we can rewrite (1.3) into its abstract form

\[
\begin{cases}
\dot{x}(t) = A x(t) + d \ddot{u}(t), \\
x(0) = d \dot{u}(0) \\
y(t) = h^* x(t) + u(t)
\end{cases}
\]

(2.1)
where \( u \in C^2[0, \infty) \), \( u(0) = 0 \), and with linear unbounded state operator \( \mathcal{A} \)

\[
\mathcal{A} \begin{bmatrix} \nu \\ \Phi \\ \Psi \end{bmatrix} = \begin{bmatrix} g\Phi'(0) \\ \Psi \\ [g(\cdot + \mu)\Phi'(\cdot)]' \end{bmatrix},
\]

\[
D(\mathcal{A}) = \left\{ \begin{bmatrix} \nu \\ \Phi \\ \Psi \end{bmatrix} \in H : \Phi \in H^2(0, L), \, \Psi \in H^1_L(0, L), \, \Psi(0) = \nu \right\},
\]

control vector \( d \in H \setminus D(\mathcal{A}) \) and observation vector \( h \in D(\mathcal{A}) \):

\[
d = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \quad h = \frac{1}{g} \begin{bmatrix} 0 \\ -\ln(\cdot + \mu) + \ln(L + \mu) \\ 0 \end{bmatrix}.
\]

**Theorem 2.1.** \( \mathcal{A} \) has a countable spectrum consisting entirely of purely imaginary single eigenvalues \( \lambda_{\pm n} \sim \pm j \frac{\mu}{\beta - \alpha} \), \( n \in \mathbb{N} \) where \( \alpha := 2 \sqrt{\frac{\mu}{g}} \), \( \beta := 2 \sqrt{\frac{L + \mu}{g}} \). The set of corresponding normalized eigenvectors forms an orthonormal basis of \( H \). Consequently, \( \mathcal{A} \) generates a unitary group \( \{ S(t) \}_{t \in \mathbb{R}} \) on \( H \).

**Proof.** The operator \( \mathcal{A} \) is invertible with the inverse

\[
\mathcal{A}^{-1} = rh^* - hr^* + \mathfrak{b}, \quad r := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = -\mu Ah \in H.
\]

\( b = -b^* \) is a rank two operator and thus it is compact, while the operator \( \mathfrak{b}, \)

\[
\mathfrak{b} \begin{bmatrix} \nu \\ \Phi \\ \Psi \end{bmatrix} := -\frac{1}{g} \int^L \left[ \frac{1}{\tau + \mu} \int_0^\tau \Psi(s) ds \right] d\tau,
\]

acts in \( \{0\} \oplus H^1_L(0, L) \oplus L^2(0, L) \) and consists of two compact operators: the operator of canonical injection \( H^1_L(0, L) \ni \Phi \mapsto \Phi \in L^2(0, L) \) and the operator

\[
\mathcal{K} \in \mathcal{L}(L^2(0, L), H^1_L(0, L)), \quad \mathcal{K} \Psi := \frac{1}{g} \int^L \left[ \frac{1}{\tau + \mu} \int_0^\tau \Psi(s) ds \right] d\tau.
\]

To demonstrate that the canonical injection \( H^1_L(0, L) \hookrightarrow L^2(0, L) \) is compact it suffices to show that it maps any sequence, weakly convergent to 0 in \( H^1_L(0, L) \), into a sequence which strongly converges to 0 in \( L^2(0, L) \). Let \( \Phi_n \rightharpoonup 0 \) as \( n \to \infty \) in \( H^1_L(0, L) \). Fix arbitrarily \( \xi_0 \in [0, L] \). In particular, for

\[
h_{\xi_0}(\xi) := \frac{1}{g} \begin{cases} 
\ln(L + \mu) - \ln(\xi_0 + \mu), & \text{if } 0 \leq \xi \leq \xi_0 \\
\ln(L + \mu) - \ln(\xi + \mu), & \text{if } \xi_0 < \xi \leq L
\end{cases}, \quad h_{\xi_0} \in H^1_L(0, L)
\]
one has
\[ \langle h_{\xi_0}, \Phi_n \rangle_{H^1_L(0,L)} = -\int_{\xi_0}^{L} \Phi_n'(\xi) d\xi = \Phi_n(\xi_0) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \]
and
\[ |\Phi_n(\xi_0)| = |\langle h_{\xi_0}, \Phi_n \rangle_{H^1_L(0,L)}| \leq \left\| h_{\xi_0} \right\|_{H^1_L(0,L)} \sup_{n \in \mathbb{N}} \| \Phi_n \|_{H^1_L(0,L)} < \infty , \]
because any weakly–convergent sequence is bounded. Since \( \xi_0 \) was chosen arbitrarily then, by the Lebesgue dominated convergence theorem,
\[ \| \Phi_n \|_{L^2(0,L)}^2 = \int_0^L \Phi_n^2(\xi) d\xi \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty . \]

Similarly, to demonstrate that \( K \) is compact we may show that \( K \) takes any sequence, weakly convergent to 0 in \( L^2(0,L) \), into a sequence which strongly converges to 0 in \( H^2_L(0,L) \). Let \( \Psi_n \rightarrow 0 \) as \( n \rightarrow \infty \) in \( L^2(0,L) \). Recall [22, Chapter 4, Section 4, Theorem 10, p. 220] that:
\[ \Psi_n \rightarrow 0 \text{ in } L^2(0,L) \iff \left[ \sup_{n \in \mathbb{N}} \| \Psi_n \|_{L^2(0,L)} < \infty \land \lim_{n \rightarrow \infty} \int_0^t \Psi_n(\tau) d\tau = 0 \quad \forall t \in [0,L] \right] . \]
Owing to this and the Schwarz inequality
\[ \left[ \int_0^\xi \Psi_n(s) ds \right]^2 \leq \xi \| \Psi_n \|_{L^2(0,L)}^2 \Rightarrow \frac{1}{g(\xi + \mu)} \left[ \int_0^\xi \Psi_n(s) ds \right]^2 \leq \frac{1}{g} \| \Psi_n \|_{L^2(0,L)}^2 \leq \frac{1}{g} \sup_{n \in \mathbb{N}} \| \Psi_n \|_{L^2(0,L)}^2 < \infty , \]
and, by the Lebesgue dominated convergence theorem,
\[ \| K \Psi_n \|_{H^2_L(0,L)}^2 = \int_0^L g(\xi + \mu) \left[ \frac{1}{g(\xi + \mu)} \int_0^\xi \Psi_n(s) ds \right]^2 d\xi = \int_0^L \frac{1}{g(\xi + \mu)} \left[ \int_0^\xi \Psi_n(s) ds \right]^2 d\xi \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty . \]

By straightforward calculations one establishes that \( h = -h^* \), whence \( A^{-1} \) is a skew–adjoint compact operator.

It follows from Hilbert’s spectral theorem [20] (applied to the self–adjoint operator \( jA^{-1} \)) that \( A \) has a countable spectrum consisting entirely of purely imaginary simple (but not necessarily single) eigenvalues and a set of corresponding eigenvectors which forms an orthonormal basis of \( H \). Thus the state operator \( A \) generates an unitary group on \( H \) (of almost periodic motions).

The eigenproblem for \( A \) takes the form
\[ A \begin{bmatrix} \nu \\ \Phi \\ \Psi \end{bmatrix} = \begin{bmatrix} g\Phi'(0) \\ \Psi \\ [g(\cdot + \mu)\Phi' (\cdot)]' \end{bmatrix} = \lambda \begin{bmatrix} \nu \\ \Phi \\ \Psi \end{bmatrix} \in D(A) . \]
Equivalently, $\Psi = \lambda \Phi \in H^1_L(0, L)$, $\nu = \lambda \Phi(0)$ and $\Phi \in H^2(0, L)$ satisfies the boundary–value problem

\[
\begin{cases}
g(\xi + \mu)\Phi''(\xi) + g\Phi'(\xi) - \lambda^2 \Phi(\xi) = 0 \\
\Phi(L) = 0 \\
g\Phi'(0) = \lambda^2 \Phi(0)
\end{cases}
\]

This problem is solved in Appendix A where, in particular, it is shown that eigenvalues $\lambda_{\pm n}$ of $A$ are single and $\lambda_{\pm n} \sim \pm j \frac{n\pi}{\beta - \alpha}$, $n \in \mathbb{N}$.

**Corollary 2.1.** A unique weak solution to the state equation in (2.1) is

\[
x(t) = S(t)\dot{u}(0) + \int_0^t S(t - \tau)d\ddot{u}(\tau) \, d\tau.
\]

### 3. Solution of Problem 1.1

#### 3.1. The system transfer function

Applying the Laplace transformation to (2.1) we find the system transfer function:

$$
\hat{g}(s) = s^2 h'(sI - A)^{-1}d + 1 = s^2 \Phi(0) + 1
$$

where $\Phi$ = second component of $(sI - A)^{-1}d$ is a (unique) solution of the boundary-value problem

\[
\begin{cases}
g(\xi + \mu)\Phi''(\xi) + g\Phi'(\xi) - s^2 \Phi(\xi) \equiv 1 \\
\Phi(L) = 0 \\
g\Phi'(0) - s^2 \Phi(0) = 1
\end{cases}
\]

It is shown in Appendix B that solving (3.1), we get

$$
\hat{g}(s) = \frac{2}{s^2 \alpha^2 [I_0(s\beta)K_2(s\alpha) - K_0(s\beta)I_2(s\alpha)]},
$$

where $I_n$ denotes the modified Bessel function of the first kind and $n$–th order whilst $K_n$ stands for the modified Bessel function of the second kind and $n$–th order, $n \in \{0\} \cup \mathbb{N}$.

**Remark 3.1.** Observe that $\beta \to 2\sqrt{L/g}$ and $\alpha \to 0$ as $\mu \to 0$ whence, upon recalling (B.3) of Appendix B, one concludes that then $\hat{g}(s)$ tends to $1/I_0\left(2s\sqrt{L/g}\right)$, i.e., to the transfer function of a chain without a load as given in [13, Corollary 2.1].

#### 3.2. Determination of the control

If $y$ is a given (planned) output trajectory with the Laplace transform $\hat{y}$ then the Laplace transform $\hat{u}$ of the control $u$, leading to such an output trajectory, is

\[
\hat{u}(s) = \left[\frac{\hat{y}(s)}{\hat{g}(s)}\right].
\]

A simple direct recovering of a formal inverse of the input–output operator in a convolution form is not possible because $1/\hat{g}$, is not a Laplace transform of a distribution with a support
in $[0, \infty)$. Nevertheless, we can represent (3.2) in the form

$$\hat{u}(s) = \frac{1}{2} \left[ I_0(s) e^{-s\beta} \right] \left[ \alpha^2 K_2(s\alpha) e^{\alpha s} \right] \left[ s^2 e^{s(\beta-\alpha)} \hat{y}(s) \right]$$

where: by [4] p. 245, (1)], $\hat{k}_1$ is the Laplace transform of

$$k_1(t) = \mathbb{I}(t) [2\beta - t] \frac{1}{\pi \sqrt{2\beta t - t^2}} ,$$

by [4] p. 246, (18) with $\nu = 2$, $\hat{k}_2$ is the Laplace transform of

$$k_2(t) = \frac{2(\alpha + t)^2 - \alpha^2}{\sqrt{t^2 + 2\alpha t}} , \quad t \geq 0 ,$$

[4] p. 246, (16)], $\hat{k}_3$ is the Laplace transform of

$$k_3(t) = \frac{1}{\sqrt{t^2 + 2\beta t}} , \quad t \geq 0$$

and, by [4] p. 246, (3) with $a = 0, b = 2\alpha$ and $n = 2$, $\hat{k}_4$ is the Laplace transform of

$$k_4(t) = \mathbb{I}(t) [2\alpha - t] \frac{2(t - \alpha)^2 - \alpha^2}{\pi \sqrt{2\alpha t - t^2}} .$$

