ON THE CIRCLE CRITERION FOR BOUNDARY CONTROL SYSTEMS IN FACTOR FORM: LYAPUNOV APPROACH

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ABSTRACT. A Lur'e feedback control system consisting of a nonlinear static sector type controller and a linear, infinite–dimensional system of boundary control in factor form is considered. Some criteria of absolute weak or strong asymptotic stability of the null equilibrium point are derived using Lyapunov functionals. The construction of such quadratic form functional is reduced to solving a Lur'e system of equations. The solvability of the latter system is investigated. Here, the main results are results similar to the Kalman – Yacubovič lemma which are generalizations of results due to a) A.V. Balakrishnan [2] and b) J.C. Oostveen and R.F. Curtain [27]. These results are illustrated in detail by electrical transmission lines: 1) of the distortionless loaded $RLCG$–type and 2) of the unloaded RC–type.

CONTENTS

1. INTRODUCTION

This paper uses some results on abstract linear systems in factor form, obtained by the authors in earlier papers [16], [17] and shortly recalled in Section 2. These results combined

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with the input–output approach using *passivity* concepts lead in [18] to a circle criterion for the nonlinear Lur'e type feedback system described by (3.4) below. The present paper shows that Lyapunov state space theory together with the abstract results of Section 2 is also useful for getting similar stability conditions.

Some absolute stability criteria are derived in Section 3. They are based on using a candidate quadratic form Lyapunov functional. A rather sophisticated procedure of evaluating the derivative of the quadratic form along the system trajectories is studied and successfully applied to get a novel so–called Lur'e system. A weak version of LaSalle's invariance principle is then used to investigate the global attractivity of the state trajectories to a unique equilibrium at the origin. Moreover the abstract results of Section 2 enable us to strengthen the attractivity properties such that even global strong asymptotic stability can be obtained under some more restrictive assumptions. An important consequence is that the problem of constructing a quadratic form Lyapunov functional is reduced to solving a Lur'e system of equations as given by (3.2) or (3.7). This reduction is famous in finite–dimensional system theory and leads to a variety of results commonly known as the Kalman–Yacubovič–Popov lemma. Only a few particular cases of these results, surveyed in Section 7.1, have been recently investigated in the infinite-dimensonal systems literature. The main difficulty in getting a generalization of the Kalman–Yacubovič–Popov lemma is due to the fact that boundary control and/or observation involve unbounded linear operators, which lead to some difficult mathematical questions. When writing this paper it was observed that its specific problem was not met by the Lur'e system results of many papers. A careful examination of the assumptions conditioning these results made it clear that the latter had none or very limited utility for the stability question of this paper's Lur'e type nonlinear feedback system.

The solvability of the Lur'e system mentioned above is analyzed in Section 4 and related to results of papers by Balakrishnan [2] and Nudel'man and Schwartzman [26]. Despite the fact that some main steps of the classical Kalman proof are repeated here, it is up to the knowledge of the authors original. Modulo some auxiliary results the proof consists of three parts. The first part describes spectral factorization using Szegö's theorem. The next part describes the solution of the realization problem (4.14) under the assumption that the open–loop system operator has a system of eigenvectors that is a state space Riesz basis. The final part handles the solvability of the Lur'e system. A result due to Oostveen and Curtain [27] is hereby usefully adapted to get solvability conditions for the specific system at hand.

Section 5 presents an exhaustive illustration of the results above for the example of a loaded distortionless electric $RLCG$–transmission line. A salient feature of its presentation is that the realizability problem is solved by two different methods: a direct method and a Riesz basis method.

Section 6 handles the example of an unloaded electric RC–transmission line. The latter has two features that are worth mentioning: 1) as here the factor control vector d is not admissible, only a weaker stability result can be obtained, and 2) it is possible to verify implicitly all assumptions made in Section 4 by using the theory of spectral factorization of even entire functions that are nonnegative on $j\mathbb{R}$.

Some prospects for further investigations are presented in the concluding Subsection 7.2.

2. Preliminary data

In a Hilbert space H with a scalar product $\langle\cdot,\cdot\rangle_{\mathrm{H}}$ consider the SISO model of boundary control in factor form [16],

$$\left\{\begin{array}{rcl}\dot{x}(t) & = & A[x(t)+u(t)d] \\ y & = & c^{\#}x\end{array}\right\} . \tag{2.1}$$

We assume that $A:\ (D(A)\subset \mathrm{H})\longrightarrow \mathrm{H}$ generates a linear exponentially stable (**EXS**), C_0–semigroup $\{S(t)\}_{t\geq 0}$ on H, $d\in \mathrm{H}$ is a factor control vector, $u\in \mathrm{L}^2(0,\infty)$ is a scalar control function, y is a scalar output defined by an A–bounded linear observation functional $c^{\#}$. The restriction of $c^{\#}$ to $D(A)$ is representable as $c^{\#}\big|_{D(A)}=h^*A$ for some $h\in\mathrm{H}$.

Define two operators:

$$V\in \mathbf{L}(\mathrm{H},\mathrm{L}^2(0,\infty)),\qquad (Vx)(t):=h^*S(t)x$$
$$W\in \mathbf{L}(\mathrm{L}^2(0,\infty),\mathrm{H}),\qquad Wu:=\int_0^\infty S(t)du(t)dt\ .$$

Recall that L and $R=L^*$,

$$Lf=f',\qquad D(L)=\mathrm{W}^{1,2}(0,\infty)\ ,$$
$$Rf=-f',\qquad D(R)=\{f\in \mathrm{W}^{1,2}(0,\infty):\ f(0)=0\}$$

are the generators of the semigoups of *left–* and *right–shifts* on $\mathrm{L}^2(0,\infty)$, respectively.

Definition 2.1. The observation functional $c^{\#}$ is called *admissible* if the *observability operator*

$$P=VA,\qquad D(P)=D(A)$$

is bounded.

Definition 2.2. The factor control vector $d\in\mathrm{H}$ is called *admissible* if

$$\mathrm{Range}(W)\subset D(A)\ .$$

In the sequel $\Pi^+:=\{s\in\mathbb{C}:\mathrm{Re}\,s>0\}$ denotes the open right-half complex plane, $\mathrm{H}^\infty(\Pi^+)$ is the Banach space of analytic functions f on Π^+, equipped with the norm $\|f\|_{\mathrm{H}^\infty(\Pi^+)}=\sup\limits_{s\in\Pi^+}|f(s)|$ and $\mathrm{H}^2(\Pi^+)$ is the Hardy space of functions f analytic on Π^+ such that $\sup\limits_{\sigma>0}\displaystyle\int_{-\infty}^{\infty}|f(\sigma+j\omega)|^2\,d\omega<\infty$, where $f(j\omega):=\lim\limits_{\sigma\to 0+}f(\sigma+j\omega)$ exists for almost all $\omega\in\mathbb{R}$. The space $\mathrm{H}^2(\Pi^+)$ is unitarily isomorphic with $\mathrm{L}^2(0,\infty)$ through the normalized Laplace transform. To be more precise,

$$\langle f,g\rangle_{\mathrm{L}^2(0,\infty))}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(j\omega)\overline{\hat{g}(j\omega)}d\omega$$

where $\hat{f},\hat{g}$ are the Laplace transform of f and g, respectively.
Moreover [20, p. 134] we shall frequently use the unitary operator $U\in\mathbf{L}(\mathrm{H}^2(\Pi^+))$ given by

$$(U\varphi)(s):=(1/s)\varphi(1/s)\ , \tag{2.2}$$

which for the $j\omega$–axis $\mathrm{H}^2(\Pi^+)$–norm corresponds to the change of variable $\omega\longmapsto -\omega^{-1}$. Finally we shall encounter Wiener and Callier-Desoer convolution algebras. Recall [6, pp.

652 - 653], [7, pp. 81 - 84], [8, pp. 337 - 338] that a scalar–valued Laplace transformable distribution f with support on $[0,\infty)$ is in the *Wiener class* $\mathcal{A}(\sigma)$ for some $\sigma \in \mathbb{R}$ if $f(t) = f_a(t) + f_{sa}(t)$ for $t \geq 0$ with $e^{-\sigma(\cdot)}f_a(\cdot) \in \mathrm{L}^1(0,\infty)$ and $f_{sa}(t) = \sum_{i=0}^{\infty} f_i\delta(t-t_i)$, where δ denotes the Dirac delta distribution and $t_0 = 0$ and $t_i > 0$ for $i > 0$ are such that $\sum_{i=0}^{\infty} e^{-\sigma t_i}|f_i| < \infty$. Such distribution is in the *Callier–Desoer class* $\mathcal{A}_-(0)$ if it is in $\mathcal{A}(\sigma)$ for some $\sigma < 0$. $\widehat{\mathcal{A}}(\sigma)$ and $\widehat{\mathcal{A}}_-(0)$ denote the classes of Laplace transforms of such distributions. $\mathcal{A}(\sigma)$ is a *convolution Banach algebra* with norm

$$\|f\|_{\mathcal{A}(\sigma)} := \left\|e^{-\sigma(\cdot)}f_a(\cdot)\right\|_{\mathrm{L}^1(0,\infty)} + \sum_{i=0}^{\infty} e^{-\sigma t_i}|f_i| \ .$$

For more information see [7] or [8].

Lemma 2.1. If $c^{\#}$ is admissible then $\overline{P}$, the closure of P has the form

$$\mathrm{Range}(V) \subset D(L), \qquad \overline{P} = LV$$

In particular for all $x_0 \in \mathrm{H}$, $(\overline{P}x_0)(t) = \dfrac{d}{dt}[h^*S(t)x_0] \in \mathrm{L}^2(0,\infty)$ with Laplace transform $\left(\widehat{\overline{P}x_0}\right)(s) = c^{\#}(sI-A)^{-1}x_0 \in \mathrm{H}^2(\Pi^+)$. Moreover if d is admissible then the *reachability operator* $Q = AW$ belongs to $\mathbf{L}(\mathrm{L}^2(0,\infty),\mathrm{H})$.

Lemma 2.2. If the *compatibility condition*

$$d \in D(c^{\#}) \tag{2.3}$$

holds then the function

$$\hat{g}(s) := sc^{\#}(sI-A)^{-1}d - c^{\#}d = sh^*A(sI-A)^{-1}d - c^{\#}d \tag{2.4}$$

is well–defined and analytic on the complex right half–plane Π^+.
If in addition to (2.3), $c^{\#}$ is admissible then:

- **(i)** $\hat{g}(s) = s\left(\widehat{\overline{P}d}\right)(s) - c^{\#}d$ with $\widehat{\overline{P}d} \in \mathrm{H}^\infty(\Pi^+) \cap \mathrm{H}^2(\Pi^+)$.
- **(ii)** The convolution operator K with kernel $\overline{P}d$, i.e., $Ku := \overline{P}d \star u$ belongs to $\mathbf{L}(\mathrm{L}^2(0,\infty))$ and it maps the domain of R into itself.

Lemma 2.2 lead to the following result [17, Theorem 4.1].

Lemma 2.3. If (2.3) holds, $c^{\#}$ is admissible and

$$\hat{g} \in \mathrm{H}^\infty(\Pi^+) \tag{2.5}$$

then the *input–output* operator F,

$$F = -KR - c^{\#}dI, \qquad D(F) = D(R)$$

is bounded and its closure $\overline{F}$ is given by

$$\mathrm{Range}(K) \subset D(R), \qquad \overline{F} = -RK - c^{\#}dI \ .$$

Moreover, $\hat{g}$ is then the *transfer function* of the system (2.1).

The following auxiliary result [17, Fact 3.2, p. 8] shall be needed.

Lemma 2.4. Let $c^{\#}$ be admissible. Let $\omega < 0$ be the growth constant of the **EXS** C_0–semigroup generated by A on H. Then for any $\sigma \in (\omega, 0]$

$$e^{-\sigma t}\left(\overline{P}x_0\right)(t) \in \mathrm{L}^1(0,\infty) \cap \mathrm{L}^2(0,\infty) \qquad \forall x_0 \in \mathrm{H} \ .$$

As a consequence with $d \in \mathrm{H}$, $\widehat{\left(\overline{P}d\right)}(s) \in \widehat{\mathcal{A}}(\sigma)$ for any $\sigma \in (\omega, 0]$, and hence for such σ, is analytic and bounded in $\mathrm{Re}\, s > \sigma$ and thus also in a full neighborhood of $s = 0$.

Finally we need

Lemma 2.5. Assume that $A : (D(A) \subset \mathrm{H}) \longrightarrow \mathrm{H}$ generates an exponentially stable **EXS**, C_0–semigroup $\{S(t)\}_{t\geq 0}$ on H and assume that $A^{-1} \in \mathbf{L}(\mathrm{H})$ generates a C_0–semigroup $\{e^{tA^{-1}}\}_{t\geq 0}$ which is bounded uniformly in $t \geq 0$ with respect to the operator norm of $\mathbf{L}(\mathrm{H})$, then the semigroup $\{e^{tA^{-1}}\}_{t\geq 0}$ is *strongly asymptotically stable* (**AS**), i.e. for every $x_0 \in \mathrm{H}$, $\lim\limits_{t\to\infty} e^{tA^{-1}}x_0 = 0$.