Here $k_1, k_4 \in L^1(0, \infty)$ with supports: supp $k_1 = [0, 2\beta]$, supp $k_4 = [0, 2\alpha]$ whilst $k_2, k_3 \in L^1(0, T)$ for any $T > 0$. By the standard convolution theory $\pi \hat{k}_1(s)\hat{k}_2(s)$ is the Laplace transform of the convolution $p := \pi k_1 * k_2$, whence for $t \geq 0$

$$p(t) = \int_0^t \frac{2(\alpha - \tau)}{\sqrt{2\alpha \tau - \tau^2}} \frac{2(\alpha + t - \tau)^2 - \alpha^2}{\sqrt{(t - \tau)^2 + 2\alpha(t - \tau)}} d\tau .$$

Furthermore $p \in L^1(0, T)$ for any $T > 0$ and since $k_2(t) \sim 2(\alpha + t)$ for large $t$, then $p(t) \sim 2\pi(\alpha + t - \beta)$. Similarly, $\pi \hat{k}_3(s)\hat{k}_4(s)$ can be identified with the Laplace transform of the convolution $q := k_3 * \pi k_4$, whence for $t \geq 0$

$$q(t) = -\int_0^t \frac{2(\alpha - \tau) \sqrt{2\alpha \tau - \tau^2}}{\left( t - \tau \right)^2 + 2\beta(t - \tau)} d\tau + \int_0^t \frac{2(\alpha - \tau)}{\left( t - \tau \right)^2 + 2\beta(t - \tau)} \frac{(\tau - \alpha)^2}{\sqrt{2\alpha \tau - \tau^2}} d\tau .$$

$q \in L^1(0, T)$ for any $T > 0$ and by the Lebesgue dominated convergence theorem $\lim_{t \to \infty} q(t) = 0$. Notice that the Tauberian theorem holds for $p$ and $q$,

$$\lim_{t \to 0} p(t) = \lim_{t \to 0} \int_0^t \frac{\alpha^2 d\tau}{2\alpha^2 \beta(t - \tau)} = \frac{\alpha^2\pi}{\sqrt{2\alpha^2 \beta}} = \lim_{s \to \infty, s \in \mathbb{R}} s\hat{\beta}(s) = \lim_{s \to \infty, s \in \mathbb{R}} s\pi \hat{k}_1(s)\hat{k}_2(s) =$$

$$= \frac{\pi \alpha^2}{\sqrt{2\alpha^2 \beta}} \lim_{s \to \infty, s \in \mathbb{R}} \frac{\sqrt{s\beta} I_0(s\beta)}{e^{s\beta}} \sqrt{s\alpha} K_2(s\alpha) e^{s\alpha} .$$

The last equality is due to the asymptotic formulae (B.4) given in Appendix B. Similarly

$$\lim_{t \to 0} q(t) = \lim_{t \to 0} \int_0^t \frac{\alpha^2 d\tau}{2\alpha^2 \beta(t - \tau)} = \lim_{t \to 0} p(t) .$$
The functions $p$ and $q$ can be expressed in terms of the elliptic integrals $F$, $E$ and $\Pi$ but formulae are too complicated. However, the precise and stable procedures of numerical integration built-in Maple or Matlab are useful for computations of $p$ and $q$.

If $y \in C^2[0, \infty)$, supp $y = [\beta - \alpha, \infty)$ then the terms (1) and (2) can be identified with the Laplace transforms of $\dot{y}(\cdot + \beta - \alpha)$ and $\ddot{y}(\cdot - \beta + \alpha)$, respectively. Next, applying the standard convolution theory once more, we can find a locally integrable control resulting in such an output trajectory,

$$u(t) = \frac{1}{2\pi} \int_0^t \det \begin{bmatrix} p(\tau) & q(\tau) \\ \dot{y}(t - \tau - \beta + \alpha) & \ddot{y}(t - \tau + \beta - \alpha) \end{bmatrix} d\tau.$$  \hfill (3.3)

Actually, $u$ is a locally absolutely continuous function, vanishing at 0, because the integrand is a locally integrable function. (3.3) defines a convolution form of an inverse of the input–output map, restricted to the class of outputs as above. It should be stressed that here the support supp $y$ cannot be extended to the left if one wants to obtain a realizable control $u$, i.e., a control with supp $u = [0, \infty)$.

However, for the existence of a weak solution (2.3) with $x(0) = 0$ we need a locally integrable $\dot{u}$ with $\dot{u}(0) = 0$. This can be achieved by taking $y \in C^4[0, \infty)$ with supp $y = [\beta - \alpha, \infty)$ in (3.3), and then

$$\dot{u}(t) = \frac{1}{2\pi} \int_0^t \det \begin{bmatrix} p(\tau) & q(\tau) \\ \dot{y}(t - \tau - \beta + \alpha) & \ddot{y}(t - \tau + \beta - \alpha) \end{bmatrix} d\tau,$$

$$\ddot{u}(t) = \frac{1}{2\pi} \int_0^t \det \begin{bmatrix} p(\tau) & q(\tau) \\ \dot{y}(t - \tau - \beta + \alpha) & \ddot{y}(t - \tau + \beta - \alpha) \end{bmatrix} d\tau$$

with $u(0) = \dot{u}(0) = \ddot{u}(0) = 0$.

### 3.3. Determination of the chain position.

The analysis of the previous section can be continued to find a position of the whole chain. Indeed, by (1.2) and (2.3), the Laplace transform $\hat{\Phi}(\xi, \cdot)$ of $\Phi(\xi, \cdot)$, a position of the chain, reads as

$$\hat{\Phi}(\xi, s) = [s^2 \Phi(\xi) + 1] \hat{u}(s)$$

where $\Phi(\xi)$ is given by (B.1) and (B.2) in Appendix B. Hence with the aid of (3.2) one finds

$$\hat{\Phi}(\xi, s) = \frac{1}{2} \hat{k}_1(\xi) \hat{k}_2(\xi) \left[ s^2 e^{s(\beta_1 - \alpha)} \hat{y}(s) \right] - \frac{1}{2} \hat{k}_3(\xi) \hat{k}_4(\xi) \left[ s^2 e^{-s(\beta_1 - \alpha)} \hat{y}(s) \right]$$

where

$$\beta_1 := 2 \sqrt{\frac{\xi + \mu}{g}}, \quad \hat{k}_1(\xi) := l_0(s\beta_1) e^{-s\beta_1}, \quad \hat{k}_3(\xi) := K_0(s\beta_1) e^{s\beta_1}.$$ 

By [4] p. 245, (1) and p. 246, (16)], $\hat{k}_1$, $\hat{k}_3$ are the Laplace transforms of, respectively,

$$k_1(t) = \begin{cases} 1(t) & t \geq 0, \\ \frac{1}{\pi \sqrt{2\beta_1 t - t^2}}, & t < 0 \end{cases}, \quad k_3(t) = \frac{1}{\sqrt{t^2 + 2\beta_1 t}}, \quad t \geq 0.$$
Consequently,
\[
\phi(\xi, t) = \frac{1}{2\pi} \int_0^t \det \begin{pmatrix}
p_{\xi}(\tau) & q_{\xi}(\tau) \\
\dot{y}(t - \tau - \beta\xi + \alpha) & \dot{y}(t - \tau + \beta\xi - \alpha)
\end{pmatrix} d\tau
\]
where \( p_{\xi} = \pi k_{1\xi} \star k_2 \) \( \text{and} \) \( q_{\xi} := k_{3\xi} \star \pi k_4 \). This means that to find the position \( \phi(\xi, t) \) only, we do not need to calculate (3.4).

4. Stabilization of the chain at a final position

4.1. General considerations. The substitution \( X(t) := x(t) - d\dot{u}(t) \) transforms (2.1) into the model of boundary control in factor form:

\[
\begin{align*}
X(t) &= AX(t) + d\dot{u}(t) \\
X(0) &= 0 \\
y(t) &= h^*X(t) + u(t)
\end{align*}
\]

where the output equation remains unchanged because \( h^*d = 0 \). Recalling definitions of \( x \) and \( d \) we can see that

\[
X(t)(\xi) = \begin{pmatrix}
\phi_1(0, t) \\
\phi(\xi, t) - 1(\xi)u(t) \\
\phi(\xi, t)
\end{pmatrix} := \begin{pmatrix}
w(t) \\
\Phi(\xi, t) \\
\psi(\xi, t)
\end{pmatrix}, \quad \xi \in [0, L].
\]

If a final position of the chain is reached then next task is to maintain its final position. To achieve this goal we propose negative dynamical feedback control law of the colocated-type:

\[
\dot{u}(t) = -kd^#X(t) - \kappa u(t) = -kd^#x(t) - \kappa u(t), \quad k > 0, \quad \kappa > 0
\]

where

\[
d^#X = g(L + \mu)\Phi'(L), \quad D(d^#) = \{X \in H : \Phi' \text{ is continuous at } \theta = L\}.
\]

Observe that

\[
d \in D(d^#), \quad d^#d = 0
\]

and

\[
X \in D(A) \implies d^#X = -d^*AX \iff d^#|_{D(A)} = -d^*A = d^*A^*.
\]

Indeed, if \( X \in D(A) \) then

\[
-d^*AX = \langle AX, -d \rangle = \left\langle \begin{bmatrix} g\Phi'(0) \\ \psi \\ [g(\cdot + \mu)\Phi'(\cdot)]'
\end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1
\end{bmatrix} \right\rangle = \mu g\Phi'(0) + \int_0^L [g(\xi + \mu)\Phi'(\xi)]'d\xi = g(L + \mu)\Phi'(L).
\]

Now, by (4.4),

\[
\dot{X} = A \left[ X - kdd^#X - \kappa du \right] = A \left[ X - \kappa du - kdd^#X + kdd^#kdu \right] = A \left\{ \left[ X - \kappa du \right] - kdd^# \left[ X - \kappa du \right] \right\} = Ac \left[ X - \kappa du \right],
\]
where

\( A_c X := A (X - k d d^# X) \), \( D(A_c) = \{ X \in D(d^#) : X - k d d^# X \in D(A) \} \).

Therefore, the closed–loop feedback control system, having the structure depicted in Figure 4.1, is governed by

\[
\begin{align*}
\dot{X} &= A_c(X - \kappa u) \\
\dot{u} &= -\kappa u - k d^# X
\end{align*}
\]

and our goal reduces to showing that its RHS generates an EXS semigroup on \( H \times \mathbb{R} \).