Proof. In addition to the assumption of uniform boundedness of $\{e^{tA^{-1}}\}_{t\geq 0}$, the exponential stability of the semigroup generated by A gives $\sigma_P[(A^*)^{-1}] \cap j\mathbb{R} = \emptyset$ and $\sigma(A^{-1}) \cap j\mathbb{R} = \{0\}$ ($0 \in \sigma_C(A^{-1})$ is the only point of the spectrum of A^{-1} on $j\mathbb{R}$). Hence the conclusion follows by a result in Lyubich and Phong [25], or equivalently one in Arendt and Batty [1]. $\square$

3. Asymptotic stability of the Lur'e feedback system

Consider the Lur'e feeback control system depicted in Figure 3.1,

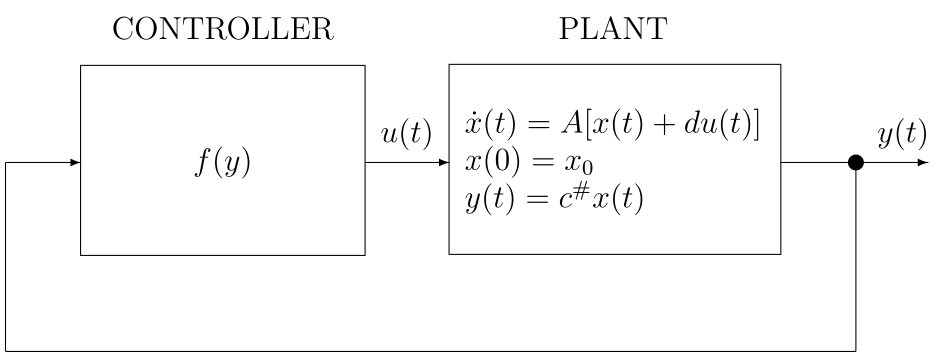

Figure 3.1. The Lur'e control system

which consists of a linear part described by (2.1), and a scalar static controller nonlinearity $f : \mathbb{R} \longrightarrow \mathbb{R}$.

Our aim in this section is to prove some criteria of global weak and strong asymptotic stability for the Lur'e feedback system. For this purpose we assume the following linear subsystem assumptions

- **(A1)** The operator A generates an **AS** C_0–semigroup $\{S(t)\}_{t\geq 0}$ on H and $0 \in \rho(A)$, where $\rho(A)$ stands for the *resolvent set* of A,
- **(A2)** The compatibility condition (2.3) holds,
- **(A3)** There exist constants k_1 and $k_2 > k_1$ such that with

$$q := k_1k_2, \qquad e := \frac{k_1 + k_2}{2} + k_1k_2c^{\#}d, \qquad \delta := (1 + k_1c^{\#}d)(1 + k_2c^{\#}d) \tag{3.1}$$

the *Lur'e system*

$$\left\{ \begin{array}{rcl} \mathcal{H}A^{-1} + \left(A^{-1}\right)^* \mathcal{H} - qhh^* & = & -gg^* \\ -\mathcal{H}d + eh & = & -\sqrt{\delta}g \end{array} \right\} \tag{3.2}$$

has a solution $(\mathcal{H}, g)$, $g \in \mathrm{H}$, $\mathcal{H} \in \mathbf{L}(\mathrm{H})$, $\mathcal{H} = \mathcal{H}^* \geq 0$.

Next for the controller two sets describe restrictions to be imposed on the static nonlinearity $f : \mathbb{R} \longrightarrow \mathbb{R}$, namely

- We define the sector

$$\mathcal{S} := \left\{ f \in \mathrm{C}(\mathbb{R}) : \; k_1 < \frac{f(y)}{y} < k_2 \qquad \forall y \in \mathbb{R} \setminus \{0\}, \; f(0) = 0 \right\} \; . \tag{3.3}$$

- We denote by $\mathcal{M}$ the class of those functions $f \in \mathrm{C}(\mathbb{R})$ which are sufficiently smooth to ensure that the solutions of the closed–loop system equations

$$\left\{ \begin{array}{rcl} A^{-1}\dot{x} & = & x + df(y) \\ y & = & c^{\#}x = c^{\#}(A^{-1}\dot{x} - df) = c^{\#}A^{-1}\dot{x} - c^{\#}df = h^*\dot{x} - c^{\#}df \end{array} \right\} \tag{3.4}$$

generate a *local dynamical system* on the state space H. Observe here that for $w \in D(A^*)$ we have

$$\frac{d}{dt}\langle w, x\rangle_{\mathrm{H}} = \frac{d}{dt}\langle A^*w, A^{-1}x\rangle_{\mathrm{H}} = \langle A^*w, A^{-1}\dot{x}\rangle_{\mathrm{H}} = \langle A^*w, x + df(y)\rangle_{\mathrm{H}}$$

then any *weak* solution of the original closed–loop system $\dot{x} = A[x + df(y)]$, $y = c^{\#}x$ satisfies (3.4) in the *classical* sense.

Theorem 3.1. Let assumptions (**A1**)÷(**A3**) hold, where moreover in (**A3**) $\mathcal{H}$ is *coercive*, i.e.

$$\mathcal{H} = \mathcal{H}^* \geq \eta I \quad \text{with} \quad \eta > 0 \; .$$

Let f belong to $\mathcal{S} \cap \mathcal{M}$. Then the origin of the space H is *globally weakly asymptotically stable*

Proof. The idea of proof is to observe that the quadratic form $V(x) = x^*\mathcal{H}x$ is a Lyapunov functional for the system (3.4). Its derivative along the solutions of (3.4) can be represented as

$$\begin{aligned} \dot{V} = &\; \dot{x}^*\mathcal{H}x + x^*\mathcal{H}\dot{x} = \dot{x}^*\mathcal{H}(A^{-1}\dot{x} - df) + (A^{-1}\dot{x} - df)^*\mathcal{H}\dot{x} = \\ = &\; \left[\begin{array}{cc} \dot{x}^* & f \end{array} \right] \left[\begin{array}{cc} \mathcal{H}A^{-1} + \left(A^{-1}\right)^* \mathcal{H} & -\mathcal{H}d \\ -d^*\mathcal{H} & 0 \end{array} \right] \left[\begin{array}{c} \dot{x} \\ f \end{array} \right] \; . \end{aligned}$$

To meet the sector conditions (3.3) we add and subtract the expression

$$[k_2 y - f(y)][f(y) - k_1 y] = [k_2 h^*\dot{x} - (k_2 c^{\#}d + 1)f][(k_1 c^{\#}d + 1)f - k_1 h^*\dot{x}] \; .$$

Now we get

$$\dot{V} = -(k_2 y - f)(f - k_1 y) + \left[\begin{array}{cc} \dot{x}^* & f \end{array} \right] \left[\begin{array}{cc} \mathcal{H}A^{-1} + \left(A^{-1}\right)^* \mathcal{H} - qhh^* & -\mathcal{H}d + eh \\ -d^*\mathcal{H} + eh^* & -\delta \end{array} \right] \left[\begin{array}{c} \dot{x} \\ f \end{array} \right] \; .$$

By (**A3**) we obtain

$$\dot{V} = -\left[g^*\dot{x} + \sqrt{\delta}f \right]^2 - (k_2 y - f)(f - k_1 y) \leq 0 \; , \tag{3.5}$$

and V is a Lyapunov functional for the system (3.4), independently of $f \in \mathcal{S} \cap \mathcal{M}$.

Since the operator $\mathcal{H}$ is *coercive* all solutions are bounded because they remain in the level set of V which are bounded positively invariant sets. Hence, the solutions of (3.4) generate a *dynamical system* on H. Moreover, the null equilibrium point is *stable* in the sense of Lyapunov. These facts follow from the estimate

$$\eta \left\|x(t,x_0)\right\|_{\mathrm{H}}^2 \leq V[x(t,x_0)] \leq V(x_0) \leq \left\|x_0\right\|_{\mathrm{H}}^2 \left\|\mathcal{H}\right\|_{\mathbf{L}(\mathrm{H})} \tag{3.6}$$

which originally holds on the right maximal interval of existence of the solution $x(\cdot,x_0)$ but then extends to all $t \geq 0$.

By the *weak invariance principle* [3] all solutions weakly tend to an *invariant* set contained in the set of those initial conditions x_0 which give rise to trajectories on which V is constant, i.e., to the set

$$E := \{x_0 \in \mathrm{H} : \ V[x(t,x_0)] = V(x_0) \qquad \forall t \in \mathbb{R}\} \ .$$

Recall that on an invariant set the flow of a dynamical system acts onto, so we took $t \in \mathbb{R}$ rather then $t \geq 0$, which would be the case of a positively invariant set. Now by (3.5) if $x_0 \in E$ then necessarily $y(t) \equiv 0$ on $[0,\infty)$. However, this implies through the control law $u(t) = f[y(t)]$ that the trajectory dictated by $x(\cdot,x_0)$ is a weak solution of the linear open–loop system $\dot{x} = Ax$, whence $x(t,x_0) = S(t)x_0$. Observe that then (3.6) yields

$$\eta \left\|x_0\right\|_{\mathrm{H}}^2 \leq V(x_0) = \langle S(t)x_0, \mathcal{H}S(t)x_0\rangle_{\mathrm{H}} \qquad \forall t \geq 0$$

and consequently, by (**ii**), we have $x_0 = 0$. Finally $E = \{0\}$ and the null equilibrium point is *globally weakly attracting* which jointly with its stability yields global weak asymptotic stability. □

Remark 3.1. Assumption (**A3**) of Theorem 3.1 can be replaced by

(iii') There exist constants k_1 and $k_2 > k_1$ such that the system

$$\left\{ \begin{array}{rcll} \langle Ax, \mathcal{H}x\rangle_{\mathrm{H}} + \langle x, \mathcal{H}Ax\rangle_{\mathrm{H}} & = & q\,(h^*Ax)^2 - (g^*Ax)^2 & \forall x \in D(A) \\ -\mathcal{H}d + eh & = & -\sqrt{\delta}g & \end{array} \right\} \tag{3.7}$$

has a solution $(\mathcal{H}, g)$, $g \in \mathrm{H}$, $\mathcal{H} \in \mathbf{L}(\mathrm{H})$, $\mathcal{H} = \mathcal{H}^* \geq \eta I$ with $\eta > 0$.

For the next result we need to reconsider the factor control vector $d \in \mathrm{H}$. Recall that Definition 2.2 defines the admissibility of d under the assumption that the C_0–semigroup $\{S(t)\}_{t\geq 0}$ generated by A is **EXS**. Below we shall need its extension to the case that semigroup is only **AS**. To do this reconsider the mapping

$$W : \mathrm{L}^2(0,\infty) \ni u \longmapsto Wu \in \mathrm{H}, \qquad Wu := \int_0^\infty S(t)du(t)dt \ .$$

Definition 3.1. Let A generate a C_0–semigroup $\{S(t)\}_{t\geq 0}$ which is **AS**. Then the factor control vector $d \in \mathrm{H}$ is said to be *admissible* if $W \in \mathbf{L}(\mathrm{L}^2(0,\infty),\mathrm{H})$ and $\mathrm{Range}(W) \subset D(A)$.

Comment 3.1. In Definition 2.2 $W \in \mathbf{L}(\mathrm{L}^2(0,\infty),\mathrm{H})$ was guaranteed by **EXS**. Hence there the *admissibility* of d reduces to $\mathrm{Range}(W) \subset D(A)$. If Definition 3.1 holds, then $W^* \in \mathbf{L}(\mathrm{H},\mathrm{L}^2(0,\infty))$ and is given by $(W^*x)(t) = d^*S^*(t)x$, $x \in \mathrm{H}$. Moreover, the *reachability operator* Q satisfies $Q := AW \in \mathbf{L}(\mathrm{L}^2(0,\infty),\mathrm{H})$ and $Q^* \in \mathbf{L}(\mathrm{H},\mathrm{L}^2(0,\infty))$, $Q^* = LW^*$, because the restriction of Q to $D(R)$ equals WR.

We are now ready for our next result, where $\mathcal{H}$ will be merely nonnegative, at the cost of restrictions on the linear subsystem factor control vector d:

(A4) The factor control vector $d \in \mathrm{H}$ is admissible according to Definition 3.1.
and on the controller sector condition:

- For sufficiently small $\varepsilon > 0$ we consider the sector

(3.8)
$$\mathcal{S}_\varepsilon := \Big\{ f \in \mathrm{C}(\mathbb{R}): \ -\infty < k_1 < \frac{1}{2}\left[k_1 + k_2 - \sqrt{(k_2-k_1)^2 - 4\varepsilon}\right] \le \frac{f(y)}{y} \le \\ \le \frac{1}{2}\left[k_1 + k_2 + \sqrt{(k_2-k_1)^2 - 4\varepsilon}\right] < k_2 < \infty \qquad \forall y \in \mathbb{R}\setminus\{0\},\ f(0) = 0 \Big\} \ .$$

Theorem 3.2. Let assumptions **(A1)**$\div$**(A4)** hold. Let f belong to $\mathcal{S}_\varepsilon \cap \mathcal{M}$. Then the origin of the space H is *globally strongly asymptotically stable.*

Proof. By the first part of the proof of Theorem 3.1 one gets (3.5). Now, due to $f \in \mathcal{S}_\varepsilon$, we have

$$\dot V = -\left[g^*\dot x + \sqrt{\delta} f\right]^2 - (k_2 y - f)(f - k_1 y) \le -\varepsilon y^2 \ . \tag{3.9}$$

A derivation of the latter estimate is presented in Appendix A. Integrating both sides of (3.9) from 0 to t we obtain

$$-V(x_0) \le V[x(t,x_0)] - V(x_0) \le -\varepsilon \int_0^t y^2(\tau)d\tau$$

whence

$$\|\mathcal{H}\|_{\mathbf{L}(\mathrm{H})}\|x_0\|_{\mathrm{H}}^2 \ge V(x_0) \ge \varepsilon \int_0^t y^2(\tau)d\tau \ .$$

This yields

$$\|y\|_{\mathrm{L}^2(0,\infty)} \le \sqrt{\frac{1}{\varepsilon}\|\mathcal{H}\|_{\mathbf{L}(\mathrm{H})}}\,\|x_0\|_{\mathrm{H}} \ .$$

By the sector conditions imposed on f we find

$$\int_0^\infty u^2(t)dt = \int_0^\infty f^2[y(t)]dt = \int_0^\infty y^2(t)\frac{f^2[y(t)]}{y^2(t)}dt \le \max\left\{k_2^2, k_1^2\right\}\|y\|_{\mathrm{L}^2(0,\infty)}^2 \ ,$$

whence

$$\|u\|_{\mathrm{L}^2(0,\infty)} \le \sqrt{\max\{k_2^2,k_1^2\}\frac{1}{\varepsilon}\|\mathcal{H}\|_{\mathbf{L}(\mathrm{H})}}\,\|x_0\|_{\mathrm{H}} \ . \tag{3.10}$$

We have proved that $y, u \in \mathrm{L}^2(0,\infty)$.

Since $d \in \mathrm{H}$ is an admissible factor control vector, then

$$x(t) = S(t)x_0 + QR_t u \qquad t \ge 0$$

where $Q \in \mathbf{L}(\mathrm{L}^2(0,\infty),\mathrm{H})$ is the reachability map of Comment 3.1 and $R_t \in \mathbf{L}(\mathrm{L}^2(0,\infty))$ denotes the reflection operator at $t > 0$,

$$(R_t u)(\tau) := \left\{\begin{array}{cc} u(t-\tau), & \tau \in [0,t) \\ 0, & \tau \ge t \end{array}\right\}, \qquad \|R_t\|_{\mathbf{L}(\mathrm{L}^2(0,\infty))} \le 1 \ .$$

There holds that $0 \le t \longmapsto x(t) \in \mathrm{H}$ is strongly continuous. Using (3.10) and recalling that **AS** of the semigroup $\{S(t)\}_{t\ge 0}$ implies by the principle of uniform boundedness its stability, we conclude that there exists a constant $\gamma > 0$, such that

$$\|x(t)\|_{\mathrm{H}} \le \gamma\|x_0\|_{\mathrm{H}} \qquad \forall x_0 \in \mathrm{H}, \quad \forall t \ge 0 \ . \tag{3.11}$$

The stability of the null equilibrium easily follows from (3.11).

Considering state–attraction to zero, there holds that $\|S(t)x_0\|_{\mathrm{H}}$ tends to zero as $t \to \infty$ for any $x_0 \in \mathrm{H}$. Hence we may without loss of generality consider $x(t) = QR_t u$. For any fixed $u \in \mathrm{L}^2(0,\infty)$ define for $t_1 > 0$

$$u_{t_1}(t) := \left\{ \begin{array}{ll} 0, & t \in [0,t_1) \\ u(t), & t \geq t_1 \end{array} \right\} .$$

One gets then, using (**A4**) and Comment 3.1, that for $t \geq t_1$

$$x(t) = QR_t u = S(t-t_1)QR_{t_1}u + QR_t u_{t_1} \ ,$$

where $\{S(t)\}_{t\geq 0}$ is **AS**,

$$\|QR_t u_{t_1}\|_{\mathrm{H}} \leq \|Q\|_{\mathbf{L}(\mathrm{L}^2(0,\infty),\mathrm{H})}\|R_t\|_{\mathbf{L}(\mathrm{L}^2(0,\infty))}\|u_{t_1}\|_{\mathrm{L}^2(0,\infty)} \leq \|Q\|_{\mathbf{L}(\mathrm{L}^2(0,\infty),\mathrm{H})}\|u_{t_1}\|_{\mathrm{L}^2(0,\infty)} \ ,$$

and $\|u_{t_1}\|_{\mathrm{L}^2(0,\infty)}$ can be made arbitrarily small for t_1 sufficiently large. Therefore a similar reasoning as in the proof of [28, Lemma 2.1.3] yields that $\lim\limits_{t\to\infty} \|x(t)\|_{\mathrm{H}} = 0$. ☐

Remark 3.2. Alternatively, the state–attraction to zero can be proved by showing that for any $u \in \mathrm{L}^2(0,\infty)$ the function $t \longmapsto QR_t u$ is bounded uniformly continuous and vanishes at infinity [18].

4. Results similar to the Kalman–Yacubovič lemma

In this section we shall study sufficient conditions for solvability of the Lur'e system of equations (3.2), or equivalently (3.7) with respect to the pair $(\mathcal{H}, g)$. In that sense they are similar to the Kalman–Yacubovič Lemma for finite-dimensional systems. A major tool of our results is spectral factorization, which we handle first. We give then a preliminary analysis and finally present our main results.

4.1. **Spectral factorization.** In the sequel $\mathrm{H}^2(\mathbb{D})$ will denote the Hardy space of analytic functions on the unit disk $\mathbb{D} := \{z \in \mathbb{C} : \ |z| < 1\}$ and square integrable on its boundary $\mathbb{T} := \partial\mathbb{D}$, and $\widehat{\mathrm{H}_{-1}}$ will be the space of functions f analytic on Π^+ such that $f\left(\dfrac{1+z}{1-z}\right) \in \mathrm{H}^2(\mathbb{D})$ i.e. $\dfrac{f(s)}{1+s} \in \mathrm{H}^2(\Pi^+)$. The following result is fundamental [20, pp. 52 - 53] and [19, Subsection 1.14]:

Theorem 4.1 (Szegö's theorem)**.** Let $h \in \mathrm{L}^1(\mathbb{T})$ be a real–valued, nonnegative function on the unit circle. A necessary and sufficient condition for the existence of $f \in \mathrm{H}^2(\mathbb{D})$ such that

$$h(e^{j\theta}) = f(e^{j\theta})f(e^{-j\theta}) = \left|f(e^{j\theta})\right|^2 \tag{4.1}$$

is $\ln h \in \mathrm{L}^1(\mathbb{T})$, or equivalently

$$\int_0^{2\pi} \ln h(e^{j\theta})d\theta > -\infty \ .$$

If the last condition is satisfied then the function

$$f(z) = \exp\left[\frac{1}{4\pi}\int_0^{2\pi} \frac{e^{j\theta}+z}{e^{j\theta}-z} \ln h(e^{j\theta})d\theta\right] \tag{4.2}$$

solves the spectral factorization problem (4.1).

Note epecially that by (4.2) in contrast to [20] the boundary value of the modulus of f reads $\left|f(e^{j\theta})\right| = \sqrt{h(e^{j\theta})}$ and not $\left|f(e^{j\theta})\right| = h(e^{j\theta})$. This spectral factor is such that it equals $\sqrt{k}$ if $h(e^{j\theta}) \equiv k > 0$, and corresponds to the *outer* function [20, p. 62] induced by the *positive* square root of h . Other spectral factors are obtained by multiplication by a constant of modulus one. Henceforth f as given by (4.1) is called *the* spectral factor.

Proposition 4.1. Let π be a real–valued, nonnegative function on the $j\omega$–axis such that the function $\omega \longmapsto \dfrac{\pi(j\omega)}{1+\omega^2}$ belongs to $\mathrm{L}^1(\mathbb{R})$. A necessary and sufficient condition for the existence of $\phi \in \widehat{\mathrm{H}_{-1}}$ such that

$$\pi(j\omega) = \phi(j\omega)\phi(-j\omega) = |\phi(j\omega)|^2 \tag{4.3}$$

is $\omega \longmapsto \dfrac{\ln \pi(j\omega)}{1+\omega^2} \in \mathrm{L}^1(\mathbb{R})$, or equivalently

$$\int_{\mathbb{R}} \frac{\ln \pi(j\omega)}{1+\omega^2} d\omega > -\infty \ . \tag{4.4}$$

If the last condition holds then the function

$$\phi(s) = \exp\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{sj\omega - 1}{j\omega - s}\frac{\ln \pi(j\omega)}{1+\omega^2} d\omega\right] \tag{4.5}$$

solves the spectral factorization problem (4.3).

Proof. The spectral factorization problem (4.3) reduces to (4.1). Indeed,

$$f(z) = \exp\left[\frac{1}{4\pi}\int_0^{2\pi} \frac{e^{j\theta}+z}{e^{j\theta}-z} \ln h(e^{j\theta}) d\theta\right] = \phi\left(\frac{1+z}{1-z}\right)$$

solves (4.1) iff, making substitutions

$$e^{j\theta} = \frac{j\omega - 1}{j\omega + 1} \iff \omega = \cot\frac{\theta}{2} \quad \text{and} \quad z = \frac{s-1}{s+1} \ ,$$

the function

$$\phi(s) = f\left(\frac{s-1}{s+1}\right) = \exp\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{sj\omega - 1}{j\omega - s} \ln h\left(\frac{j\omega+1}{j\omega-1}\right)\frac{d\omega}{1+\omega^2}\right] \ ,$$

where $\pi(j\omega) := h\left(\dfrac{j\omega+1}{j\omega-1}\right)$, solves (4.3). Observe that

$$\int_{-\infty}^{\infty} \frac{\pi(j\omega)}{1+\omega^2} d\omega = \int_{-\infty}^{\infty} \frac{h\left(\dfrac{j\omega+1}{j\omega-1}\right)}{1+\omega^2} d\omega = \frac{1}{2\pi}\int_0^{2\pi} h(e^{j\theta}) d\theta \ ,$$

i.e., the function $\omega \longmapsto \dfrac{\pi(\omega)}{1+\omega^2}$ belongs to $\mathrm{L}^1(\mathbb{R})$ iff $h \in \mathrm{L}^1(\mathbb{T})$, and

$$\int_{-\infty}^{\infty} \frac{\ln \pi(j\omega)}{1+\omega^2} d\omega = \int_{-\infty}^{\infty} \frac{\ln h\left(\dfrac{j\omega+1}{j\omega-1}\right)}{1+\omega^2} d\omega = \frac{1}{2}\int_0^{2\pi} \ln h(e^{j\theta}) d\theta \ ,$$

i.e., the function $\omega \longmapsto \dfrac{\ln \pi(j\omega)}{1+\omega^2}$ belongs to $\mathrm{L}^1(\mathbb{R})$ iff $\ln h \in \mathrm{L}^1(\mathbb{T})$. Moreover, $f \in \mathrm{H}^2(\mathbb{D})$ iff $\phi \in \widehat{\mathrm{H}_{-1}}$. $\square$

Henceforth the function $\pi(\cdot)$ of Proposition 4.1 will be called the *Popov function* and the function defined by (4.5) *the* spectral factor. Further important properties of the spectral factor are gathered in the lemma below.

Lemma 4.1. Let the Popov function $\pi(j\omega)$ belong to $\mathrm{L}^\infty(\mathbb{R})$ and consider two cases:

(a) Let in addition $\pi(j\omega)$ satisfy (4.4), then $\phi \in \mathrm{H}^\infty(\Pi^+)$; moreover if in addition $\pi(j\omega)$ has an analytic extension in a domain containing a full neighborhood of $s = 0$ which is para–Hermitian self–adjoint (i.e. $\pi(s) = \pi(-s)$), then

$$\left(s \longmapsto \frac{\phi(s) - \phi(0)}{s} \right) \in \mathrm{H}^\infty(\Pi^+) \cap \mathrm{H}^2(\Pi^+) \ . \tag{4.6}$$

(b) Let in addition $\pi(j\omega)$ be *coercive*, i.e. there exists an $\epsilon > 0$ such that

$$\pi(j\omega) \geq \epsilon \quad \text{ for all } \omega \in \mathbb{R} \ ,$$

then both ϕ and $1/\phi$ are in $\mathrm{H}^\infty(\Pi^+)$; moreover if in addition $\pi(j\omega)$ has a para–Hermitian self–adjoint analytic extension in a domain containing a full neighborhood of $s = 0$, then (4.6) holds.

Proof. As $\pi(j\omega)$ belongs to $\mathrm{L}^\infty(\mathbb{R})$ the assumption of Proposition 4.1 holds.

(a). As $\pi(j\omega)$ satisfies (4.4) the conclusion of Proposition 4.1 holds. Observe that with $s = x + jy$, $x \geq 0$,

$$\frac{js\omega - 1}{j\omega - s} = \frac{x(1+\omega^2)}{x^2 + (y-\omega)^2} + j\frac{y(\omega^2 - 1) + (1 - x^2 - y^2)\omega}{x^2 + (y-\omega)^2} \ . \tag{4.7}$$

The spectral factor ϕ is analytic on Π^+ because $\phi \in \widehat{\mathrm{H}_{-1}}$, and we have to prove that ϕ is bounded on Π^+. By (4.5) we have

$$|\phi(s)| = e^{\mathrm{Re}\,\varphi(s)}, \qquad \varphi(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{sj\omega - 1}{j\omega - s}\frac{\ln \pi(j\omega)}{1+\omega^2} d\omega \ .$$

Using (4.7) we get

$$|\phi(s)| = \exp\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{x \ln \pi(j\omega)}{x^2 + (y-\omega)^2} d\omega\right] = \exp\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\ln \pi(j(y - x\omega))}{1+\omega^2} d\omega\right] \ .$$

But

$$\ln \pi[(j(y - x\omega)] \leq \ln^+ \pi[(j(y - x\omega)] \leq \ln^+ \|\pi\|_{\mathrm{L}^\infty(\mathbb{R})} \ ,$$

where

$$\ln^+ x = \left\{ \begin{array}{cc} \ln x, & x \geq 1 \\ 0, & 0 < x < 1 \end{array} \right\} \ ,$$

and therefore

$$|\phi(s)| \leq \exp\left[\frac{\ln^+ \|\pi\|_{\mathrm{L}^\infty(\mathbb{R})}}{\pi} \int_0^\infty \frac{1}{1+\omega^2} d\omega\right] = \exp\left[\frac{1}{2}\ln^+ \|\pi\|_{\mathrm{L}^\infty(\mathbb{R})}\right] \qquad \forall s \in \Pi^+ \ .$$

We prove now (4.6). As the Popov function has a para–Hermitian self–adjoint analytic extension in a domain containing a full neighbourhood of $s = 0$ then we have there the factorization

$$\pi(s) = \phi(s)\phi(-s) \ , \tag{4.8}$$

with $\phi(s)$ *analytic* (this can be seen by considering the successive self–adjoint polynomial approximations and their factorizations of the Taylor expansion π near zero). This jointly with $\phi \in \mathrm{H}^\infty(\Pi^+)$ leads to the fact that the function $s \longmapsto \dfrac{\phi(s)-\phi(0)}{s}$ is analytic and bounded in a full neighborhood of $s=0$ and finally is in $\mathrm{H}^\infty(\Pi^+) \cap \mathrm{H}^2(\Pi^+)$.