**Theorem 4.1.** The operator \( A_c \) generates a \( C_0 \)–semigroup of contractions on \( H \).

**Proof.** We start from proving that \( A_c \) is \( \omega \)–dissipative with \( \omega = 0 \). Let \( X \in D(A_c) \) then, by (4.5) and (4.4), \( d^# z = d^# X, z := X - k d d^# X \in D(A) \), whence

\[
\langle A_c X, X \rangle + \langle X, A_c X \rangle = \langle A z, z + k d d^# z \rangle + \langle z + k d d^# z, A z \rangle =
\]

\[
= \langle A z, z \rangle + \langle z, A z \rangle + k d^# z \langle A z, d \rangle + k d^# z \langle d, A z \rangle =
\]

\[
= -2k[d^# z]^2 = -2k[d^# X]^2 \leq 0 .
\]

Next we prove that \( \mathcal{R}(\lambda I - A_c) = H \) for \( \lambda > 0 \). This requires proving that for each \( y \in H \) and \( \lambda > 0 \), the equation

\[
\lambda X - A \left[ X - k d d^# X \right] = y
\]

has a solution \( X \in D(d^#) \) such that \( X - k d d^# X \in D(A) \). Applying the resolvent of \( A \) to the both sides of (4.8) one obtains

\[
X + k \underbrace{A(\lambda I - A)^{-1} d}_{=\lambda(\lambda I - A)^{-1} d - d \in D(d^#)} d^# X = (\lambda I - A)^{-1} y .
\]

This equation may have a desired solution only if

\[
d^# X \left[ 1 + k \lambda d^# (\lambda I - A)^{-1} d \right] = d^# (\lambda I - A)^{-1} y
\]

is solvable with respect to \( d^# X \) for any \( \lambda > 0 \). Since \( A \) generates the unitary group of contractions \( \{ S(t) \} \in \mathbb{R} \) then

\[
\| (\lambda I - A)^{-1} d \| \leq \int_0^\infty \| S(t) d \| e^{-\lambda t} d t = \frac{1}{\lambda} \| d \| \quad \forall \lambda > 0
\]
and consequently, by the Schwarz inequality,
\[
\lambda d^*(\lambda I - A)^{-1}d \leq \lambda |d^*(\lambda I - A)^{-1}d| \leq \|d\|^2 \quad \forall \lambda \geq 0.
\]
Hence the transfer function of open-loop system, with new output \( y_c = d^#X \) and control \( \hat{u} \),
\[
\hat{G}(s) := sd^#(sl - A)^{-1}d = -s^2d^*(sl - A)^{-1}d + s\|d\|^2 (= d^#A(sl - A)^{-1}d) ,
\]
where \( s \in \mathbb{C} \) is not an eigenvalue of \( A \), satisfies:
\[
\hat{G}(\lambda) \geq 0 \quad \forall \lambda \geq 0.
\]
Thanks to this \( 1 + k\lambda d^#(\lambda I - A)^{-1}d = 1 + k\hat{G}(\lambda) \geq 1 \) and therefore (4.10) has a unique solution with respect to \( d^#X \),
\[
d^#X = \frac{1}{1 + k\hat{G}(\lambda)}d^#(\lambda I - A)^{-1}y, \quad \lambda > 0.
\]
Inserting this solution into (4.9) we conclude that (4.8) is uniquely solvable by
\[
X = (\lambda I - A)^{-1}y - \frac{k}{1 + k\hat{G}(\lambda)}A(\lambda I - A)^{-1}dd^#(\lambda I - A)^{-1}y
\]
and the assertion of our theorem follows from the Lumer–Phillips theorem [24, Theorem 4.3a, p. 14].

For further studies we need the detailed form of \( A_c \):
\[
A_c \begin{bmatrix} w \\ \Phi \\ \psi \end{bmatrix} = \begin{bmatrix} g\phi'(0) \\ \psi + kg(L + \mu)\phi'(L)1 \\ [g(\cdot + \mu)\phi'(\cdot)]' \end{bmatrix},
\]
\[
D(A_c) = \left\{ \begin{bmatrix} w \\ \Phi \\ \psi \end{bmatrix} \in \mathcal{H} : \begin{cases} \Phi \in \mathcal{H}^2(0, L) \\ \psi + kg(L + \mu)\phi'(L)1 \in \mathcal{H}^1(0, L) \\ \psi(0) = w \end{cases} \right\}.
\]
This comes from
\[
X - kdd^#X = \begin{bmatrix} w + kd^#X \\ \Phi \\ \psi + kd^#X1 \end{bmatrix} = \begin{bmatrix} w + kg(L + \mu)\phi'(L) \\ \Phi \\ \psi + kg(L + \mu)\phi'(L)1 \end{bmatrix} \in D(A),
\]
whence in particular
\[
\psi(0) + kg(L + \mu)\phi'(L)1(0) = w + kg(L + \mu)\phi'(L) \iff \psi(0) = w.
\]
There are various ways of proving that the semigroup generated by \( A_c \) is \( \text{EXS} \). The Lyapunov–Datko approach is relatively simple as shown in the proof of the following theorem.

**Theorem 4.2.** The \( C_0 \)-semigroup generated by \( A_c \) is \( \text{EXS} \).
Proof. Define the operator $\mathcal{H}_0$.

$$
\mathcal{H}_0 \begin{bmatrix} w \\ \Phi \\ \psi \end{bmatrix} := \begin{bmatrix} 0 \\ - \int_{\theta}^{L} \frac{\gamma(\xi)\psi(\xi)}{g(\xi + \mu)} \, d\xi \\ \gamma(\theta)\Phi'(\theta) \end{bmatrix}, \quad \gamma \in C^1[0, L].
$$

For $\Phi \in H^1_L(0, L)$ one has: $\Phi', \gamma \Phi' \in L^2(0, L)$ while for $\psi \in L^2(0, L)$ the second component of $\mathcal{H}_0 \psi$ is on $H^1_L(0, L)$, whence $\mathcal{H}_0$ is everywhere defined. In fact $\mathcal{H}_0 \in \mathbf{L}(H)$ as for any $X \in H$ there holds

$$
\|\mathcal{H}_0 X\|_H^2 = \int_0^L g(\theta + \mu) \frac{\gamma^2(\theta)\psi^2(\theta)}{g^2(\theta + \mu)} \, d\theta + \int_0^L \frac{\gamma^2(\theta)\Phi'(\theta)^2}{g(\theta + \mu)} \, d\theta \\
\leq \|X\|_H^2 \max_{\theta \in [0, L]} \frac{\gamma^2(\theta)}{g(\theta + \mu)}.
$$

We claim that $\mathcal{H}_0 = \mathcal{H}_0^*$. Indeed,

$$
\langle \begin{bmatrix} v \\ \varphi \\ \psi \end{bmatrix}_{\in H}, \mathcal{H}_0 \begin{bmatrix} w \\ \Phi \\ \psi \end{bmatrix}_{\in H} \rangle = \langle \begin{bmatrix} v \\ \varphi \\ \psi \end{bmatrix}, \begin{bmatrix} 0 \\ - \int_{\theta}^{L} \frac{\gamma(\xi)\psi(\xi)}{g(\xi + \mu)} \, d\xi \\ \gamma(\theta)\Phi'(\theta) \end{bmatrix} \rangle = \\
= \int_0^L [\gamma(\theta)\varphi'(\theta)]\psi(\theta) \, d\theta + \int_0^L \left( \gamma(\theta)\psi(\theta) \right) (g(\theta + \mu)\Phi'(\theta)) \, d\theta = \\
= \langle \begin{bmatrix} v \\ \varphi \\ \psi \end{bmatrix}, \begin{bmatrix} w \\ \Phi \\ \psi \end{bmatrix} \rangle = \langle \mathcal{H}_0 \begin{bmatrix} v \\ \varphi \\ \psi \end{bmatrix}_{\in H} \rangle.
$$

Farther,

$$
\langle A_c X, \mathcal{H}_0 X \rangle = \left\langle \begin{bmatrix} g\Phi'(0) \\ \psi + kg(L + \mu)\Phi'(L) \end{bmatrix}, \begin{bmatrix} 0 \\ - \int_{\theta}^{L} \frac{\gamma(\xi)\psi(\xi)}{g(\xi + \mu)} \, d\xi \\ \gamma(\theta)\Phi'(\theta) \end{bmatrix} \right\rangle
$$

and

$$
\langle X, \mathcal{H}_0 A_c X \rangle = \left\langle \begin{bmatrix} w \\ \Phi \\ \psi \end{bmatrix}, \begin{bmatrix} 0 \\ - \int_{\theta}^{L} \frac{\gamma(\xi)\psi(\xi)}{g(\xi + \mu)} \, d\xi \\ \gamma(\theta)\psi'(\theta) \end{bmatrix} \right\rangle.
$$
whence for $X \in D(A_c)$:
\[
\langle A_c X, H_0 X \rangle + \langle X, H_0 A_c X \rangle = \\
= \int_0^L \gamma(\theta) \frac{d}{d\theta} [\psi^2(\theta)] \, d\theta + 2 \int_0^L \Phi'(\theta) \gamma(\theta) [g(\theta + \mu) \Phi'(\theta)]' \, d\theta = \\
= \int_0^L \gamma(\theta) \frac{d}{d\theta} [\psi^2(\theta)] \, d\theta + \int_0^L \gamma(\theta) g(\theta + \mu) \frac{d}{d\theta} [\Phi'(\theta)]^2 \, d\theta + 2 \int_0^L g \gamma(\theta) [\Phi'(\theta)]^2 \, d\theta = \\
= \gamma(L) \psi^2(L) - \gamma(0) \psi^2(0) - \int_0^L \gamma'(\theta) \psi^2(\theta) \, d\theta + \gamma(L) g(L + \mu) [\Phi'(L)]^2 \\
- \gamma(0) g \mu [\Phi'(0)]^2 + \int_0^L \left\{ \frac{\gamma(\theta)}{\theta + \mu} - \gamma'(\theta) \right\} [\Phi'(\theta)]^2 g(\theta + \mu) \, d\theta = \\
= -\gamma(0) g \mu [\Phi'(0)]^2 - \gamma(0) w^2 + \int_0^L \left\{ \frac{\gamma(\theta)}{\theta + \mu} - \gamma'(\theta) \right\} [\Phi'(\theta)]^2 g(\theta + \mu) \, d\theta \\
- \int_0^L \gamma'(\theta) \psi^2(\theta) \, d\theta + \gamma(L) g(L + \mu) [\Phi'(L)]^2 [1 + k^2 g(L + \mu)] .
\]
Assume that $\gamma$ solves the initial–value problem
\[
\begin{align*}
\gamma'(-) - \frac{\gamma(\theta)}{\theta + \mu} &= 1 \\
\gamma(0) &= \mu
\end{align*}
\]
Then $\gamma(\theta) = (\theta + \mu) [\ln(\theta + \mu) - \ln \mu + 1]$ and consequently:

- By (4.11), $\|H_0\| \leq \max_{\theta \in [0, L]} \left\{ \sqrt[2]{\frac{\theta + \mu}{g}} [\ln(\theta + \mu) - \ln \mu + 1] \right\}$ where
  \[
  \delta := \ln(L + \mu) - \ln \mu > 0.
  \]
- For any $X \in D(A_c)$
  \[
  \langle A_c X, H_0 X \rangle + \langle X, H_0 A_c X \rangle \leq \\
  \leq -\mu w^2 - \int_0^L [\Phi'(\theta)]^2 g(\theta + \mu) \, d\theta - \int_0^L [\ln(\theta + \mu) - \ln \mu + 2] \psi^2(\theta) \, d\theta + \\
  + [\ln(L + \mu) - \ln \mu + 1] g(L + \mu) [\Phi'(L)]^2 [1 + k^2 g(L + \mu)] \leq \\
  \leq -\|X\|^2 + (\delta + 1) g(L + \mu) [\Phi'(L)]^2 [1 + k^2 g(L + \mu)].
  \]

The two facts above, jointly with (4.7) and (4.3), yield
\[
\langle A_c X, (\epsilon l + H_0) X \rangle + \langle X, (\epsilon l + H_0) A_c X \rangle \leq -\|X\|^2 \quad \forall X \in D(A_c)
\]
where $\epsilon := \frac{(\delta + 1) [1 + k^2 g(L + \mu)]}{2 g k}$. Since
\[
\epsilon l + H_0 \geq (\epsilon - \|H_0\|) l \geq \frac{(\delta + 1)(k \sqrt{g(L + \mu) - 1})^2}{2 g k} \geq 0 ,
\]
we have proved that the Lyapunov operator inequality
\[
\langle A_c X, \mathcal{X}X \rangle + \langle X, \mathcal{X}A_c X \rangle \leq -\|X\|^2 \quad \forall X \in D(A_c)
\]
has a solution $X \in L(H)$, $X = X^* \geq 0$, namely: $X = \epsilon I + \mathcal{H}_0$ from which $\text{EXS}$ follows by the result of Datko [10], with a remark that in the case $k \neq \frac{1}{\sqrt{g(L+\mu)}}$ (as coercivity of $\epsilon I + \mathcal{H}_0$ is then ensured) one can apply the Lumer–Phillips theorem [24, Theorem 4.3a, p. 14] as well.