(b). This follows because (4.4) holds whence the conclusions of (**a**) hold and moreover

$$\|\phi^{-1}\|^2_{\mathrm{H}^\infty(\Pi^+)} \leq \epsilon^{-1} \ .$$

□

4.2. **Preliminary analysis.** From now on we consider the feedback system of Figure 3.1 with $f(y) = \mu_0 y$ and make a preliminary analysis under the following assumptions:

(H1) The operator $A : (D(A) \subset \mathrm{H}) \longrightarrow \mathrm{H}$ generates an **EXS** linear C_0–semigroup on H;

(H2) The compatibility condition (2.3) holds;

(H3) The observation functional $c^\#$ is admissible, $c^\#|_{D(A)} = h^*A$;

(H4) The transfer function $\hat{g}$, defined by (2.4), satisfies (2.5);

(H5) There exist k_1, k_2, $k_1 < k_2$ such that

$$\begin{aligned}\pi(\omega) :=\ & 1 - (k_1+k_2)\,\mathrm{Re}[\hat{g}(j\omega)] + k_1k_2\,|\hat{g}(j\omega)|^2 = \\ =\ & \delta + 2e\,\mathrm{Re}[-\hat{g}(j\omega) - c^\# d] + q\big|\hat{g}(j\omega) + c^\# d\big|^2, \qquad \omega \in \mathbb{R}\ ,\end{aligned} \tag{4.9}$$

satisfies[1]

$$\pi(\omega) \geq 0 \qquad \forall \omega \in \mathbb{R} \tag{4.10}$$

and[2]

$$\int_{-\infty}^{\infty} \frac{\ln \pi(\omega)}{1+\omega^2} d\omega > -\infty \iff \left(\omega \longmapsto \frac{\ln \pi(\omega)}{1+\omega^2}\right) \in \mathrm{L}^1(\mathbb{R})\ ; \tag{4.11}$$

(H6) For $\mu_0 = (k_1+k_2)/2$ we have

$$\left(s \longmapsto \frac{1}{1-\mu_0 \hat{g}(s)}\right) \in \mathrm{H}^\infty(\Pi^+)\ .$$

Note that as $\hat{g} \in \mathrm{H}^\infty(\Pi^+)$ one gets $\pi \in \mathrm{L}^\infty(\mathbb{R})$ and consequently the function $\omega \longmapsto \dfrac{\pi(j\omega)}{1+\omega^2}$ is in $\mathrm{L}^1(\mathbb{R})$. It follows from (4.10) and (4.11) and Proposition 4.1 that the spectral factorization (4.3) holds with $\phi \in \widehat{\mathrm{H}_{-1}}$, given by (4.5). Moreover by Lemma 4.1(**a**), $\phi \in \mathrm{H}^\infty(\Pi^+)$. Furthermore, as $\hat{g}(s) + c^\# d = s\big(\widehat{Pd}\big)(s)$, it follows by Lemma 2.4 that the Popov function has a para–Hermitian self–adjoint analytic extension in a domain containing a full neighborhood of $s=0$ which reads

$$\begin{aligned}\pi(s) :=\ & 1 - \frac{(k_1+k_2)}{2}[\hat{g}(s) + \hat{g}(-s)] + k_1k_2\hat{g}(s)\hat{g}(-s) = \\ =\ & \delta - es\left[\big(\widehat{Pd}\big)(s) - \big(\widehat{Pd}\big)(-s)\right] - qs^2\big(\widehat{Pd}\big)(s)\big(\widehat{Pd}\big)(-s)\ .\end{aligned} \tag{4.12}$$

---

[1]If $k_1k_2 < 0$ then the frequency–domain inequality (4.10) means geometrically that the plot of the transfer function $\hat{g}(j\omega)$ is located in the circle with centre at $(k_1^{-1} + k_2^{-1})/2$ and radius $(k_2^{-1} - k_1^{-1})/2$. In particular, this yields $\hat{g} \in \mathrm{H}^\infty(\Pi^+)$.

[2](4.11) means that the rate of which π approaches its zeros (asymptotic zeros included) is not to fast (e.g. for the function $\pi(\omega) = e^{-\omega^2}$ all assumptions of Proposition 4.1 besides (4.11) are met and therefore it is not factorizable), however the number of zeros may be countably infinite. The latter holds for the example studied in Section 5.

Hence again by Lemma 4.1(**a**)

$$\left(s \longmapsto \frac{\phi(s)-\phi(0)}{s} = \frac{\phi(s)-\sqrt{\delta}}{s}\right) \in \mathrm{H}^\infty(\Pi^+) \cap \mathrm{H}^2(\Pi^+) \ , \tag{4.13}$$

where by (4.12) and (4.8) we got: $\delta \geq 0$ and $\phi(0) = \sqrt{\delta}$.

Henceforth given (**H1**)$\div$(**H5**), we call *realization problem* that of finding a $g \in \mathrm{H}$ satisfying the identity:

$$\frac{\phi(s)-\sqrt{\delta}}{s} = g^* A(sI - A)^{-1} d, \qquad s \in \Pi^+ \ , \tag{4.14}$$

such that the observation functional g^*A is admissible.

The following auxiliary result will be useful:

Lemma 4.2. Let (**H1**)$\div$(**H6**) hold. Then

(**1**) h^* is a linear bounded observation functional, which is admissible with respect to the semigroup $\{e^{tA_0}\}_{t\geq 0}$ generated by the operator

$$A_0 := A^{-1} - \frac{\mu_0}{1+\mu_0 c^\# d} dh^* \in \mathbf{L}(\mathrm{H}) \ . \tag{4.15}$$

(**2**) If $g \in \mathrm{H}$ is a solution of the realization equation (4.14), then, g^* is admissible with respect to the semigroup $\{e^{tA_0}\}_{t\geq 0}$ if and only if g^*A is admissible for the semigroup generated by A.

(**3**) If $g \in \mathrm{H}$ is a solution of the realization problem (4.14), then g^* is admissible for the semigroup $\{e^{tA_0}\}_{t\geq 0}$.

Proof. Obviously (**3**) follows by (**2**). Thus we must prove (**1**) and (**2**). A preliminary exploration is made first.

Observe that the static loop return difference in Figure 3.1, namely $1 + \mu_0 c^\# d \neq 0$. Indeed, if $\mu_0 = 0$ then this is obvious. If $\mu_0 \neq 0$ then $1 + \mu_0 c^\# d = 0$ would imply

$$\pi(0) = \delta = \Big(1 - \frac{k_1}{\mu_0}\Big)\Big(1 - \frac{k_2}{\mu_0}\Big) = \Big(1 - \frac{2k_1}{k_1+k_2}\Big)\Big(1 - \frac{2k_2}{k_1+k_2}\Big) = -\Big(\frac{k_2-k_1}{k_1+k_2}\Big)^2 < 0$$

which leads to a contradiction with (4.10). Hence, A_0 is well–defined and clearly belongs to $\mathbf{L}(\mathrm{H})$. A_0 is the inverse of the closed–loop generator arising from (2.1) by taking a linear feedback law $u(t) = \mu_0 y(t)$ in Figure 3.1, i.e.

$$A_0^{-1} = A(I + \mu_0 dc^\#)$$

with

$$D(A) \subset D(c^\#) \ , \quad D(A_0^{-1}) \subset D(c^\#) \quad \text{and} \quad [I + \mu_0 dc^\#] D(A_0^{-1}) = D(A) \ ,$$

where the last relation is a linear bijection as on $D(c^\#)$, $[I + \mu_0 dc^\#]^{-1} = I - \mu_0 (1 + \mu_0 c^\# d)^{-1} dc^\#$. Consider now also the generator $A^{-1} \in \mathbf{L}(\mathrm{H})$ and recall that $\rho(\cdot)$ stands for the resolvent set of an operator. Then the idea of open–loop versus closed–loop resolvent identity and (4.15) give the following identities in $\mathbf{L}(\mathrm{H})$ valid for every $s \in \rho(A) \cap \rho(A_0^{-1})$, viz.

$$\begin{aligned} (sA^{-1} - I)^{-1} - (sA_0 - I)^{-1} &= -(sA^{-1} - I)^{-1} \Big[\frac{s\mu_0}{1+\mu_0 c^\# d} dh^*\Big] (sA_0 - I)^{-1} \\ &= (sA_0 - I)^{-1} \Big[\frac{s\mu_0}{1+\mu_0 c^\# d} dh^*\Big] (sA^{-1} - I)^{-1} \ . \end{aligned} \tag{4.16}$$

(1). Premultiplying and postmultying the first identity of (4.16) by respectively h^* and $x_0 \in \mathrm{H}$, and using (2.4) gives

$$h^*(sA^{-1} - I)^{-1}x_0 = \frac{1 - \mu_0\hat{g}(s)}{1 + \mu_0 c^{\#}d}\, h^*(sA_0 - I)^{-1}x_0$$

or equivalently similarly as in [18, p. 12]

$$h^*(sA_0 - I)^{-1}x_0 = \frac{1 + \mu_0 c^{\#}d}{1 - \mu_0\hat{g}(s)}\, h^*A(sI - A)^{-1}x_0 = \frac{1 + \mu_0 c^{\#}d}{1 - \mu_0\hat{g}(s)}\, \widehat{\overline{P}x_0} \ , \tag{4.17}$$

where by (**H3**) and Lemma 2.1: $\overline{P} \in \mathbf{L}(\mathrm{H}, \mathrm{L}^2(0,\infty))$ is the observability map associated with $c^{\#}$, while by (**H6**): $\dfrac{1}{1 - \mu_0\hat{g}} \in \mathrm{H}^\infty(\Pi^+)$. Hence the operator

$$\mathrm{H} \ni x_0 \longmapsto h^*(sA_0 - I)^{-1}x_0 \in \mathrm{H}^2(\Pi^+)$$

belongs to $\mathbf{L}(\mathrm{H}, \mathrm{H}^2(\Pi^+))$. Recall now the operator $U \in \mathbf{L}(\mathrm{H}^2(\Pi^+))$ given by (2.2) and note here that $\varphi(s) = h^*(sA_0 - I)^{-1}x_0$ gives $(U\varphi)(s) = -h^*(sI - A_0)^{-1}x_0$. Hence by composition, one gets that the operator

$$\mathrm{H} \ni x_0 \longmapsto h^*(sI - A_0)^{-1}x_0 \in \mathrm{H}^2(\Pi^+)$$

belongs to $\mathbf{L}(\mathrm{H}, \mathrm{H}^2(\Pi^+))$. Thus h^* is admissible with respect to the semigroup $\{e^{tA_0}\}_{t\geq 0}$.