4.2. **Exponential stabilization.** Consider the system with the state operator $\mathcal{A}_c$, the factor control vector $d$, controlled by $f$, and with the output $y_c = d^#X$:

\begin{align}
\dot{X} &= \mathcal{A}_c(X + df) \\
y_c &= d^#X .
\end{align}

Observe that

\begin{align}
X \in D(\mathcal{A}_c) \implies X \in D(d^#), & \quad X - kdd^#X \in D(\mathcal{A}), \\
d^*\mathcal{A}_cX = d^*\mathcal{A}(X - kdd^#X) = -d^#(X - kdd^#X) = -d^#X \implies d^#|_{D(\mathcal{A}_c)} = -d^*\mathcal{A}_c .
\end{align}

Let $\{S_c(t)\}_{t \geq 0}$ be the $\text{EXS}$ semigroup of contractions generated by $\mathcal{A}_c$ as stated in Theorems 4.1 and 4.2. The following facts hold – see, e.g., [18, Section 2].

**F1** $X_0 \in H$, $f \in W^{1,2}[0, \infty)$ then

\begin{align}
X(t) = S_c(t)X_0 + \mathcal{A}_c \int_0^t S_c(t - \tau)df(\tau)d\tau
\end{align}

is a weak solution of (4.12) with initial condition $X(0) = X_0$.

**F2** $X_0 \in D(\mathcal{A}_c)$, $f \in W^{2,2}[0, \infty)$, $f(0) = 0$ and $f'(0) = 0$ then $x$ is a classical solution, and it may be represented as

\begin{align}
X(t) = S_c(t)X_0 + \int_0^t S_c(t - \tau)df(\tau)d\tau - df(t) .
\end{align}

Moreover, then

\begin{align}
y_c(t) = d^#X(t) = (\Psi_cX_0)(t) - \int_0^t d^*S_c(t - \tau)df(\tau)d\tau + d^*df(t) ,
\end{align}

where $\Psi_c$ denotes the observability operator, $\Psi_c : H \ni X_0 \mapsto d^#S_c(\cdot)X_0 \in L^2(0, \infty)$, and by $\text{EXS}$, $\Psi_c$ is well–defined on $D(\mathcal{A}_c)$, a dense subspace of $H$. The output $y_c$ is then a continuous function of $t \geq 0$.

**F3** The observation functional $d^#$ is called (infinite–time) admissible if $\Psi_c$ is bounded on $D(\mathcal{A}_c)$. If the latter holds then $\Psi_c$ is closable, its closure $\overline{\Psi}_c$ belongs to $L(H, L^2(0, \infty))$ and

\begin{align}
(\overline{\Psi}_cX_0)(t) = -\frac{d [d^*S_c(t)X_0]}{dt}, \quad X_0 \in H .
\end{align}

Actually, $\overline{\Psi}_cX_0 \in L^1(0, \infty) \cap L^2(0, \infty)$ for any $X_0 \in H$ [14, Appendix C]. Furthermore, if $X_0 \in H$ and $f \in W^{1,2}_0[0, \infty)$ then

\begin{align}
y_c = \overline{\Psi}_cX_0 + \mathcal{F}_c f ,
\end{align}

where $\mathcal{F}_c$ stands for the input–output operator, $\mathcal{F}_c : L^2(0, \infty) \to L^2(0, \infty)$, defined on $W^{1,2}_0[0, \infty)$, a dense subspace of $L^2[0, \infty)$, as

\begin{align}
(\mathcal{F}_c f)(t) := \int_0^t (\overline{\Psi}_c d) (t - \tau)f(\tau)d\tau = \frac{d}{dt} \int_0^t (\overline{\Psi}_c d) (t - \tau)f(\tau)d\tau .
\end{align}
If, in addition, the transfer function of the system \((4.12) \div (4.13)\)
\[
\hat{g}_c(s) := s \left( \hat{\Psi} d \right)(s) = sd^#(sl - A_c)^{-1}d = d^#A_c(sl - A_c)^{-1}d
\]
belongs to \(H^\infty(\mathbb{C}^+)\) then, by the Paley–Wiener theory for the standard Laplace transformation, \(F_c\) is also bounded on its domain of definition, whence closable, its closure \(\overline{F_c}\) belongs to \(L(L^2(0, \infty))\) and
\[
(\overline{F_c}f)(t) := \frac{d}{dt} \int_0^t (\overline{\Psi} d)(t - \tau)f(\tau)d\tau,
\]
\[
(\overline{F_c}f)(s) = \hat{g}_c(s)\hat{f}(s).
\]
Consequently,
\[
y_c = \overline{\Psi} c X_0 + \overline{F_c}f, \quad \hat{y}_c = \overline{\Psi} c X_0 + \overline{F_c}f \iff \hat{y}_c(s) = d^#(sl - A_c)^{-1}X_0 + \hat{g}_c(s)\hat{f}(s), \quad s \in \mathbb{C}^+.
\]
(F5) Define the operator \(Wf := \int_0^\infty S_c(t)df(t)dt\). By EXS, \(W \in L(L^2(0, \infty), H)\).

The factor control vector \(d\) is called (infinite–time) admissible if the range of \(W\) is contained in \(D(A_c)\). If the latter holds, then by the closed–graph theorem: \(Q_c := A_cW \in L(L^2(0, \infty), H)\). Moreover, then for every \(X_0 \in H\) and every \(f \in L^2(0, \infty)\)
\[
X(t) = S_c(t)X_0 + Q_cR_tf = \text{the RHS of } (4.14),
\]
where \(R_t\) denotes the operator of reflection at \(t \geq 0\) defined as
\[
(R_tf)(\tau) := \begin{cases} f(t - \tau), & \tau \in [0, t) \\ 0, & \tau \geq t \end{cases}
\]
is a weak solution of \((4.12)\). Finally, the operator \(f \mapsto Q_cR_tf\) is everywhere defined and bounded operator from \(L^2(0, \infty)\) into \(\text{BUC}_0([0, \infty), H)\).

The above facts (F1) \div (F5) apply because here we have:

✔ In view of the Lyapunov criterion of admissibility \([12, \text{Theorem 3, p. } 322], [16, \text{Theorem 1.1, p. } 88]\) and \([18, \text{Lemma } 2.1]\), \((4.7)\) immediately implies that \(d^#\) is admissible (with respect to the semigroup \(\{S_c(t)\}_{t \geq 0}\)). By duality \([16, \text{Theorem } 4.3, \text{p. } 110]\), \(d\) is an admissible factor control vector iff \(d^*A_c^*\) is an admissible observation functional, but this fact again follows from \((4.7)\). Indeed,
\[
\langle A_cX, X \rangle + \langle X, A_cX \rangle = -2k[d^#X]^2 = -2k[d^*A_cX]^2 \quad \forall X \in D(A_c) \quad \overset{A_cX=z}{\iff}
\]
\[
\langle z, A_c^{-1}z \rangle + \langle A_c^{-1}z, z \rangle = -2k[d^*z]^2 \quad \forall z \in H \quad \overset{(A_c^{-1})^{-1}z=Y}{\iff}
\]
\[
\langle (A_c^{-1})^* z, z \rangle + \langle z, (A_c^{-1})^* z \rangle = -2k[d^*z]^2 \quad \forall z \in H \quad \overset{(A_c^{-1})^{-1}z=Y}{\iff}
\]
\[
\langle Y, A_c^*Y \rangle + \langle A_c^*Y, Y \rangle = -2k[d^*A_c^*Y]^2 \quad \forall Y \in D(A_c^*) ,
\]
which, again by the Lyapunov criterion of admissibility, implies that \(d^*A_c^*\) is an admissible observation functional.

✔ The system \((4.12) \div (4.13)\) corresponds to the heavy chain system depicted in Figure 4.1 with proportional control loop only, marked on this figure by a dashed box, whence its is clear that
\[
\hat{g}_c(s) = \frac{\hat{G}(s)}{1 + k\hat{G}(s)}.
\]
Since
\[ \left| \frac{z}{1 + k z} \right|^2 = \frac{|z|^2}{1 + 2k \text{Re} z + k^2 |z|^2} \leq \frac{1}{k^2}, \quad \text{Re} z \geq 0, \]
the rational complex function \( z \mapsto \frac{z}{1 + k z} \) is bounded by \( 1/k \) on \( \mathbb{C}^+ \cup j\mathbb{R} \). Hence \( \| \hat{G}_c \|_{H^\infty(\mathbb{C}^+)} = 1/k \), provided that the open-loop transfer function \( \hat{G} \) maps \( \mathbb{C}^+ \cup j\mathbb{R} \) into itself. However, this is the case of our \( \hat{G} \). Indeed,
\[
2 \text{Re} \hat{G}(s) = \hat{G}(s) + \overline{\hat{G}(s)} = d^* \hat{A}^2 \left[ (-\bar{s} I - \hat{A})^{-1} - (s I - \hat{A})^{-1} \right] d
\]
and, by the resolvent identity,
\[
\text{Re} \hat{G}(s) = \text{Re} s \ d^* \hat{A}^2 (-\bar{s} I - \hat{A})^{-1} (s I - \hat{A})^{-1} = \text{Re} s \ d^* \left[ \hat{A} (s I - \hat{A})^{-1} \right]^* \left[ \hat{A} (s I - \hat{A})^{-1} \right] d,
\]
whence \( \text{Re} \hat{G}(s) \geq 0 \) if \( \text{Re} s \geq 0 \).

Now we pass to the analysis of (4.6). Its idea is as follows: from the first equation of (4.6), we express \( X \) and \( d^* X \) as operator functions of \( X_0 \in \mathcal{H} \) and \( u \in L^2(0, \infty) \). This enables us to eliminate \( d^* X \) from the second equation of (4.6), yielding an equation determining the closed-loop variable \( u \), provided that \( u \in L^2(0, \infty) \) for each \( u_0 \in \mathbb{H} \) and \( X_0 \in \mathcal{H} \). It is more convenient to write down this equation in terms of the Laplace transform – see (4.15) below. Then Lemma 4.1 below is pivotal to ensure that its solution \( u \in L^2(0, \infty) \) is unique and has desired properties. Going back to the first equation of (4.6) we can conclude that the state trajectory \( X \), corresponding to this control \( u \), jointly with \( u \) is a pair being the unique weak solution of the interconnected system (4.6), which leads to its well-posedness. Let us give details of this circular reasoning. By \( (F4) \) and \( (F5) \) with \( f = -\kappa u \), an initial state \( X_0 \) and a \( L^2(0, \infty) \)–control \( u \) give rise to a weak solution of the first equation of (4.6)
\[ X(t) = S_c(t) - \kappa Q_c R_t u, \quad X \in \text{BUC}_0([0, \infty), \mathcal{H}) \]
and to the output
\[ \hat{y}_c(s) = d^* \hat{X}(s) = (\overline{\Psi_c X_0}) (s) - \kappa \hat{g}_c(s) \hat{u}(s), \]
which enters the second equation of (4.6) as a forcing term \( -k \hat{y}_c \). Thus if the second equation of (4.6) evolves from an initial state \( u_0 \) and this forcing term then, by the classical existence theory for a scalar ordinary differential equation, one concludes that the second equation of (4.6) has a unique solution \( u \in W^{1,2}([0, \infty)) \), provided that the equation
\[ (4.15) \quad \left[ s + \kappa - \kappa k \hat{g}_c(s) \right] \hat{u}(s) = \left[ s + \frac{\kappa}{1 + k \hat{G}(s)} \right] \hat{u}(s) = u_0 - k (\overline{\Psi_c X_0}) (s) \]
has a unique solution \( \hat{u} \in H^2(\mathbb{C}^+) \). The existence of such a solution can be established using the following lemma.