(2). Let $g \in \mathrm{H}$ be a solution of the realization equation (4.14). Premultiplication and postumultiplication of the first identity of (4.16) by respectively g^* and $x_0 \in \mathrm{H}$ give successively

$$\begin{aligned} g^*(sA^{-1} - I)^{-1}x_0 - g^*(sA_0 - I)^{-1}x_0 \ &= -sg^*(sA^{-1} - I)^{-1}d\, \frac{\mu_0}{1 + \mu_0 c^{\#}d} h^*(sA_0 - I)^{-1}x_0 \\ &= \frac{\mu_0}{1 - \mu_0\hat{g}(s)}\left[\phi(s) - \sqrt{\delta}\right] h^*A(sI - A)^{-1}x_0 \ , \end{aligned}$$

where the last equality follows by (4.14) and the first equality of (4.17). There results

$$g^*(sA_0 - I)^{-1}x_0 = g^*A(sI - A)^{-1}x_0 + \frac{\mu_0}{1 - \mu_0\hat{g}(s)}\left[\phi(s) - \sqrt{\delta}\right]\widehat{\overline{P}x_0} \ , \tag{4.18}$$

where one uses (**H3**) and Lemma 2.1 with $\overline{P} \in \mathbf{L}(\mathrm{H}, \mathrm{L}^2(0,\infty))$. Now $\phi \in \mathrm{H}^\infty(\Pi^+)$, whence by (**H6**) the two terms between square brackets of the last summand in (4.18) are in $\mathrm{H}^\infty(\Pi^+)$. Hence the operator

$$\mathrm{H} \ni x_0 \longmapsto \frac{\mu_0}{1 - \mu_0\hat{g}(s)}\left[\phi(s) - \sqrt{\delta}\right]\widehat{\overline{P}x_0} \in \mathrm{H}^2(\Pi^+)$$

belongs to $\mathbf{L}(\mathrm{H}, \mathrm{H}^2(\Pi^+))$. Note also that a subsequent application of the unitary operator $U \in \mathbf{L}(\mathrm{H}^2(\Pi^+)$ given by (2.2), will map $\varphi(s) = g^*(sA_0 - I)^{-1}x_0$ into $(U\varphi)(s) = -g^*(sI - A_0)^{-1}x_0$. These informations show finally that by (4.18) and composition, g^*A is admissible for the semigroup generated by A iff g^* is admissible with respect to the semigroup $\{e^{tA_0}\}_{t\geq 0}$. □

Remark 4.1. (4.17) represents the Laplace transformed action of the closed–loop observability map of $c^{\#}$ with respect to the semigroup generated by A_0^{-1}, while (4.18) gives the same for the observability map of state–feedback control induced by the spectral factor. The proof above shows that this follows by the important identity (4.16). In fact it is the source of all linear feedback wisdom. To discover more note that as soon as (2.3) holds and

$c^\#$ is admissible, then the open–loop transfer function (2.4) reads $\hat{g}(s) = c^\#(sA^{-1} - I)^{-1}d$. Postmultiplying the first identity of (4.16) by d gives

$$(sA^{-1} - I)^{-1}d - (sA_0 - I)^{-1}d = -(sA^{-1} - I)^{-1}d \ \frac{s\mu_0}{1 + \mu_0 c^\# d} \ h^*(sA_0 - I)^{-1}d \ .$$

Then with $h^*(sA_0 - I)^{-1}d$ and $\hat{g}(s) = c^\#(sA^{-1} - I)^{-1}d$ well–defined, $c^\#(sA_0 - I)^{-1}d$ is well–defined. Assume that $(1 - \mu_0 \hat{g}(s))^{-1}$ is well–defined and consider the transfer function

$$\hat{g}_c(s) := \frac{\mu_0}{1 + \mu_0 c^\# d} \ c^\#(sA_0 - I)^{-1}d \ . \tag{4.19}$$

It turns out that

$$\hat{g}_c(s) = \frac{\mu_0 \hat{g}(s)}{1 - \mu_0 \hat{g}(s)} \tag{4.20}$$

i.e. one gets by (4.19) an *explicit form* of the system *closed–loop transfer function* in Figure 3.1. To see this premultiply and postmultiply the second identity in (4.16) by respectively $c^\#$ and d. One gets

$$c^\#(sA^{-1} - I)^{-1}d - c^\#(sA_0 - I)^{-1}d = c^\#(sA_0 - I)^{-1}d \ \frac{\mu_0}{1 + \mu_0 c^\#} \ sh^*(sA^{-1} - I)^{-1}d \ ,$$

which gives exactly

$$\hat{g}(s) - \frac{1 + \mu_0 c^\# d}{\mu_0} \ \hat{g}_c(s) = \hat{g}_c(s)\big[\hat{g}(s) + c^\# d\big] \ ,$$

from which (4.20) follows. The *explicit form* of the system *closed–loop sensitivity transfer function* (*inverse of the return–difference*) in Figure 3.1 reads then easily

$$\hat{g}_s(s) := \frac{1}{1 - \mu_0 \hat{g}(s)} = \frac{1}{1 + \mu_0 c^\# d} \ \big[1 + \mu_0 \left(c^\#(sA_0 - I)^{-1}d + c^\# d\right)\big] \ ,$$

with correct static value for $s = 0$.

We are now ready for our main results.

4.3. **Main results.** The theorem below is a generalization of the results due to Nudel'man and Schwartzman [26, Theorem 4, p. 570], and A.V. Balakrishnan [2, Theorem 2.1, p. 179]. See also Subsection 7.1.

Theorem 4.2. Let assumptions (**H1**)÷(**H6**) hold. Moreover assume that:

(H7) The vector $d \in \mathrm{H}$ is *cyclic* for A^{-1}, i.e., $\{A^{-k}d\}_{k=0}^\infty$ is a *complete* system in H;

(H8) The operator A has a system of eigenvectors $\{\phi_k\}_{k\in\mathbb{N}}$, corresponding to the eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$, which forms a *Riesz basis* of H;

(H9) There holds:

$$\left\{\frac{\alpha_k}{\lambda_k \langle d, \psi_k\rangle_{\mathrm{H}}}\right\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{N}) \tag{4.21}$$

and

$$\sum_{k=1}^\infty \Big|\frac{\alpha_k}{\langle d, \psi_k\rangle_{\mathrm{H}}}\Big|^2 \big|\hat{f}(-\overline{\lambda_k})\big|^2 < \infty \qquad \forall \hat{f} \in \mathrm{H}^2(\Pi^+) \tag{4.22}$$

where

$$\alpha_k := \operatorname*{Res}_{s=\lambda_k}\Big[\frac{\phi(s) - \sqrt{\delta}}{s}\Big] = \frac{1}{\lambda_k} \operatorname*{Res}_{s=\lambda_k} \phi(s) \ , \tag{4.23}$$

$\{\psi_k\}_{k\in\mathbb{N}}$ is the system of eigenvectors of A^* which is biorthogonal with respect to $\{\phi_k\}_{k\in\mathbb{N}}$.

Then the system (3.2), or equivalently (3.7), has a solution $(\mathcal{H}, g)$, $\mathcal{H} \in \mathbf{L}(\mathrm{H})$, $\mathcal{H} = \mathcal{H}^* \geq 0$, $g \in \mathrm{H}$ and the observation functional g^*A is admissible with respect to the semigroup generated by A.

Proof. By (**H1**)÷(**H6**) all the results of preliminary subsection 4.2 hold and in particular the conclusions of Lemma 4.2. For convenience recall that the *realization problem* (4.14) is to find $g \in \mathrm{H}$ from the identity:

$$\frac{\phi(s) - \sqrt{\delta}}{s} = g^*A(sI - A)^{-1}d, \qquad s \in \Pi^+ \ ,$$

such that the observation functional g^*A is admissible.

We were not able to find a general solution to the realization problem using only assumptions (**H1**)÷(**H5**) and (**H7**), which is the case for a finite–dimensional version of (3.2) or (3.7).

However, a systematic procedure of solving the realization problem can be given provided that additionally assumptions (**H7**)÷(**H9**) hold. Then each element $x \in \mathrm{H}$ has the expansions

$$x = \sum_{k=1}^{\infty} \langle x, \psi_k\rangle_{\mathrm{H}} \phi_k, \qquad x = \sum_{k=1}^{\infty} \langle x, \phi_k\rangle_{\mathrm{H}} \psi_k \ . \tag{4.24}$$

Both the transfer function $\hat{g}$ and the spectral factor ϕ are meromorphic functions. Hence $\dfrac{\phi(s) - \sqrt{\delta}}{s}$ is meromorphic too and by (4.14) we get

$$\sum_{k=1}^{\infty} \frac{\alpha_k}{s - \lambda_k} = g^*A(sI - A)^{-1}d = \sum_{k=1}^{\infty} \frac{\lambda_k}{s - \lambda_k} \langle d, \psi_k\rangle_{\mathrm{H}} \langle \phi_k, g\rangle_{\mathrm{H}} \qquad \forall s \in \mathbb{C}, \quad s \neq \lambda_k$$

where the α_k for $k \in \mathbb{N}$ are given by (4.23). The vector d is cyclic with respect to A^{-1} iff (A^{-1}, d) is *approximately controllable* or equivalently (A, d) is approximately controllable, or equivalently $\langle d, \psi_k\rangle_{\mathrm{H}} \neq 0$ for all $k \in \mathbb{N}$. Thus by (**H7**) one gets

$$g^*A\phi_k = \langle A\phi_k, g\rangle_{\mathrm{H}} = \langle \lambda_k \phi_k, g\rangle_{\mathrm{H}} = \frac{\alpha_k}{\langle d, \psi_k\rangle_{\mathrm{H}}}, \qquad g^*\phi_k = \langle \phi_k, g\rangle_{\mathrm{H}} = \frac{\alpha_k}{\lambda_k \langle d, \psi_k\rangle_{\mathrm{H}}}$$

and $g \in \mathrm{H}$ can be recovered using (4.24), provided that $\{g^*\phi_k\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{N})$, i.e., when (4.21) holds.

Applying the spectral criterion of admissibility in [15] we get that g^*A is admissible iff

$$\sum_{k=1}^{\infty} |g^*A\phi_k|^2 \left|\hat{f}(-\overline{\lambda_k})\right|^2 < \infty \qquad \forall \hat{f} \in \mathrm{H}^2(\Pi^+) \ ,$$

i.e., when (4.22) is satisfied.

We show now the existence of an appropriate solution $(\mathcal{H}, g)$ of the Lur'e system (3.2). We start by noting that, since g^*A and $h^*A = c^{\#}|_{D(A)}$ are both admissible and by (**H1**), there exist unique $\mathcal{H}_g = \mathcal{H}_g^* \geq 0$, $\mathcal{H}_g \in \mathbf{L}(\mathrm{H})$ and $\mathcal{H}_h = \mathcal{H}_h^* \geq 0$, $\mathcal{H}_h \in \mathbf{L}(\mathrm{H})$ such that

$$(A^*)^{-1}\mathcal{H}_g + \mathcal{H}_g A^{-1} = -gg^*, \qquad (A^*)^{-1}\mathcal{H}_h + \mathcal{H}_h A^{-1} = -hh^* \ .$$

Hence $\mathcal{H} := \mathcal{H}_g - q\mathcal{H}_h$ is a bounded self–adjoint solution of the first equation of the system (3.2).

We show now that the pair $(\mathcal{H}, g)$ solves also the second equation of (3.2). Note that by (**H1**) the resolvent $[sI - A^{-1}]^{-1}$ of A^{-1} is regular on $j\mathbb{R} \setminus \{0\}$. Therefore premultiplying the first equation of (3.2) by $d^*[-j\omega I - (A^*)^{-1}]^{-1}$ and postmultiplying it by $[j\omega I - A^{-1}]^{-1}d$ yields

$$\begin{aligned} &d^*[-j\omega I - (A^*)^{-1}]^{-1}[\mathcal{H}A^{-1} + (A^*)^{-1}\mathcal{H}][j\omega I - A^{-1}]^{-1}d = \\ &= d^*[-j\omega I - (A^*)^{-1}]^{-1}[qhh^* - gg^*][j\omega I - A^{-1}]^{-1}d = \\ &= q\left|h^*[j\omega I - A^{-1}]^{-1}d\right|^2 - \left|g^*[j\omega I - A^{-1}]^{-1}d\right|^2 \qquad \forall \omega \in \mathbb{R} \setminus \{0\} \ . \end{aligned}$$

Using the operator identities:

$$\begin{aligned} [-j\omega I - (A^*)^{-1}]^{-1}(A^*)^{-1} &= -j\omega[-j\omega I - (A^*)^{-1}]^{-1} - I \ , \\ A^{-1}[j\omega I - A^{-1}]^{-1} &= j\omega[j\omega I - A^{-1}]^{-1} - I \end{aligned}$$

one obtains

$$-2\,\mathrm{Re}\, d^*\mathcal{H}[j\omega I - A^{-1}]^{-1}d = q\left|h^*[j\omega I - A^{-1}]^{-1}d\right|^2 - \left|g^*[j\omega I - A^{-1}]^{-1}d\right|^2 \quad \forall \omega \in \mathbb{R} \setminus \{0\}.$$

Let $\Delta := -\mathcal{H}d + eh + \sqrt{\delta}g$. Then upon substituting Δ by its expression one gets

$$\begin{aligned} &2\,\mathrm{Re}\big[\Delta^*(j\omega I - A^{-1})^{-1}d\big] = q\left|h^*(j\omega I - A^{-1})^{-1}d\right|^2 - \left|g^*(j\omega I - A^{-1})^{-1}d\right|^2 + \\ &+2e\,\mathrm{Re}\big[h^*(j\omega I - A^{-1})^{-1}d\big] + 2\sqrt{\delta}\,\mathrm{Re}\big[g^*(j\omega I - A^{-1})^{-1}d\big] = \\ &= \delta + 2e\,\mathrm{Re}\,[h^*(j\omega I - A^{-1})^{-1}d] + q\,|h^*(j\omega I - A^{-1})^{-1}d|^2 - \\ &-|\sqrt{\delta} - g^*(j\omega I - A^{-1})^{-1}d|^2 \qquad \forall \omega \in \mathbb{R}, \quad \omega \neq 0 \ . \end{aligned}$$

Applying the transformation $\mathbb{C} \ni \zeta \longmapsto s^{-1} \in \mathbb{C}$ and (2.4) we get

$$\begin{aligned} h^*(\zeta I - A^{-1})^{-1}d &= -sh^*A(sI - A)^{-1}d = -\hat{g}(s) - c^{\#}d \ , \\ g^*(\zeta I - A^{-1})^{-1}d &= -sg^*A(sI - A)^{-1}d \ . \end{aligned}$$

Now, using (4.9) and (4.14) we obtain

$$2\,\mathrm{Re}[\Delta^*\left(j\omega I - A^{-1}\right)^{-1}d] = 0 \qquad \forall \omega \in \mathbb{R} \setminus \{0\} \ .$$