**Lemma 4.1.** The complex functions: \( s \mapsto \frac{1}{s + \frac{\kappa}{1 + k \hat{G}(s)}} := \Omega(s) \) and \( s \mapsto s \Omega(s) \) belong to \( H^\infty(\mathbb{C}^+) \).
Proof. Since \( \text{Re} \hat{G}(s) = 0 \) for \( \text{Re} \, s = 0 \), we have \( \hat{G}(j\omega) = j \text{Im} \hat{G}(j\omega) \), where \( \text{Im} \hat{G}(j\omega) \) varies from \(-\infty\) to \(+\infty\) if \( \omega \) varies between two consecutive eigenfrequencies \( \omega_k \) and \( \omega_{k+1} \), and the imaginary axis is, under \( \hat{G} \), mapped into the Nyquist curve \( \{ \hat{G}(j\omega) \}_{\omega \in \mathbb{R}, j\omega \notin \sigma(A)} \). Hence, the Nyquist curve of \( \frac{1}{1+k_\hat{G}} \) is contained in the Nyquist curve of \( \frac{1}{1+k_\hat{G}} \) being the circle \( |z - \frac{1}{2}| = \frac{1}{2} \). Consequently, the equation

\[
\frac{-s}{\kappa} = \frac{1}{1+k\hat{G}(s)}
\]

has no solution in \( s \in \mathbb{C}^+ \cup j\mathbb{R} \), because its LHS takes values in \( s \in \mathbb{C}^- \cup j\mathbb{R} \). Hence the functions \( \Omega \) and \( s \mapsto s\Omega(s) \) are both bounded on any bounded subset of \( s \in \mathbb{C}^+ \cup j\mathbb{R} \). On the other side,

\[
\left| s + \frac{\kappa}{1+k\hat{G}(s)} \right| \geq |s| - \kappa \quad \text{for} \quad s \in \mathbb{C}^+ \cup j\mathbb{R}, \ |s| \geq 2\kappa
\]

whence

\[
|\Omega(s)| \leq \frac{1}{\kappa}, \quad |s\Omega(s)| \leq 2 \quad \text{for} \quad s \in \mathbb{C}^+ \cup j\mathbb{R}, \ |s| \geq 2\kappa
\]

\( \square \)

Since, by \((F3), \overline{X}_0 \in L^1(0, \infty)\), the RHS of \((4.15)\) is in \( H^\infty(\mathbb{C}^+) \). Thus, by Lemma \(4.1, (4.15)\) has a unique solution \( \hat{u} \in H^\infty(\mathbb{C}^+) \) such that \( s \mapsto s\hat{u}(s) \in H^\infty(\mathbb{C}^+) \) which implies \( \hat{u} \in H^2(\mathbb{C}^+) \). Consequently, for each \((X_0, u_0)\) there exists a unique weak solution to the first equation of \((4.6)\) and a unique \( W^{1,2}([0, \infty)) \)-solution \( u \) of the second equation of \((4.6)\), whence \((4.6)\) has a unique weak solution for every initial condition. This implies, via Ball’s theorem with \( f \equiv 0 \) [2, p. 371] and [24, p. 259] that the RHS of \((4.6)\) generates a \( C_0 \)-semigroup on \( \mathbb{H} \times \mathbb{R} \).

Actually this semigroup is \( \text{EXS} \), which can be shown using Datko’s theorem. For that we have to prove that not only \( \hat{u} \in H^2(\mathbb{C}^+) \) but also \( \hat{X} \in H^2(\mathbb{C}^+, \mathbb{H}) \) for any initial condition. For the latter observe that

\[
\hat{X}(s) = (sI - A_c)^{-1}X_0 - \kappa A_c (sI - A_c)^{-1}d\hat{u}(s) = \begin{align*}
&= (sI - A_c)^{-1}X_0 - \kappa (sI - A_c)^{-1} \int_0^s [s\hat{u}(s) + \kappa d\hat{u}] \\
&\quad + \int_0^s [s\hat{u}(s) + \kappa d\hat{u}].
\end{align*}
\]

Components \(1\) and \(3\) are in \( H^2(\mathbb{C}^+, \mathbb{H}) \) by \( \text{EXS} \) of the semigroup \( \{S_c(t)\}_{t \geq 0} \) and because \( \hat{u} \in H^2(\mathbb{C}^+) \). \(2\) is in \( H^2(\mathbb{C}^+, \mathbb{H}) \) too, as \( s \mapsto s\hat{u}(s) \) is in \( H^2(\mathbb{C}^+) \) whilst, again by \( \text{EXS} \): \( S_c(\cdot)d \in L^1(0, \infty; \mathbb{H}) \), whence \( s \mapsto (sI - A_c)^{-1}d \in H^\infty(\mathbb{C}^+, \mathbb{H}) \). The results of this section may be abbreviated as the following theorem.

**Theorem 4.3.** The RHS of \((4.6)\) generates an \( \text{EXS} \) \( C_0 \)-semigroup on \( \mathbb{H} \times \mathbb{R} \).
4.3. **Realization of the stabilizing feedback.** Going back to nonabstract notation we obtain the velocity proportional–integral control law

\[
\dot{u}(t) = -kg(L + \mu)\phi_\xi(L, t) - \kappa u(t) = -kg(L + \mu)\tan[\alpha(L, t)] - \kappa u(t) 
\approx -kg(L + \mu)\alpha(L, t) - \kappa u(t)
\]

where \(\alpha(L, t)\) is the angle between the chain and the 0\(\xi\)–axis at \(\xi = L\). Thus our control law (4.16) is physically realizable, provided that the platform velocity \(\dot{u}(t)\) can be made a linear combination of measurement of the angle \(\alpha(L, t)\) and the platform position \(u(t)\). Contemporary control equipment allows to realize this task as it will be seen in the next section.

5. Example

5.1. **General remarks.** The result of Section 4.3 has been addressed to Andrzej Turnau from the Institute of Automatics, AGH University of Science and Technology, to whom I am deeply indebted out of his enthusiasm for verifying my results by laboratory experiments. Currently, as the supervisor of Piotr Bałys’ MSc Thesis, he suggested to apply a linear

![Figure 5.1. Laboratory experimental setup](image-url)
robot module NSK XY-HRS060EH202\textsuperscript{1} for a realization of the controlled moving platform [3, Section 5].

In the laboratory experiments a motorcycle chain of length $L = 1.89\text{[m]}$ and of weight $M = 1.33\text{[kg]}$, loaded by an iron ball of mass $m = 1.45\text{[kg]}$ was used as the heavy chain and the punctual load, respectively. Constructing details of the test bed are depicted in Figures 5.1, 5.2 and 5.3. In addition, Figure 5.3 shows the principle of the angle $\alpha(L, t)$

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure5_2.png}
\caption{Fastening the chain: front view.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure5_3.png}
\caption{Fastening the chain: lateral view.}
\end{figure}

measurement by an encoder. For more details on both the construction of test bed and the on–line, real–time computer control system we refer to [3]. The mass of the chain is comparable with the punctual load mass and therefore cannot be neglected. Further, $\mu = 2.060526316\text{[m]}$, $\alpha = 0.9166099656\text{[s]}$, $\beta = 1.269179275\text{[s]}$.

\textsuperscript{1}NSK–Ltd. is the Japanese company producing hardware for the control of mechanical systems.
In our laboratory test bed the velocity control of the linear robot module is available using the method of *pulse trains* [3. Section 2.3, pp. 15 - 16], whence \( \dot{u} \) has to be computed. To verify whether a control is efficient one ought to execute it and compare the theoretical position \( u \) with an experimental, obtained from the measurement due to the position encoder built–in the module.

### 5.2. Solution of an exemplary motion planning problem

To achieve the *delayed ninth order polynomial* output trajectory

\[
y(t) := \begin{cases} 
0 & \text{if } t \in [0, \beta - \alpha] \\
y_1(t - \beta + \alpha) & \text{if } t \in [\beta - \alpha, T + \beta - \alpha] \\
2L/7 = u_\infty & \text{if } t \in [T + \beta - \alpha, \infty) 
\end{cases}
\]

where \( 2L/7 = u_\infty = 0.54[\text{m}] \) is smaller than 0.60[\text{m}] – the range of distances within which the module NSK XY-HRS060EH202 can move, and

\[
y_1(t) := \frac{2Lt^5}{T^9} [70t^4 - 315t^3T + 540t^2T^2 - 420tT^3 + 126T^4], \\
y_1(t) = \frac{180Lt^4}{T^9} [t^4 - 4t^3T + 6t^2T^2 - 4tT^3 + T^4], \\
y_1(t) = \frac{720Lt^3}{T^9} [2t^4 - 7t^3T + 9t^2T^2 - 5tT^3 + T^4] \\
y_1(t) = \frac{720Lt^2}{T^9} [14t^4 - 42t^3T + 45t^2T^2 - 20tT^3 + 3T^4] \\
y_1(t) = \frac{4320Lt}{T^9} [14t^4 - 35t^3T + 30t^2T^2 - 10tT^3 + T^4]
\]

we apply the control \( \ddot{u} \) or \( \dddot{u} \) given by (3.3). In the sequel we shall assume \( T = 10[\text{s}] \). In computer implementation of (3.3) and (3.4) the simplest *method of rectangles* with constant base \( \Delta \) (uniform step) was used to approximate the convolutions:

\[
u(n\Delta) = \frac{\Delta}{2\pi} \left[ \sum_{k=0}^{n} p(k\Delta)\dddot{y}(n\Delta - k\Delta + \beta - \alpha) - \sum_{k=0}^{n} q(k\Delta)\dddot{y}(n\Delta - k\Delta - \beta + \alpha) \right], \\
\ddot{u}(n\Delta) = \frac{\Delta}{2\pi} \left[ \sum_{k=0}^{n} p(k\Delta)\dddot{y}(n\Delta - k\Delta + \beta - \alpha) - \sum_{k=0}^{n} q(k\Delta)\dddot{y}(n\Delta - k\Delta - \beta + \alpha) \right], \\
\dddot{u}(n\Delta) = \frac{\Delta}{2\pi} \left[ \sum_{k=0}^{n} p(k\Delta)\dddot{y}(n\Delta - k\Delta + \beta - \alpha) - \sum_{k=0}^{n} q(k\Delta)\dddot{y}(n\Delta - k\Delta - \beta + \alpha) \right]
\]

with a remark that in Matlab both formulae can easily be realized employing the \texttt{conv} command. Observe that

\[
y^{(n)}(t) := \begin{cases} 
0 & \text{if } t \in [0, \beta - \alpha] \\
y_1^{(n)}(t - \beta + \alpha) & \text{if } t \in [\beta - \alpha, T + \beta - \alpha] \\
0 & \text{if } t \in [T + \beta - \alpha, \infty) 
\end{cases}, \quad n = 1, 2, 3, 4.
\]

The corresponding plots of the kernel functions \( p \) and \( q \) are depicted in Figures 5.4 and 5.5. They have been found using quadrature procedures built–in both Matlab and Maple.
Observe that $p$ and $q$ have narrow (logarithmic) peaks around their discontinuities at $t = 2\beta = 2.538358550$ and $t = 2\alpha = 1.833219931$, respectively.

Figures 5.6, 5.7 and 5.8 present the results of computations of $u$ for $\Delta = 0.02[s]$ – in red, for $\Delta = 0.05[s]$ – in yellow, and for $\Delta = 0.1[s]$ – in cyan and comparison of $u$ with $y$ – in black.

Despite of this an experiment on a real laboratory test bed has been done in which the theoretically calculated velocity control $\dot{u}$, corresponding to $\Delta = 0.05[s]$, was executed from Matlab workspace. This control resulted in experimental position of the platform measured
Figure 5.7. Influence of the step size $\Delta$ on stability of computations of $\dot{u}$ for $\Delta = 0.02$[s] – in red, for $\Delta = 0.05$[s] – in yellow, and for $\Delta = 0.1$[s] – in cyan and comparison of $\dot{u}$ with $\dot{y}$.

Figure 5.8. Influence of the step size $\Delta$ on stability of computations of $\ddot{u}$ for $\Delta = 0.02$[s] – in red, for $\Delta = 0.05$[s] – in yellow, and for $\Delta = 0.1$[s] – in cyan and comparison of $\ddot{u}$ with $\ddot{y}$.

by the position encoder and compared in Figure 5.9 with the theoretical position of the platform. This laboratory experiment and the next ones, described in Section 5.3, have been done by two MSc students: Mateusz Brandys and Michał Zborowski to whom the author is deeply indebted. A maximal inaccuracy did not exceed approximately 1% which may be regarded as an excellent agreement.

Unfortunately our laboratory setup does not permit yet to measure $y$ directly (it is being planned to arrange such a measurement in the future). Instead of this, we have compared the theoretically determined positions of the chain, depicted in Figure 5.11, with a real

\footnote{Currently a digital video camera recording the chain positions was installed.}
observation showing an agreement between theory and practice: the absence of oscillations around the final position in our experiment was regarded as a characteristic feature confirming the results obtained by means of the method presented in Section 3.3.

5.3. **Experiments with colocated controller (4.16)**. In order to verify how good is the control law (4.16), the chain hanging silently in the position $0.3\,[\text{m}]$ was excited by a rapid hitting the loading ball with an iron hammer. This experiment realizes almost ideal elastic collision of two masses in which the momentum of the hammer is converted into the ball momentum, thanks to which an initial velocity is being given only to the ball rather than
to the whole chain. Hence we may regard that then the closed-loop system (4.6) evolves from an initial condition \[ X(0) \mathcal{T} u(0) = \begin{bmatrix} c & 0 & 0 & u_0 \end{bmatrix} = \begin{bmatrix} cr^T & u_0 \end{bmatrix} \] where \( c \) is a positive (negative) constant, if the ball is hit from the left (right). The theoretical analysis of Section 4 shows that, after some transient oscillations, the closed-loop system should return to the initial position, here 0.3[m]. This is, with a remarkable precision, confirmed by laboratory experiments, the results of which are depicted in Figures 5.12 to 5.20.
some imperfections in realization of the mathematical model of both the open–loop system and the controller, the lack of symmetry in measurements and a possible lack of symmetry in geometric configuration of the laboratory setup, generally, the platform does not exactly return to its initial position. However, the difference between initial and final positions is less than 1% and can be reduced by increasing $\kappa$. Nevertheless, while keeping the value of $k$ constant, increasing of $\kappa$ deteriorates the damping of oscillations.
Figure 5.17. Real transient position of the platform as a reaction on hitting the ball from left: the case $k = 0.07741$, $\kappa = 2$.

Figure 5.18. Real transient position of the platform as a reaction on hitting the ball from right: the case $k = 0.07741$, $\kappa = 2$.

Figure 5.19. Real transient position of the platform as a reaction on hitting the ball from left: the case $k = 0.25803$, $\kappa = 2$.

Figure 5.20. Real transient position of the platform as a reaction on hitting the ball from right: the case $k = 0.25803$, $\kappa = 2$. 
6. Discussion and conclusions

Surprisingly, using directly the definition (4.1) of the state vector \( X \) to the original chain dynamics (1.1) one obtains the additive abstract model

\[
\begin{aligned}
X(t) &= AX(t) + b\dot{u}(t), \\
X(0) &= 0 \\
y(t) &= h^*X(t) + u(t)
\end{aligned}
\]

(6.1)

where \( b \notin H \) because \( 1 \notin H_L \). This illustrates some differences between the abstract models in factor and in additive forms. The additive form of state equations involves an additional

\[
\langle A^1x_1, A^1x_2 \rangle_H^1 \leq \langle A^1x_1, A^1x_2 \rangle_H^1.
\]

To get consistency of these two approaches we should have \( A_e^{-1}b = d \) where \( A_e^{-1} \in L(H) \) is the extension of \( A^{-1} \in L(H) \). Recall that

\[
A^1x = bx + \|x\| = rh^*x - hr^*x + \|x\| = \Phi(0)r - \mu
\]

and observe that \( b \in \mathbb{R} \oplus H^1(0, L) \oplus L^2(0, L) \), whence \( A_e^{-1}b = -1(0) + \|x\| = -r + \|x\| = d \). Here \( b \) is a natural extension of \( b \in L(H) \) to \( b \in L(\mathbb{R} \oplus H^1(0, L) \oplus L^2(0, L)) \) and \( b_e = d + r \).

The representations (3.3) and (3.4) are numerically more convenient than those of [27].

The essence of the idea of determination of the chain position which was given in Section 3.3 is to eliminate \( u \) from (2.1) with \( \ddot{u}(0) = 0 \) which is generally possible. Indeed, since \( h \in D(A) \) and \( h^*d = 0 \) then by definition of a weak solution (2.3) one has

\[
\dot{\dot{y}}(t) = \frac{d}{dt}[h^*x(t)] + \dot{\ddot{u}}(t) = -(Ah)^*x(t) + \ddot{u}(t) = -\int_0^t (Ah)^*S(t - \tau)\dot{u}(\tau) \, d\tau + \ddot{u}(t)
\]

and the convolution in the RHS is absolutely continuous. Thus for almost all \( t \geq 0 \)

\[
\ddot{u}(t) = \ddot{y}(t) + \frac{d}{dt}[(Ah)^*x(t)]
\]

Elimination of \( \ddot{u} \) from (2.1) leads to

\[
Ex(t) = Ax(t) + d\dot{y}(t), \quad E := I - d(\mathcal{A}h)^* \in L(H),
\]

i.e., to an infinite-dimensional singular or descriptor system because \( d \in \mathcal{N}(E) \) and \( \mathcal{R}(E) \subset \{ x \in H : (Ah)^*x = 0 \} \). Now, if we assume that the output \( y \) plays a role of a new control then it can be observed that, in an accordance with [23] Definition 3.1 and Remarks 3.4], this is a resolvent linear system. However, since

\[
(sE - A)^{-1}d = \frac{1}{\gamma(s)}(sl - A)^{-1}d,
\]

this system is not a distributional resolvent system [23] Definition 3.2] as \( 1/\gamma(s) \) grows faster than polynomially along the positive real semi-axis.

As we already said \( \text{EXS} \) of the semigroup \( \{ S_c(t) \}_{t \geq 0} \), generated by \( A_c \), can be proved by other means. In the second part of this paper [15], Theorem 4.2 is proved using a spectral approach and some relationships between \( \text{EXS} \) and exact observability. In comparison of those two attempts, the Lyapunov approach is relatively simple provided that a quadratic Lyapunov functional available is know. This is the case of a heavy chain closed–loop control system as shown in Section 4.1. It is rather clear that \( k \) has to be positive, to ensure \( \text{EXS} \) of the semigroup \( \{ S_c(t) \}_{t \geq 0} \), however it should be stressed that exponential stabilization
cannot be achieved by using the purely proportional control law only, i.e., by applying (4.2) with \( \kappa = 0 \). Indeed, in this case one has
\[
\dot{u}(t) = -kd^\#X \Rightarrow u(t) = u_0 - k \int_0^t d^\#S_c(\tau)X_0d\tau = u_0 + k \int_0^t \frac{d}[d^*S_c(\tau)X_0]}{d\tau}d\tau = u_0 + kd^*S_c(t)X_0 - kd^*X_0 \rightarrow u_0 - kd^*X_0 \text{ as } t \rightarrow \infty,
\]
whence the final position \( u(\infty) := \lim_{t \rightarrow \infty} u(t) \) of the platform equals its initial position \( u_0 \) iff \( d^*X_0 \). In the case of initial conditions generated by hammer hits, (4.6) evolves from an initial condition \( X_0 = X(0)^T = \begin{bmatrix} c & 0 & 0 \end{bmatrix} = cr^T \) where \( c \) is a positive (negative) constant, if the ball is hit from the left (right). Since then \( d^*r = -\mu \neq 0 \), one has \( u(\infty) - u_0 = kc\mu \) and the platform position cannot be maintained in its initial position as confirmed by laboratory experiments, the results of which are depicted in Figures 6.1, 6.2, 6.3 and 6.4. In particular, one can see that if \( c \) is negative (the ball hit from right) this difference is very small – less than \( 0.002 \text{[m]} \), because here \( u(\infty) = u_0 - k\mu |c| \) (Figures 6.2, 6.4). Observe that in our experiments \( u_0 \) slightly dominated \( k\mu |c| \). However, the drift of the platform position significantly increases if \( c \) is positive, because now \( u(\infty) = u_0 + k\mu |c| \) (Figures 6.1, 6.3).

The following result show that the global strong asymptotic stability (GAS) is robust with respect to a kind of nonlinear perturbations.

**Theorem 6.1.** Let \( W(s) := \frac{1}{1 + k\hat{G}(s)} \). If there exist \( K > 0 \) and \( q_0 \in \mathbb{R} \) such that for some \( \varepsilon > 0 \) the frequency–domain inequality of the Popov–type is satisfied
\[
\frac{1}{K} + q_0 \Re W(j\omega) + \frac{\Im W(j\omega)}{\omega} \geq \varepsilon \quad \forall \omega \in \mathbb{R},
\]
then the null equilibrium of the Lur’e system of indirect control:

\begin{equation}
\begin{aligned}
\dot{x} &= (A - kbd\#)X - bf(u) \\
\dot{u} &= -f(u) - kd\#X
\end{aligned}
\end{equation}

(6.3)
arising from (6.1) by applying the generalized control law (4.2)

\[ \dot{u} = -f(u) - kd\#X, \]

is GAS for any locally Lipschitz function \( f \) satisfying the sector condition

\[ 0 < \frac{f(u)}{u} \leq K \quad \forall u \neq 0, \quad f(0) = 0. \]

\textbf{Proof.} The assertion is a direct consequence of [8, Theorem 5.3] with the following identifications\(^3\): \( A := A - kbd\#, \quad B := -b, \quad k := 1, \quad \rho := 0, \quad C_\Lambda := -kd\#, \quad s_0 := 0, \quad D := -1 \)
\((\text{giving } C_\Lambda(sI - A)^{-1}B + D = W(s)), \quad Q := 1, \quad P := K^{-1}, \quad \varphi := f, \quad \Phi(u) := \int_0^u f(\xi)d\xi, \quad \psi := q_0).\)

\textbf{Remark 6.1.} A result very close to Theorem 6.1 has been derived in [17, Theorem 4.1] by applying the method of Lyapunov functionals to the Lur’e system of indirect control (6.3) written in its factor form

\[ \begin{aligned}
\dot{x} &= A_c[X - df(u)] \\
\dot{u} &= -f(u) - kd\#X
\end{aligned} \]

Geometrically, (6.2) means that the parametric curve \( \{(x(j\omega), y(j\omega))\}_{\omega \geq 0} \), where \( x(j\omega) := \text{Re} W(j\omega) \) and \( y(j\omega) := \text{Im} W(j\omega) \), is located strictly in the half–plane \( K^{-1} + q_0 x \geq -y \).

\(^3\)They may be obtained by comparing (6.3) with [5] (5.4)].
It is not difficult to see that for any positive $\kappa = K$ one can choose sufficiently large positive $q_0$ for which (6.2) holds, as confirmed by Figures 6.5 and 6.6.

Figure 6.5. Plot of the curve $\{(x(j\omega), y(j\omega))\}_{\omega \geq 0}$ (in red) lies strictly in the half–plane $K^{-1} + q_0 x \geq -y$: $k = 0.07741$, $K = \kappa = 10$, $q_0 = 1$.  

Figure 6.6. Plot of the curve $\{(x(j\omega), y(j\omega))\}_{\omega \geq 0}$ (in red) lies strictly in the half–plane $K^{-1} + q_0 x \geq -y$: $k = 0.02580$, $K = \kappa = 20$, $q_0 = 1.25$.