Recall that the system $\{\phi_k\}_{k\in\mathbb{N}}$ is a Riesz basis of H with corresponding biorthogonal system $\{\psi_k\}_{k\in\mathbb{N}}$, iff there exist an isomorphism $T \in \mathbf{L}(\mathrm{H})$ and an orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$ such that

$$\phi_k = Te_k, \qquad \psi_k = (T^*)^{-1}e_k, \qquad k \in \mathbb{N} \ .$$

Then

$$\|Tx\|_{\mathrm{H}} \leq \|T\|_{\mathbf{L}(\mathrm{H})}\|x\|_{\mathrm{H}}, \qquad \|T^{-1}x\|_{\mathrm{H}} \leq \|T^{-1}\|_{\mathbf{L}(\mathrm{H})}\|x\|_{\mathrm{H}} \qquad \forall x \in \mathrm{H}$$

and Parseval's theorem gives

$$\frac{1}{\|T^{-1}\|^2_{\mathbf{L}(\mathrm{H})}} \sum_{k=1}^{\infty} |\langle x, \psi_k\rangle_{\mathrm{H}}|^2 \leq \|x\|^2_{\mathrm{H}} \leq \|T\|^2_{\mathbf{L}(\mathrm{H})} \sum_{k=1}^{\infty} |\langle x, \psi_k\rangle_{\mathrm{H}}|^2 \qquad \forall x \in \mathrm{H} \ . \tag{4.25}$$

Thus, inserting $x = e^{tA^{-1}}x_0$ into (4.25) gives

$$\begin{aligned}\left\|e^{tA^{-1}}x_0\right\|^2_{\mathrm{H}} = & \left\|\sum_{k=1}^{\infty}\langle e^{tA^{-1}}x_0,\psi_k\rangle_{\mathrm{H}}\phi_k\right\|^2_{\mathrm{H}} \le \|T\|^2_{\mathbf{L}(\mathrm{H})}\sum_{k=1}^{\infty}\left|\langle e^{tA^{-1}}x_0,\psi_k\rangle_{\mathrm{H}}\right|^2 = \\ = & \|T\|^2_{\mathbf{L}(\mathrm{H})}\sum_{k=1}^{\infty}\left|\langle x_0, e^{t(A^*)^{-1}}\psi_k\rangle_{\mathrm{H}}\right|^2 = \|T\|^2_{\mathbf{L}(\mathrm{H})}\sum_{k=1}^{\infty}e^{2t/\operatorname{Re}\lambda_k}\left|\langle x_0,\psi_k\rangle_{\mathrm{H}}\right|^2 \le \\ \le & \|T\|^2_{\mathbf{L}(\mathrm{H})}\sum_{k=1}^{\infty}\left|\langle x_0,\psi_k\rangle_{\mathrm{H}}\right|^2 \le \|T\|^2_{\mathbf{L}(\mathrm{H})}\|T^{-1}\|^2_{\mathbf{L}(\mathrm{H})}\|x_0\|^2_{\mathrm{H}} \quad \forall x\in\mathrm{H},\ \forall t\ge 0 \ ,\end{aligned}$$

i.e., the semigroup $\{e^{tA^{-1}}\}_{t\ge 0}$ is uniformly bounded (stable). Hence by Lemma 2.5, the semigroup $\{e^{tA^{-1}}\}_{t\ge 0}$ is **AS**.

Observe now

$$\begin{aligned}\Delta^* e^{tA^{-1}}de^{-j\omega t} = & \ \Delta^*(j\omega I - A^{-1})(j\omega I - A^{-1})^{-1}e^{tA^{-1}}de^{-j\omega t} = \\ = & \ j\omega e^{-j\omega t}\Delta^*(j\omega I - A^{-1})^{-1}e^{tA^{-1}}d - e^{-j\omega t}\Delta^*(j\omega I - A^{-1})^{-1}A^{-1}e^{tA^{-1}}d = \\ = & \ \frac{d}{dt}\left[-e^{-j\omega t}\Delta^*(j\omega I - A^{-1})^{-1}e^{tA^{-1}}d\right] \ .\end{aligned}$$

Integrating from 0 to T and using the **AS** of the semigroup $\{e^{tA^{-1}}\}_{t\ge 0}$ gives

$$\Delta^*(j\omega I - A^{-1})^{-1}d = \int_0^{\infty}\Delta^* e^{tA^{-1}}de^{-j\omega t}dt \ .$$

Hence,

$$\begin{aligned}2\operatorname{Re}[\Delta^*(j\omega I - A^{-1})^{-1}d = & \int_0^{\infty}\Delta^* e^{tA^{-1}}de^{-j\omega t}dt + \overline{\int_0^{\infty}\Delta^* e^{tA^{-1}}de^{-j\omega t}dt} = \\ = \int_0^{\infty}\Delta^* e^{tA^{-1}}de^{-j\omega t}dt + & \int_0^{\infty}\Delta^* e^{tA^{-1}}de^{j\omega t}dt = \int_{-\infty}^{\infty}\Delta^* e^{|t|A^{-1}}de^{-j\omega t}dt \ .\end{aligned}$$

The latter means that $2\operatorname{Re}[\Delta^*(j\omega I - A^{-1})^{-1}d$ can be regarded as the Fourier transform of the continuous, decaying (for $|t|$ increasing) function $\mathbb{R}\ni t\longmapsto \Delta^* e^{|t|A^{-1}}d$. Now, the injectivity of the Fourier transform in the class of distributions of slow growth [37, p. 185] yields

$$\Delta^* e^{|t|A^{-1}}d = 0 \qquad \forall t\in\mathbb{R}$$

and consequently

$$\Delta^* e^{tA^{-1}}d = 0 \qquad \forall t\ge 0 \ ,$$

where the left-hand side is an analytic function of $t\ge 0$. Repeated differentiations at $t = 0+$ give $\Delta^* A^{-k}d$, $k = 0,1,2,3,\ldots$, i.e., the vector Δ is orthogonal to the subspace spanned by system $\{A^{-k}d\}_{k=0}^{\infty}$. Since, by assumption (**vi**) of Theorem 4.2 this system is complete we obtain $\Delta = 0$ and the second equation of the Lur'e system (3.2) is satisfied.

We show finally that $\mathcal{H}\ge 0$. Adding

$$-\frac{\mu_0}{1+\mu_0 c^{\#}d}hd^*\mathcal{H} - \frac{\mu_0}{1+\mu_0 c^{\#}d}\mathcal{H}dh^*$$

to both sides of the first equation of (3.2) and using the second equation of (3.2) we get

$$\begin{aligned}
&\mathcal{H}A_0 + A_0^*\mathcal{H} = \\
&= (q - \frac{2eq_1}{\sqrt{\delta}})hh^* - gg^* - q_1(hg^* + gh^*) = \\
&= -[g + q_1 h][g + q_1 h]^* + \Big[q - \frac{2eq_1}{\sqrt{\delta}} + q_1^2\Big]hh^*, \qquad q_1 := \frac{\sqrt{\delta}\mu_0}{1 + \mu_0 c^{\#} d} \ .
\end{aligned}$$

Hence, recalling the definitions of e, q and δ we obtain the system

(4.26)
$$\left\{ \begin{array}{rcl} \mathcal{H}A_0 + A_0^*\mathcal{H} & = & -[g + q_1 h][g + q_1 h]^* - q_0 hh^* \\ -\mathcal{H}d + eh & = & -\sqrt{\delta} g \end{array} \right\}, \quad q_0 := \frac{(k_2 - k_1)^2}{4(1 + \mu_0 c^{\#} d)^2} > 0 \ .$$

By Lemma 4.2: $(g + q_1 h)^*$, h^* are admissible with respect to the semigroup $\{e^{tA_0}\}_{t\geq 0}$ and therefore $\mathcal{H} \geq 0$. □

Our next criterion of solvability of the Lur'e system of equations (3.2) is partially based on the results of Oostveen and Curtain [27, Theorem 19 and Corollary 20]. See also Subsection 7.1.

Theorem 4.3. Let assumptions (**H1**)÷(**H6**) hold. Moreover assume that:

(H10) The operator $A : (D(A) \subset \mathrm{H}) \longrightarrow \mathrm{H}$ is such that the semigroup generated by A^{-1} is uniformly bounded;

(H11) In assumption (**H5**) condition (4.11) is strengthened to

$$\pi(\omega) \geq \varepsilon > 0 \qquad \forall \omega \in \mathbb{R} \ , \tag{4.27}$$

where π is given by (4.9).

(H12) The operator A_0 defined by (4.15) generates an **AS** semigroup on H.

Then the system (3.2) has a solution $(\mathcal{H}, g)$, $\mathcal{H} \in \mathbf{L}(\mathrm{H})$, $\mathcal{H} = \mathcal{H}^* \geq 0$, $g \in \mathrm{H}$ and g^*A is admissible with respect to the semigroup generated by A.

Proof. By (**H1**)÷(**H6**) all the results of preliminary subsection 4.2 are valid and in particular the conclusions of Lemma 4.2; moreover by (**H11**) and Lemma 4.1(**b**) the spectral factor ϕ of π is such that both ϕ and $1/\phi$ are in $\mathrm{H}^\infty(\Pi^+)$.

Now by (4.27) $\pi(0) = \delta > 0$. Hence system (3.2) has a solution $(\mathcal{H}, g)$, $\mathcal{H} \in \mathbf{L}(\mathrm{H})$, $\mathcal{H} = \mathcal{H}^* \geq 0$, $g \in \mathrm{H}$ and g^*A is admissible iff the operator equation

$$(A^{-1})^*\mathcal{H} + \mathcal{H}A^{-1} + \frac{1}{\delta}(-d^*\mathcal{H} + eh^*)^*(-d^*\mathcal{H} + eh^*) - qhh^* = 0 \tag{4.28}$$

has a solution $\mathcal{H} \in \mathbf{L}(\mathrm{H})$, $\mathcal{H} = \mathcal{H}^* \geq 0$, such that g^*A is admissible, where

$$g = \frac{1}{\sqrt{\delta}}(\mathcal{H}d - eh) \ . \tag{4.29}$$

Upon changing $\mathcal{H}$ into $-X$, (4.28) coincides with the *operator Riccati equation* discussed in Oostveen and Curtain [27, Formula (1) with A replaced by A^{-1} and $B = d$, $C = h^*$, $N = e$, $Q = q$, $R = \delta$], see also (7.7) in Subsection 7.1.4, where their results are shortly reported.

By assumption (**H1**) and (**H10**) and by Lemma 2.5 the semigroup $\{e^{tA^{-1}}\}_{t\geq 0}$ is **AS**.

The admissibility of $c^{\#}|_{D(A)} = h^*A$ with respect to the semigroup generated by A, guaranteed by (**H3**), is equivalent[3] to the admissibility of h^* with respect to the semigroup $\{e^{tA^{-1}}\}_{t\geq 0}$. Indeed, by the unitary operator $U \in \mathbf{L}(\mathrm{H}^2(\Pi^+))$ defined in (2.2), $\varphi(s) = h^*A(sI-A)^{-1}x_0$ is mapped into $(U\varphi)(s) = -h^*(sI-A^{-1})^{-1}x_0$.

Similarly, the transfer function defined in [27, Definition 13] (see also (7.5) in Subsection 7.1.4)

$$G(s) = C(sI-A)^{-1}B = h^*(sI-A^{-1})^{-1}d = -s^{-1}h^*A(s^{-1}I-A)^{-1}d \tag{4.30}$$

is tied with our transfer function $\hat{g}$ (see (2.4)) by

$$G(s^{-1}) = -\hat{g}(s) - c^{\#}d \tag{4.31}$$

and thus due to (**H2**) and (**H4**) we get $G \in \mathrm{H}^\infty(\Pi^+)$. Moreover the coercivity assumption imposed on the Popov function defined in [27, Formulae (19) and (21)] (see also (7.6) in Subsection 7.1.4) reduces here to

$$\delta + 2e\,\mathrm{Re}[h^*(j\omega I - A^{-1})^{-1}d] + q\left|h^*(j\omega I - A^{-1})^{-1}d\right|^2 \geq \varepsilon > 0 \qquad \forall \omega \in \mathbb{R}\setminus\{0\}\ .$$

By (4.30) and (4.31) the latter holds iff

$$\delta - 2e\,\mathrm{Re}[\hat{g}(j\omega) + c^{\#}d] + q\left|\hat{g}(j\omega) + c^{\#}d\right|^2 \geq \varepsilon > 0 \qquad \forall \omega \in \mathbb{R}\ ,$$

i.e., upon recalling the definitions of δ, e and q, iff

$$\pi(\omega) = 1 - (k_1+k_2)\,\mathrm{Re}\,\hat{g}(j\omega) + k_1k_2|\hat{g}(j\omega)|^2 \geq \varepsilon > 0 \qquad \forall \omega \in \mathbb{R}\ .$$

Now this holds by assumption (**H11**). Denote by $\Psi \in \mathbf{L}(\mathrm{H}, \mathrm{L}^2(0,\infty))$ the observability map associated with the pair $(\{e^{tA^{-1}}\}_{t\geq 0}, h^*)$ and by $\mathbb{F}$ – the input–output map dictated by the triple (A^{-1}, h^*, d). In the frequency–domain $\mathbb{F}$ is given by the transfer function defined in (4.30) and (4.31). Then by M.Weiss [35, Theorem 2.15] there holds that

$$\mathcal{H} := -q\Psi^*\Psi - \Psi^*(q\mathbb{F} + eI)\mathcal{R}^{-1}(q\mathbb{F}^* + eI)\Psi\ ,$$

where

$$\mathcal{R} \in \mathbf{L}(\mathrm{L}^2(0,\infty)), \qquad \mathcal{R} := \delta I + e\mathbb{F} + e\mathbb{F}^* + q\mathbb{F}^*\mathbb{F}\ ,$$

and (by (4.27)) $\mathcal{R}^{-1} \in \mathbf{L}(\mathrm{L}^2(0,\infty))$, is a self–adjoint bounded solution of the operator Riccati equation (4.28).