A yet another abstract model of the stabilized system can be obtained by combining the third equation of the system (1.1) with the control law (4.16) in order to eliminate $u$ from (1.1). This yields:

\[ \begin{align*}
\phi_{tt}(\xi, t) &= [g(\xi + \mu) \phi_x(\xi, t)]_\xi, \\
\phi_{tt}(0, t) &= g \phi_x(0, t), \\
\phi_t(L, t) &= -kg(L + \mu) \phi_x(L, t) - \kappa \phi(L, t)
\end{align*} \]

Choosing the vector of system variables

\[ \begin{bmatrix} w(t) \\ \phi(\theta, t) \\ \psi(\theta, t) \end{bmatrix} = \begin{bmatrix} \phi_t(0, t) \\ \phi(\theta, t) \\ \phi_t(\theta, t) \end{bmatrix} \]

as a vector in the space $\mathbb{R} \oplus H^1(0, L) \oplus L^2(0, L)$, equipped with an equivalent scalar product

\[ \left[ \begin{bmatrix} w \\ \phi \\ \psi \end{bmatrix}, \begin{bmatrix} W \\ \Phi \\ \Psi \end{bmatrix} \right] = \mu w W + \int_0^L g(\xi + \mu) \phi'(\xi) \Phi'(\xi) d\xi + \frac{\kappa}{k} \phi(L) \Phi(L) + \int_0^L \psi(\theta) \Psi(\theta) d\theta, \]

we get

\[ \begin{align*}
\dot{X}(t) &= \sigma X(t) \\
\tau X(t) &= \psi(L, t)
\end{align*} \]
where
\[
\begin{bmatrix}
w \\
\phi \\
\psi
\end{bmatrix}
= 
\begin{bmatrix}
g\phi'(0) \\
\psi \\
[g(\cdot + \mu)\phi]'
\end{bmatrix},
\quad
D(\sigma) = \left\{ \begin{bmatrix}
\phi \\
\psi \\
w
\end{bmatrix} \in H^2(0,1) \oplus H^1(0,1) \oplus \mathbb{R} : \psi(0) = w \right\},
\]
\[
\begin{bmatrix}
w \\
\phi \\
\psi
\end{bmatrix}
= -kg(L + \mu)\phi'(L) - \kappa\phi(L),
\quad
D(\tau) = \left\{ \begin{bmatrix}
\phi \\
\psi \\
w
\end{bmatrix} : \phi \text{ is differentiable at } L \right\}.
\]

A vector, called the factor control vector, \(d \in H\) exists such that
\[
d \in D(\sigma), \quad \sigma d = 0, \quad \tau d = -1.
\]

Elementary calculations yield:
\[
d = \frac{1}{\kappa} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.
\]

Now
\[
0 = \tau X(t) - \psi(L, t) = \tau [X(t) - d\psi(L, t)]
\]
and therefore, with \(N := \sigma|_{\ker \tau}\),
\[
\begin{bmatrix}
w \\
\phi \\
\psi
\end{bmatrix}
= \begin{bmatrix}
g\phi'(0) \\
\psi \\
[g(\cdot + \mu)\phi]'
\end{bmatrix},
\]
\[
D(N) = \left\{ \begin{bmatrix}
w \\
\phi \\
\psi
\end{bmatrix} \in H^2(0,1) \oplus H^1(0,1) \oplus \mathbb{R} : \psi(0) = w, \ k g(L + \mu)\phi'(L) + \kappa\phi(L) = 0 \right\}
\]
we get the final abstract dynamical model in the factor colocated form
\[
(6.5) \quad X(t) = N [X(t) + kdd^#X(t)], \quad t \geq 0
\]
where
\[
d^# := \frac{1}{\kappa} \psi(L), \quad D(d^#) = \left\{ \begin{bmatrix}
w \\
\phi \\
\psi
\end{bmatrix} : \psi \text{ is continuous at } L \right\}.
\]

Here, the operator \(N : (D(N) \subset H) \rightarrow H\) is skew–adjoint with compact inverse \(N^{-1}\) and finally \(d^#\) is an unbounded linear functional associated with \(d\) with the following properties:
\[
D(N) \subset D(d^#), \quad d^#|_{D(N)} = d^*N, \quad d \in D(d^#) \setminus D(N), \quad d^#d = 0.
\]

(6.5) seems to be a realization of (4.6) on a smaller state space \(H = \mathbb{R} \oplus H^1(0, L) \oplus L^2(0, L)\), consisting of three components only, and therefore it is expected that the RHS of (6.5) generates an \(\text{EXS}\) semigroup on this state space.

A number of strongly stabilized heavy chain systems modelling an overhead crane has been proposed in the literature [7, 28, 8, 5, 11]. It is explained in Appendix C why \(\text{EXS}\) cannot be achieved in those systems with the except for the paper by B. Rao [28], who obtained \(\text{EXS}\) for an abstract system closely related to, but not equivalent with (6.4) – see Appendix C for more details.
In [9], a continuation of [7], the exponential stabilization is achieved via a back-stepping approach, but a disadvantage of this paper is that it (implicitly) deals with \( g = \infty \) which is mathematically interesting but physically a nonsense. In [29] the back-stepping approach was applied to a more realistic heavy-chain system, nevertheless an abstract model setting proposed in [30] requires some corrections. To the author’s knowledge the last paper is the only one in which formal results were verified by laboratory experiments, though a limited number of technical details about a realization of the laboratory test bed has been presented.

In a more recent paper [21] the exponential stabilization is obtained by applying a dynamic mechanical damper located at an interior point on a chain.

Let us give some prospects for possible future investigations. A generalization of the results to encounter \( \tan[\alpha(L, t)] \) instead of \( \alpha(L, t) \) in the control law (4.16) seems to be quite natural. A challenging direction of theoretical considerations is to develop some optimal control ideas to find a control law (or a feedback controller if possible) which can move the chain from a starting position to the desired final position, e.g., in minimal time.

The laboratory set-up can be improved by introducing higher resolution encoders for more precise angle and position measurements. A digital video camera can be applied not only to register the experiments [32], but also as a measuring device.

Appendix A. Solution of the eigenproblem (2.2)

The change of variables \( \xi = \frac{\theta^2 g - 4 \lambda^2 \mu}{4 \lambda^2} = \theta = 2 \lambda \sqrt{\frac{\xi + \mu}{g}} \), \( \Xi(\theta) = \Phi[\xi(\theta)] \) reduces the first equation of (2.2) to the modified Bessel differential equation [1, 9.6.1]:

\[
\theta^2 \Xi''(\theta) + \theta \Xi'(\theta) - (\theta^2 + n^2) \Xi(\theta) = 0
\]

with \( n = 0 \). Recall that \( I_n \) – the modified Bessel function of the first kind and \( n \)-th order and \( K_n \) – the modified Bessel function of the second kind and \( n \)-th order, \( n \in \{0\} \cup \mathbb{N} \), are its two linearly independent solutions with the Wronskian [1, 9.6.15]

\[
W(\{I_n, K_n\}) = I_n(z)K_{n+1}(z) + I_{n+1}(z)K_n(z) = z^{-1}.
\]

Hence a general solution of the first equation in (2.2) reads as

\[
\Phi(\xi) = C_1 I_0 \left( 2 \lambda \sqrt{\frac{\xi + \mu}{g}} \right) + C_2 K_0 \left( 2 \lambda \sqrt{\frac{\xi + \mu}{g}} \right).
\]

Inserting (A.2) into boundary conditions and making use of the recurrence relations [1, 9.6.26 and 9.6.27] valid for \( n \in \mathbb{N} \):

\[
\begin{align*}
I_{n+1}(z) &= -2nz^{-1}I_n(z) + I_{n-1}(z) \\
I_1(z) &= I_0'(z)
\end{align*}
\]

\[
\begin{align*}
K_{n+1}(z) &= 2nz^{-1}K_n(z) + K_{n-1}(z) \\
K_1(z) &= -K_0'(z)
\end{align*}
\]

The formula [30, (36), p. 405] cannot define a scalar product on a state space proposed by the authors. This is because it employs the functional of taking derivative at a point of a \( H^1(0, L) \)-function, but such a linear functional is neither everywhere defined nor continuous on \( H^1(0, L) \).

Nowadays, the access to a video movie via \( \text{http://www.LSR.uni-saarland.de/rd/crane06.htm} \) announced in [29, p. 661], has been transferred to \( \text{http://cds.acin.tuwien.ac.at/fileadmin/cds/data/video/Kette.mpeg} \).
one obtains the linear homogeneous system determining unknown constants $C_1, C_2$

\[
\begin{bmatrix}
I_0(\lambda \beta) & K_0(\lambda \beta) \\
I_2(\lambda \alpha) & K_2(\lambda \alpha)
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix}, \quad \alpha := 2 \sqrt{\frac{\mu}{g}}, \quad \beta := 2 \sqrt{\frac{L + \mu}{g}}.
\]

A nonzero solution of (A.4) exists iff $\lambda$ is a solution of the characteristic equation

\[
\Delta(\lambda) := I_0(\lambda \beta)K_2(\lambda \alpha) - K_0(\lambda \beta)I_2(\lambda \alpha) = 0.
\]

Applying the identities [1, 9.6.3 and 9.6.4, both with $z = jx$] we get

\[
I_n(jx) = (-j)^n J_n(-x), \quad K_n(jx) = \frac{\pi}{2} j^n \left[ -Y_n(-x) + j J_n(-x) \right], \quad x \in \mathbb{R},
\]

where $J_n$ and $Y_n$ stand for the Bessel functions of $n$–th order and the first and second kind, respectively, jointly with [1, 9.1.35 and 9.1.36 with $m = 1$]

\[
J_n(-x) = (-1)^n J_n(x), \quad Y_n(-x) = (-1)^n \left[ Y_n(x) + 2 j J_n(x) \right], \quad x > 0,
\]

we conclude that

\[
\Delta(j \omega) = \frac{\pi}{2} \left[ J_0(\beta |\omega|)Y_2(\alpha |\omega|) - J_2(\alpha |\omega|)Y_0(\beta |\omega|) \right], \quad \omega \in \mathbb{R}.
\]

Since all eigenvalues are purely imaginary then, they are of the form $\lambda = j \omega$ where $\omega$ solves the equation $\Delta(j \omega) = 0$. But for $x \in \mathbb{R}$ the recurrence relations hold [1, 9.1.27 and 9.1.28]

\[
\begin{align*}
J_{n+1}(z) &= 2nz^{-1} J_n(z) - J_{n-1}(z) \\
J_1(z) &= -J_0'(z)
\end{align*}
\]

\[
\begin{align*}
Y_{n+1}(z) &= 2nz^{-1} Y_n(z) - Y_{n-1}(z) \\
Y_1(z) &= -Y_0'(z)
\end{align*}
\]

and therefore the asymptotic real zeroes of the equation $\Delta(j \omega) = 0$ coincide with the asymptotic real zeroes of the cross–product

\[
J_0(\alpha |\omega|)Y_0(\beta |\omega|) - J_0(\beta |\omega|)Y_0(\alpha |\omega|) = 0.
\]

Hence, by [1, 9.5.27 and 9.5.28 with $z = \alpha |\omega|$, $\lambda = \frac{\beta}{\alpha} > 1$], one concludes that the eigenvalues $\lambda_{\pm n}$ satisfy $\lambda_{\pm n} \sim \pm j \frac{n \pi}{\beta - \alpha}$, $n \in \mathbb{N}$. It remains to show that any eigenvalue $\lambda_{\pm n} = j \omega_{\pm n}$ is single. Observe that

\[
K_0(\beta \lambda_{\pm n}) = K_0(j \beta \omega_{\pm n}) = \frac{\pi}{2} \left[ Y_0(-\beta \omega_{\pm n}) + j J_0(-\beta \omega_{\pm n}) \right] = -\frac{\pi}{2} \left[ Y_0(\beta \omega_{\pm n}) + j J_0(\beta \omega_{\pm n}) \right].
\]

It is well–known that $J_0$ and $Y_0$ do not have common zeroes, whence $K_0(\beta \lambda_{\pm n}) \neq 0$ and consequently

\[
\begin{bmatrix}
I_0(\beta_{\pm n}) & K_0(\beta \lambda_{\pm n}) \\
I_2(\alpha \lambda_{\pm n}) & K_2(\alpha \lambda_{\pm n})
\end{bmatrix} = 1.
\]

This means that any eigenspace of $A$ corresponding to $\lambda_{\pm n}$ is one–dimensional, whence $\lambda_{\pm n}$ is a single eigenvalue because a normal operator cannot have generalized eigenvectors.

---

6Where, in addition, the identity [1, 9.1.3] has been used to eliminate a Hankel function normally appearing in [1, 9.6.4].
Appendix B. Solution of (3.1)

A unique solution of this system is

(B.1) \[ \Phi(\xi) = -\frac{1}{s^2} + C_1 I_0 \left( 2s \sqrt{\frac{\xi + \mu}{g}} \right) + C_2 K_0 \left( 2s \sqrt{\frac{\xi + \mu}{g}} \right) \]

where \( C_1 \) and \( C_2 \) satisfy

(B.2) \[ \begin{bmatrix} I_2(s\alpha) & K_2(s\alpha) \\ I_0(s\beta) & K_0(s\beta) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ s^{-2} \end{bmatrix} \]

Hence, by (A.3),

\[ s^2 C_1 = \frac{K_2(s\alpha)}{\Delta(s)} = \frac{2}{s\alpha} \frac{K_1(s\alpha) + K_0(s\alpha)}{\Delta(s)}, \quad s^2 C_2 = \frac{-I_2(s\alpha)}{\Delta(s)} = \frac{2}{s\alpha} \frac{I_1(s\alpha) - I_0(s\alpha)}{\Delta(s)} \]

and consequently, making use of (A.1) with \( n = 0 \), we finally obtain

\[ \hat{g}(s) = s^2 \Phi(0) + 1 = \frac{2}{s\alpha} \frac{[K_1(s\alpha) I_0(s\alpha) + I_1(s\alpha) K_0(s\alpha)]}{s\alpha \Delta(s)} = \frac{2}{s^2 \alpha^2 \Delta(s)} \]

Let us recall also the asymptotics, following from [11] 9.6.7 ÷ 9.6.9,

(B.3) \[ I_n(z) \sim \frac{z^n}{2^n n!}, \quad \forall n \in \{0\} \cup \mathbb{N}; \quad K_n(z) \sim \frac{(n-1)!}{z^n} \frac{2^{n-1}}{\pi} \quad \forall n \in \mathbb{N}, \quad K_0(z) \sim -\ln z \]

for small \( |z| \) and the asymptotics for large positive real \( z \), following from [11] 9.7.1 and 9.7.2:

(B.4) \[ \sqrt{z} e^{-z} I_n(z) \sim 1/\sqrt{2\pi}, \quad \sqrt{z} e^{-z} K_n(z) \sim \sqrt{\pi}/2 \]

Appendix C. Unified approach to some earlier investigations

Conrad and Mifdal [5] have considered a boundary feedback stabilization problem of an overhead crane depicted in Figure 1.1, but with counter-flowing and normalized spatial variable \( \theta \), governed by the system of equations

(C.1) \[
\begin{cases}
\phi_{tt}(\theta, t) = (a \phi_\theta)_\theta(\theta, t), & t \geq 0, \quad \theta \in [0, 1] \\
-a(0) \phi_\theta(0, t) + m_\phi \phi_{tt}(0, t) = -\alpha_1 \phi(0, t) - \beta_1 \phi_\theta(0, t), & t \geq 0 \\
a(1) \phi_\theta(1, t) + m_\phi \phi_{tt}(1, t) = 0, & t \geq 0 \\
\phi(\theta, 0) = \phi_0(\theta), \quad \phi_\theta(\theta, 0) = \phi_1(\theta), & \theta \in [0, 1]
\end{cases}
\]

with constants \( m_\phi > 0 \) (mass of the platform), \( m > 0, \alpha_1 > 0 \) and \( \beta_1 \geq 0 \). The boundary damping is represented by the term \( -\beta_1 \phi_\theta(0, t) \). A model with \( \beta_1 \phi_\theta(0, t) \) replaced by a nonlinear damping \( f[\phi_\theta(0, t)] \) has been analyzed in [11]. Choosing the system variables

\[
\begin{bmatrix}
\phi(\theta, t) \\
\psi(\theta, t) \\
z(t) \\
w(t)
\end{bmatrix}
= \begin{bmatrix}
\phi(\theta, t) \\
\phi_\theta(\theta, t) \\
\phi(0, t) \\
\phi_\theta(1, t)
\end{bmatrix}
\]
one obtains the first order dynamics
\[ \begin{align*}
\phi_t(\theta, t) &= \psi(\theta, t), \\ 
\psi_t(\theta, t) &= [a(\theta) \phi_\theta(\theta, t)]_\theta, \\ 
\dot{z}(t) &= -\frac{\alpha_1}{m_P} \phi(0, t) + \frac{a(0)}{m_P} \phi_\theta(0, t) - \frac{\beta_1}{m_P} z(t), \\ 
\dot{w}(t) &= -\frac{a(1)}{m} \phi(1, t)
\end{align*} \]
for \(0 \leq \theta \leq 1, \ t \geq 0\).

In the Hilbert space \(H = H^1(0, 1) \oplus L^2(0, 1) \oplus \mathbb{R}^2\) equipped with the energetic scalar product
\[
\left\langle \begin{bmatrix} \phi \\ \psi \\ z \\ w \end{bmatrix}, \begin{bmatrix} \Phi \\ \Psi \\ Z \\ W \end{bmatrix} \right\rangle := \int_0^1 a(\theta) \phi'(\theta) \Phi'(\theta) d\theta + \alpha_1 \phi(0) \Phi(0) + \frac{a(0)}{m_P} \phi_\theta(0) \Phi_\theta(0) + \beta_1 \dot{z} + \dot{w},
\]
\(\text{(C.2)}\) can be rewritten into its abstract form:
\[
\dot{x}(t) = (N - bb^*) x(t),
\]
where \(b \in H \setminus D(N)\) and \(N : (D(N) \subset H) \to H\) is a skew-adjoint operator \((N = -N^*)\) with a compact resolvent. Here we have:
\[
b = \begin{bmatrix} 0 \\ 0 \\ \sqrt{\beta_1/m_P} \\ 0 \end{bmatrix}; \quad N \begin{bmatrix} \phi \\ \psi \\ z \\ w \end{bmatrix} = \begin{bmatrix} 
\psi \\ (a\phi')' \\ -\frac{\alpha_1}{m_P} \phi(0) + \frac{a(0)}{m_P} \phi_\theta(0) \\ -\frac{a(1)}{m} \phi'(1)
\end{bmatrix},
\]
with the domain
\[
D(N) = \left\{ \begin{bmatrix} \phi \\ \psi \\ z \\ w \end{bmatrix} \in H : \phi \in H^2(0, 1), \ \psi \in H^1(0, 1), \ \psi(0) = z, \ \psi(1) = w \right\},
\]
and it can be easily established that \(b \notin D(N)\), \(N = -N^*\) and \(N\) has a compact resolvent.

It is demonstrated in [5, Subsection 4.2] that the null equilibrium is strongly asymptotically stable but not exponentially stable.

In the case of \(\alpha_1 = 0, \ m_P > 0, \ m > 0\) and \(\beta_1 \geq 0\) the energetic scalar product degenerates to a nonnegative quadratic form and the system has equilibria different from zero, located in the linear subspace spanned by the eigenvector
\[
e := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in D(N).
\]
Moreover, there holds: \(\text{span}\{1\} \oplus H^1_0(0, 1) = H(0, 1)\), where \(H^1_0(0, 1)\) stands for the subspace of \(H^1(0, 1)\) consisting of functions vanishing at 0, and the orthogonal sum is taken
in \( H^1(0, 1) \) equipped with the scalar product
\[
\langle \phi, \Phi \rangle_{H(0, 1)} = \int_0^1 a(\theta)\phi'(\theta)\Phi'(\theta)d\theta + \phi(0)\Phi(0).
\]

This suggests that a reasonable way to get the system dynamics description outside the subspace consisting of equilibria is to project the original state space \( H^1(0, 1) \oplus L^2(0, 1) \oplus \mathbb{R}^2 \) onto the new one \( H = H^1_0(0, 1) \oplus L^2(0, 1) \oplus \mathbb{R}^2 \), using the orthoprojector \( I - ee^* \). This orthoprojector acts as follows
\[
\begin{bmatrix}
\Phi \\
\Psi \\
Z \\
W
\end{bmatrix} = (I - ee^*)
\begin{bmatrix}
\phi \\
\psi \\
z \\
w
\end{bmatrix} =
\begin{bmatrix}
\phi - 1\phi(0) \\
\psi \\
z \\
w
\end{bmatrix}.
\]

The energetic scalar product for the new state space is
\[
\langle \begin{bmatrix}
\phi \\
\psi \\
z \\
w
\end{bmatrix}, \begin{bmatrix}
\Phi \\
\Psi \\
Z \\
W
\end{bmatrix} \rangle := \int_0^1 a(\theta)\phi'(\theta)\Phi'(\theta)d\theta + \psi(\theta)\psi(\theta)d\theta + mzZ + MwW .
\]

In the new coordinates the system state operator is again representable in the form \( N - bb^* \) with the same vector \( b \), but with modified \( N \):
\[
b = \begin{bmatrix}
0 \\
0 \\
\sqrt{\frac{\beta_1}{m_P}} \\
0
\end{bmatrix}, \quad
N \begin{bmatrix}
\phi \\
\psi \\
z \\
w
\end{bmatrix} =
\begin{bmatrix}
\psi - Z1 \\
(a\phi')' \\
\frac{a(0)\phi'(0)}{m_P} \\
\frac{a(1)\phi'(1)}{m}
\end{bmatrix}, \quad
- N^* \begin{bmatrix}
\Phi \\
\Psi \\
Z \\
W
\end{bmatrix},
\]
\[
D(N) = \left\{ \begin{bmatrix}
\phi \\
\psi \\
z \\
w
\end{bmatrix} \in [H^2(0, 1) \cap H^1_0(0, 1)] \oplus H^1(0, 1) \oplus \mathbb{R}^2 : \psi(0) = Z, \psi(1) = W \right\}.
\]

B. d’Andréa–Novel et al. [8] investigated the nonlinear model\(^7\) with \( m_P > 0, \, m = 0, \, \alpha_1 > 0 \) and \( \beta_1 \geq 0 \). In this case the state space modifies to be \( H = H^1(0, 1) \oplus L^2(0, 1) \oplus \mathbb{R} \). It is still equipped with energetic scalar product which reduces to
\[
\langle \begin{bmatrix}
\phi \\
\psi \\
z
\end{bmatrix}, \begin{bmatrix}
\Phi \\
\Psi \\
Z
\end{bmatrix} \rangle := \int_0^1 a(\theta)\phi'(\theta)\Phi'(\theta)d\theta + \alpha_1\phi(0)\Phi(0) + \int_0^1 \psi(\theta)\psi(\theta)d\theta + mzZ .
\]

The associated linear model has again the form [C.3], now with
\[
N \begin{bmatrix}
\phi \\
\psi \\
z
\end{bmatrix} = \begin{bmatrix}
\psi \\
(a\phi')' \\
-\frac{\alpha_1}{m_P}\phi(0) + \frac{a(0)}{m_P}\phi'(0)
\end{bmatrix}, \quad
b = \begin{bmatrix}
0 \\
0 \\
\sqrt{\frac{\beta_1}{m_P}}
\end{bmatrix},
\]

\(^7\)We remark that the authors of [8] use the same orientation of the spatial variable as ours.
where the domain of $N$ is defined as

$$D(N) = \left\{ \begin{bmatrix} \phi \\ \psi \\ z \end{bmatrix} \in H : \phi \in H^2(0,1), \psi \in H^1(0,1), \psi(0) = z, \phi'(1) = 0 \right\}.$$  

This notation significantly simplifies that of [8], where the state space vector consists of four components and the system acts on a subspace of the original state space $H = H^1(0,1) \oplus L^2(0,1) \oplus \mathbb{R}^2$.

The degenerate case: $\alpha_1 > 0$, $m_P = 0$, $m > 0$, $\beta_1 \geq 0$, considered by Rao [28], is essentially different. An important result due of [28, Theorem 5, (i), p. 315 with $f(s) = \beta_1 s$] states that the system (C.1) gives rise to an EXS semigroup on the subspace of $H^1(0,1) \times L^2(0,1) \times \mathbb{R}^2$, consisting of vectors whose third component is a value of the first function component at $\theta = 1$, equipped with the same scalar product.

References


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