Consider now $g \in \mathrm{H}$ given by (4.29). It turns out that the spectral factor of π reads

$$\phi(s) = sg^*A(sI-A)^{-1}d + \sqrt{\delta}\ ,$$

i.e. g solves the realization equation (4.14). To see this premultiply and postmultiply equation (4.28) by respectively $d^*\left[-(j\omega)^{-1}I - (A^*)^{-1}\right]^{-1}$ and $\left[(j\omega)^{-1}I - A^{-1}\right]^{-1}$d and subtract δ from both sides. Tedious but straightforward manipulations deliver ultimately

$$\left|j\omega g^*A(j\omega I - A)^{-1}d + \sqrt{\delta}\right|^2 =$$
$$= \delta - 2e\,\mathrm{Re}\left[j\omega h^*A(j\omega I - A)^{-1}d\right] + q\left|j\omega h^*A(j\omega I - A)^{-1}d\right|^2 = \pi(\omega)\ .$$

Consider now the generator $A_0 \in \mathrm{L}(\mathrm{H})$ given by (4.15) and observe that by Lemma 4.2(**2**), the observation functional g^*A is admissible with respect to the semigroup generated by A iff the same holds for g^* with respect to the semigroup $\{e^{tA_0}\}_{t\geq 0}$. Hence we are done if the latter holds and $\mathcal{H} \geq 0$.

[3]Similarly, d is an admissible factor control vector with respect to the semigroup generated by A iff d is an admissible control vector with respect to the semigroup $\{e^{tA^{-1}}\}_{t\geq 0}$. However, here we do not assume admissibility of d.

Now by Lemma 4.2(**1**), h^* is admissible for the semigroup $\{e^{tA_0}\}_{t\geq 0}$, which by (**H12**) is **AS**. Thus by [13, Theorems 3 and 4] the operator Lyapunov equation

$$A_0^*X + XA_0 = -hh^* \ ,$$

has a unique solution $\mathcal{H}_h \in \mathrm{L(H)}$, $\mathcal{H}_h = \mathcal{H}_h^* \geq 0$. Observe now that the pair $(\mathcal{H}, g)$ solves (4.26). Hence the operator $\mathcal{H} \in \mathrm{L(H)}$, with $\mathcal{H} = \mathcal{H}^*$, is a solution of the Lyapunov equation

$$A_0^*X + XA_0 = -ww^* - q_0hh^*, \qquad w := g + q_1h \in \mathrm{H} \ .$$

Define now $\mathcal{H}_0 := \mathcal{H} - q_0\mathcal{H}_h$. Obviously $\mathcal{H}_0 \in \mathrm{L(H)}$ with $\mathcal{H}_0 = \mathcal{H}_0^*$, is a solution of the Lyapunov equation

$$A_0^*X + XA_0 = -ww^*, \qquad w := g + q_1h \in \mathrm{H} \ , \tag{4.32}$$

which turns out to be nonnegative. Indeed premultiplication and postmultiplication of (4.32) by respectivelely $x^*e^{tA_0^*}$ and $e^{tA_0}x$ gives for any fixed $x \in \mathrm{H}$

$$-\frac{d}{dt}\left[x^*e^{tA_0^*}\mathcal{H}_0e^{tA_0}x\right] = -x^*e^{tA_0^*}\left[A_0^*\mathcal{H}_0 + \mathcal{H}_0A_0\right]e^{tA_0}x = |w^*e^{tA_0}x|^2 \qquad \forall t \geq 0 \ .$$

Moreover integration of both sides from 0 to t yields

$$-x^*e^{tA_0^*}\mathcal{H}_0e^{tA_0}x + x^*\mathcal{H}_0x = \int_0^t |w^*e^{\tau A_0}x|^2 d\tau \qquad \forall t \geq 0 \ .$$

Now as e^{tA_0} is **AS**, the left-hand side above converges to $x^*\mathcal{H}_0x \in \mathbb{R}$ as $t \to \infty$, whence the right–hand side must do the same. Hence for any $x \in \mathrm{H}$

$$x^*\mathcal{H}_0x = \int_0^\infty |w^*e^{\tau A_0}x|^2 d\tau \geq 0 \ ,$$

where the right–hand side is an improper Riemann integral. Thus $\mathcal{H}_0 \geq 0$ and therefore the existence of the solution $\mathcal{H}_0$ of the Lyapunov equation (4.32) implies by [13, Theorem 3] that w^* (as given in (4.32)) is admissible for the semigroup $\{e^{tA_0}\}_{t\geq 0}$. As h^* had already this property, the same holds for g^*. Finally $\mathcal{H} = \mathcal{H}_0 + q_0\mathcal{H}_h$ is nonnegative as the sum of two such operators and we are done. □

5. Example 1: Distortionless loaded $RLCG$–transmission line

In this section we discuss an electrical transmission line as a plant in Figure 3.1 illustrating hereby the results of the previous sections.

The *distortionless* transmission line is a $RLCG$ line for which $\alpha := R/L = G/C$. Following [17, Subsection 5.1] consider such line loaded by a resistance R_0. By using the Hilbert space $\mathrm{H} = \mathrm{L}^2(-r, 0) \oplus \mathrm{L}^2(-r, 0)$ with $r = \sqrt{LC}$ equipped with the standard scalar product, one gets its dynamics desribed by an abstract model in factor form as in (2.1). More precisely:

- The state space operator A takes the form

  $$Ax = x', \qquad D(A) = \left\{x \in \mathrm{W}^{1,2}(-r, 0) \oplus \mathrm{W}^{1,2}(-r, 0) : \ x(0) = C_Sx(-r)\right\} \ ,$$

  where

  $$C_S = \begin{bmatrix} 0 & 1 \\ -b & 0 \end{bmatrix}, \qquad b = \frac{\kappa}{\rho^2}, \qquad \kappa = \frac{R_0 - z}{R_0 + z}, \qquad z = \sqrt{\frac{L}{C}}, \qquad \rho = e^{\alpha r} \ .$$

The operator A generates a C_0–semigroup $\{S(t)\}_{t\geq 0}$ on H (or even a C_0–group if $\det C_S \neq 0$). This semigroup is **EXS** iff $|\lambda(C_S)| < 1$ or equivalently $|b| < 1$ [12, pp. 148 - 154], which is the case. Thus assumption (**H1**) of Theorem 4.2 holds.

- The observation functional $c^{\#}$ is given by

$$c^{\#}x = c_0^T x(-r), \qquad D(c^{\#}) = \{x \in \mathrm{H} : \ c_0^T x \text{ is right-continuous at } -r\} \ , \tag{5.1}$$

where

$$c_0 = \begin{bmatrix} 0 \\ a \end{bmatrix}, \qquad a = \frac{1+\kappa}{\rho} \geq 0 \ .$$

It is representable on $D(A)$ as

$$c^{\#}\big|_{D(A)} = h^* A, \qquad h = \vartheta \begin{bmatrix} b\mathbf{1} \\ -\mathbf{1} \end{bmatrix} \in \mathrm{H}, \qquad \vartheta := \frac{a}{1+b} \ ,$$

where $\mathbf{1}$ denotes the constant function taking the value 1 on $[-r, 0]$. The admissibility of $c^{\#}$ was implicitly discussed in [14, p. 363]. The Lyapunov proof of this fact is presented in [17]. Thus assumption (**H3**) of Theorem 4.2 holds.

- The factor control vector is identified as

$$d = \frac{-1}{1+b} d_0, \qquad d_0 = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \in \mathrm{H} \ ,$$

where d is admissible [17]. It is also proved therein that the Lyapunov operator equation which proves the admissibility of d has a unique coercive solution. Thus the system is *exactly controllable*. This implies that the pair (A, d) is approximately controllable, which is equivalent to the cyclicity of d with respect to A^{-1}. Thus assumption (**H7**) of Theorem 4.2 holds.

The system dynamics can also be described by

$$\left\{ \begin{array}{rcl} w(t) & = & C_S w(t-r) + u(t) b_0 \\ y(t) & = & c_0^T w(t-r) \end{array} \right\}, \qquad b_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ . \tag{5.2}$$

The compatibility condition (2.3) holds with $c^{\#}d = -\vartheta$ and by (2.4) the transfer function reads

$$\hat{g}(s) = \frac{a e^{-sr}}{1 + b e^{-2sr}} \ . \tag{5.3}$$

This is confirmed by applying the Laplace transform directly to (5.2). Moreover,

$$\|\hat{g}\|_{\mathrm{H}^\infty(\Pi^+)} = \frac{a}{1-|b|} \ .$$

and thus (2.5) is satisfied. The situation is even better, namely we have that g is in the Callier–Desoer algebra $\mathcal{A}_-(0)$. All these results and many others can be found in [17]. In particular assumptions (**H2**) and (**H4**) of Theorem 4.2 hold.

The closed–loop semigroup generator corresponding to the linear feedback law $f(y) = \mu y$ takes the form

$$A_\mu x = x', \qquad D(A_\mu) = \left\{ x \in \mathrm{W}^{1,2}(-r,0) \oplus \mathrm{W}^{1,2}(-r,0) : \ x(0) = \left[C_S + \mu b_0 c_0^T\right] x(-r) \right\} \ .$$

Indeed, $D(A_\mu)$ consists of these x for which $x + \mu d c^{\#} x \in D(A)$. The latter holds if $x \in \mathrm{W}^{1,2}(-r,0) \oplus \mathrm{W}^{1,2}(-r,0)$ and $x(0) + \mu d c^{\#} x = C_S \left[x(-r) + \mu d c^{\#} x\right]$, or equivalently, if $x(0) = \left[C_S + \mu b_0 c_0^T\right] x(-r)$. The semigroup generated on $\mathrm{H} = \mathrm{L}^2(-r,0) \oplus \mathrm{L}^2(-r,0)$ by A_μ

is **EXS** iff all eigenvalues of the matrix $C_S + \mu b_0 c_0^T$ are in the open unit disk [12]. This is the case if

$$|\mu| < \frac{1+b}{a} \quad . \tag{5.4}$$

Stability condition (5.4) yields the *Hurwitz sector* which has to be compared with a sector (k_1, k_2) generated by the frequency–domain inequality (4.10). It is clear that by (5.4) the upper limit for k_2 is $\dfrac{1+b}{a}$ and the lower limit for k_1 is $-\dfrac{1+b}{a}$.

5.1. **Analysis of the case $b \leq 0$.** Substituting $k_2 = -k_1 = \dfrac{1+b}{a}$ into (4.9) gives

$$\pi(j\omega) = 1 - \left(\frac{1+b}{a}\right)^2 |\hat{g}(j\omega)|^2 = \frac{-4b\sin^2\omega r}{(1-b)^2 + 4b\cos^2\omega r} \geq 0 \qquad \forall \omega \in \mathbb{R}$$

and therefore the Hurwitz sector (5.4) agrees with the sector implied by (4.10). Now

$$\pi(j\omega) \geq \frac{-4b\sin^2\omega r}{(1-b)^2} \qquad \forall \omega \in \mathbb{R} \ ,$$

and by [11, 865.64]

$$\int_{-\infty}^{\infty} \frac{\ln \pi(j\omega)}{1+\omega^2} d\omega \geq \ln\left[\frac{-4b}{(1-b)^2}\right] \int_{-\infty}^{\infty} \frac{1}{1+\omega^2} d\omega + \int_{-\infty}^{\infty} \frac{\ln \sin^2\omega r}{1+\omega^2} d\omega > -\infty \ .$$

Hence condition (4.11) holds provided that $b \neq 0$, and by Proposition 4.1 there exists a spectral factor $\phi \in \widehat{\mathrm{H}_{-1}}$. Thus assumption (**H5**) of Theorem 4.2 is valid. We have even more, by Lemma 4.1(**a**) the spectral factor is in $\mathrm{H}^\infty(\Pi^+)$. It can be obtained from (4.5), however we shall use an elementary method to recover it. To do this rewrite π in the form

$$\pi(j\omega) = \frac{-2b(1-\cos 2\omega r)}{(1-b)^2 + 4b\cos^2\omega r} \ ,$$

and apply the identity

$$\begin{aligned}&(\alpha + \beta e^{-sr} + \gamma e^{-2sr})(\alpha + \beta e^{sr} + \gamma e^{2sr}) = \\ &= (\alpha^2+\beta^2+\gamma^2) + \beta(\alpha+\gamma)e^{-sr} + \beta(\alpha+\gamma)e^{sr} + \alpha\gamma e^{-2sr} + \alpha\gamma e^{2sr}\end{aligned} \tag{5.5}$$

to get the spectral factor only for the denominator. It requires solving the system of equations

$$\alpha^2+\beta^2+\gamma^2 = -2b, \qquad 2\beta(\alpha+\gamma) = 0, \qquad 2\alpha\gamma = 2b \ .$$

The solution for $\alpha = \sqrt{-b}$, $\beta = 0$ and $\gamma = -\sqrt{-b}$ leads to the spectral factorization

$$\pi(j\omega) = \frac{\sqrt{-b}(1-e^{-2j\omega r})}{1+be^{-2j\omega r}} \frac{\sqrt{-b}(1-e^{2j\omega r})}{1+be^{2j\omega r}} \ ,$$

with spectral factor

$$\phi(s) = \frac{\sqrt{-b}(1-e^{-2sr})}{1+be^{-2sr}} \ . \tag{5.6}$$

For this factor one has $\phi(0) = \sqrt{\delta} = 0$. If $b = 0$ then $\pi \equiv 0$ and the *trivial case* of Proposition 4.1 is met. It was not mentioned in Proposition 4.1 that in this case we trivially get the null spectral factor, which then coincides with (5.6).

To determine the vector $g \in \mathrm{H}$ we have to solve the realization problem (4.14). Here $\delta = 0$ because we have $k_2 = 1/\vartheta$ and $c^{\#}d = -\vartheta$. A solution will be sought in the form

$$g = \text{constant} \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} .$$

On substituting g and

$$(A(sI - A)^{-1}d)(\theta) = \frac{1}{1 + be^{-2sr}} \begin{bmatrix} e^{-sr+s\theta} \\ e^{s\theta} \end{bmatrix}, \qquad s \in \rho(A), \quad \theta \in [-r, 0] \tag{5.7}$$

into (4.14) one has

$$g = \sqrt{-b} \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} . \tag{5.8}$$

The solution is unique because the system is exactly controllable. Now using (5.8) we find

$$g^* Ax = \sqrt{-b} \left[\int_{-r}^{0} x_1'(\theta) d\theta + \int_{-r}^{0} x_2'(\theta) d\theta \right] = -\sqrt{-b}(1 + b)x_1(-r) \quad \forall x \in D(A) . \tag{5.9}$$

Assuming a solution of (3.7) of the form

$$(\mathcal{H}x)(\theta) = Hx(\theta), \qquad H \in \mathbf{L}(\mathbb{R}^2), \quad H = H^*$$

and taking (5.1) with $q = -k_2^2 < 0$, and (5.9) into account we can reduce (3.7) to a *discrete matrix Lyapunov equation*

$$C_S^T H C_S - H = -(1 + b)^2 \operatorname{diag}\{-b, 1\} .$$

Its solution is $H = (1 + b) \operatorname{diag}\{-b, 1\}$. Since $H > 0$, provided that $b \neq 0$, then by **AS** a unique solution $\mathcal{H}$ of (3.7) is coercive. For $b = 0$ we have only $\mathcal{H} \geq 0$.

Elementary calculations show that the second equation of the system (3.7) is also satisfied for $e = 1/\vartheta$.

Assumption (**H6**) of Theorem 4.2 clearly holds, because here $\mu_0 = (k_1 + k_2)/2 = 0$.

Note that we have solved the realization problem directly, i.e. without using assumptions (**H8**), (**H9**) of Theorem 4.2. Below, we confirm this solution using spectral analysis.

The eigenproblem for A takes the form

$$\left\{ \begin{array}{rcl} x'(\theta) & = & \lambda x(\theta) \\ x(0) & = & C_S x(-r) \end{array} \right\}$$

and therefore the general form of an eigenvector is $x(\theta) = e^{\lambda\theta}x(0)$, where $x(0)$ is a nonzero solution of the linear homogeneous equation $[I - \xi C_S]x(0) = 0$ with $\xi := e^{-\lambda r}$. Such a solution exists iff $\det(I - \xi C_S) = 0$, or equivalently, iff ξ is a root of the *characteristic polynomial* $1 + \xi^2 b$.

For $b = 0$ there is no eigenvalue, and the resolvent of A is not compact.

For $b < 0$ there are two series of eigenvalues, corresponding to two roots $\xi^{\pm} = \dfrac{\pm 1}{\sqrt{-b}} = e^{-\lambda_n^{\pm} r}$ of the characteristic polynomial, where

$$\lambda_n^+ = \frac{\ln \sqrt{-b}}{r} + j\frac{2n\pi}{r}, \qquad \lambda_n^- = \frac{\ln \sqrt{-b}}{r} + j\frac{2n\pi + \pi}{r}, \qquad n \in \mathbb{Z} .$$

The corresponding eigenvectors $\{x_n^+\}_{n\in\mathbb{Z}}$, $\{x_n^-\}_{n\in\mathbb{Z}}$ can be written as

$$\left[\begin{array}{cc} x_n^+ & x_n^- \end{array}\right] = e^{\dfrac{\ln\sqrt{-b}}{r}\theta}\left[\begin{array}{cc} x^+(0) & x^-(0) \end{array}\right]\left[\begin{array}{cc} y_n^+ & 0 \\ 0 & y_n^- \end{array}\right]$$

where $\{y_n^+\}_{n\in\mathbb{Z}}$, $y_n^+(\theta) := \dfrac{1}{\sqrt{r}}e^{j\dfrac{2n\pi}{r}\theta}$ is the classical Fourier orthonormal basis of eigenvectors of a skew–adjoint operator B^+ acting in $\mathrm{L}^2(-r,0)$,

$$B^+y = y', \qquad D(B^+) = \{y \in \mathrm{W}^{1,2}(-r,0):\ y(0) = y(-r)\}\ ,$$

while $\{y_n^-\}_{n\in\mathbb{Z}}$, $y_n^-(\theta) := \dfrac{1}{\sqrt{r}}e^{j\dfrac{2n\pi+\pi}{r}\theta}$ stands for the classical Fourier orthonormal basis of eigenvectors of a skew–adjoint operator B^- acting in $\mathrm{L}^2(-r,0)$,

$$B^-y = y', \qquad D(B^-) = \{y \in \mathrm{W}^{1,2}(-r,0):\ y(0) = -y(-r)\}\ .$$

Finally, $x^\pm(0)$ is a solution of the equation $[I - \xi^\pm C_S]x(0) = 0$. In particular, for

$$x^+(0) = \left[\begin{array}{c} 1 \\ \sqrt{-b} \end{array}\right], \qquad x^-(0) = \frac{1}{2}\left[\begin{array}{c} 1 \\ -\sqrt{-b} \end{array}\right]$$

the system $\{x_n^+\}_{n\in\mathbb{Z}}$, $\{x_n^-\}_{n\in\mathbb{Z}}$ is the image of the system $\left\{\left[\begin{array}{c} y_n^+ \\ 0 \end{array}\right]\right\}_{n\in\mathbb{Z}}$, $\left\{\left[\begin{array}{c} 0 \\ y_n^- \end{array}\right]\right\}_{n\in\mathbb{Z}}$, being an orthonormal basis in $\mathrm{H} = \mathrm{L}^2(-r,0) \oplus \mathrm{L}^2(-r,0)$, under a bounded linear isomorphism. This implies that the system $\{x_n^+\}_{n\in\mathbb{Z}}$, $\{x_n^-\}_{n\in\mathbb{Z}}$,

$$x_n^+(\theta) = e^{\lambda_n^+\theta}\left[\begin{array}{c} \dfrac{1}{\sqrt{r}} \\ \dfrac{\sqrt{-b}}{\sqrt{r}} \end{array}\right], \qquad x_n^-(\theta) = e^{\lambda_n^-\theta}\left[\begin{array}{c} \dfrac{1}{2\sqrt{r}} \\ \dfrac{-\sqrt{-b}}{2\sqrt{r}} \end{array}\right] \tag{5.10}$$

of eigenvectors of A forms a Riesz basis in H and assumption (**H8**) of Theorem 4.2 is met. The biorthogonal Riesz basis of eigenvectors of the adjoint operator A^*,

$$A^*p = -p', \qquad D(A^*) = \{p \in \mathrm{W}^{1,2}(-r,0) \oplus \mathrm{W}^{1,2}(-r,0):\ p(-r) = C_S^T p(0)\}\ ,$$

corresponding to the conjugate eigenvalues $\overline{\lambda_n^+} = \lambda_{-n}^+$, $\overline{\lambda_n^-} = \lambda_{-n-1}^-$ is

$$p_n^+(\theta) = e^{-\overline{\lambda_n^+}\theta}\left[\begin{array}{c} \dfrac{1}{2\sqrt{r}} \\ \dfrac{1}{2\sqrt{-br}} \end{array}\right], \qquad p_n^-(\theta) = e^{-\overline{\lambda_n^-}\theta}\left[\begin{array}{c} \dfrac{1}{\sqrt{r}} \\ \dfrac{-1}{\sqrt{-br}} \end{array}\right]\ .$$

Now we have

$$\alpha_n^\pm := \operatorname*{Res}_{s=\lambda_n^\pm}\left[\frac{\phi(s) - \sqrt{\delta}}{s}\right] = \operatorname*{Res}_{s=\lambda_n^\pm}\left[\frac{\sqrt{-b}(1 - e^{-2sr})}{s(1 + be^{-2sr})}\right] = \frac{\sqrt{-b}(1+b)}{2rb\lambda_n^\pm}\ ,$$

and

$$\langle d, p_n^+\rangle_{\mathrm{H}} = \frac{1}{2\lambda_n^+\sqrt{-br}}, \qquad \langle d, p_n^-\rangle_{\mathrm{H}} = \frac{-1}{\lambda_n^-\sqrt{-br}}\ .$$

Since $\langle d, p_n^+\rangle_{\mathrm{H}} \neq 0$ and $\langle d, p_n^-\rangle_{\mathrm{H}} \neq 0$ the system is approximately controllable, which confirms that assumption (**H7**) of Theorem 4.2 holds (moreover we know that the system is exactly controllable). Therefore

$$g^* x_n^+ = \frac{\alpha_n^+}{\lambda_n^+ \langle d, p_n^+\rangle_{\mathrm{H}}} = -\frac{1+b}{\sqrt{r}\lambda_n^+}, \qquad g^* x_n^- = \frac{\alpha_n^-}{\lambda_n^- \langle d, p_n^-\rangle_{\mathrm{H}}} = \frac{1+b}{2\sqrt{r}\lambda_n^-}$$

with $\{g^* x_n^+\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, $\{g^* x_n^-\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, whence the realizability problem has a solution $g \in \mathrm{H}$ such that

$$g^* A x_n^+ = -\frac{1+b}{\sqrt{r}}, \qquad g^* A x_n^- = \frac{1+b}{2\sqrt{r}} \ .$$

The latter can be confirmed by inserting (5.10) into (5.9).

Since $g^* A x_n^{\pm}$ and $\operatorname{Re}\lambda_n^{\pm}$ do not depend on $n \in \mathbb{Z}$, the spectral admissibility condition holds iff

$$\sum_{n=-\infty}^{\infty} \left[\left| \operatorname{Re}\lambda_n^+ \hat{f}(-\overline{\lambda_n^+}) \right|^2 + \left| \operatorname{Re}\lambda_n^- \hat{f}(-\overline{\lambda_n^-}) \right|^2 \right] < \infty \qquad \forall \hat{f} \in \mathrm{H}^2(\Pi^+) \ ,$$

i.e., when the sequences $\{\lambda_n^{\pm}\}_{n\in\mathbb{Z}}$ are of the Carleson–Newman type, which is the case as they are uniformly separated and located on a vertical line parallel to the imaginary axis. Thus assumption (**H9**) of Theorem 4.2 holds.

5.2. **Analysis of the case $b > 0$.** Here for $k_1 = -\dfrac{1+b}{a}$ we cannot take $k_2 = \dfrac{1+b}{a}$. Thus the Hurwitz sector (5.4) is essentially larger than the sector implied by (4.10) and another choice of k_1, k_2 has to be proposed. Assuming $k_1 = -\dfrac{1+b}{a}$ we search for the maximal allowed value of k_2 for which (4.10) holds. Since

$$\pi(j\omega) = 1 - (k_1 + k_2)\operatorname{Re}[\hat{g}(j\omega)] + k_1 k_2 \left|\hat{g}(j\omega)\right|^2 =$$
$$= \frac{(1+b)^2\cos^2\omega r + (1-b)^2\sin^2\omega r + (1+b)^2\cos\omega r - k_2 a(1+b)\cos\omega r - k_2 a(1+b)}{(1-b)^2 + 4b\cos^2\omega r}$$

then treating the numerator as a polynomial of $\cos\omega r$ we find the maximal allowed value of k_2 for which the frequency domain inequality (4.10) holds, viz.

$$k_2 = \frac{1+b}{a} - \frac{8b}{a(1+b)} \ . \tag{5.11}$$

Then

$$\pi(j\omega) = \frac{4b(1+\cos\omega r)^2}{(1-b)^2 + 4b\cos^2\omega r} = \frac{6b + 8b\cos\omega r + 2b\cos 2\omega r}{(1-b)^2 + 4b\cos^2\omega r} \geq 0 \ .$$

Observe that now

$$\pi(j\omega) \geq \frac{16b\cos^4\dfrac{\omega r}{2}}{(1+b)^2} \qquad \forall \omega \in \mathbb{R}$$

and from [11, 865.65] we know that

$$\int_{-\infty}^{\infty} \frac{\ln\pi(j\omega)}{1+\omega^2} d\omega \geq \ln\left[\frac{16b}{(1+b)^2}\right] \int_{-\infty}^{\infty} \frac{1}{1+\omega^2} d\omega + 2\int_{-\infty}^{\infty} \frac{\ln\cos^2\dfrac{\omega r}{2}}{1+\omega^2} d\omega > -\infty \ .$$

Condition (4.11) is satisfied, whence assumption (**H5**) of Theorem 4.2 holds. By Proposition 4.1 there exists a spectral factor $\phi \in \widehat{\mathrm{H}_{-1}}$. Actually by Lemma 4.1 it belongs to

$\mathrm{H}^\infty(\Pi^+)$. As in the previous case formula (4.5) will not be used to determine ϕ, but an elementary method based on identity (5.5) will give its denominator. Simple calculations yield the spectral factor (here $\alpha = \sqrt{b}$, $\beta = 2\sqrt{b}$ and $\gamma = \sqrt{b}$)

$$\phi(s) = \frac{\sqrt{b}(1+e^{-sr})^2}{1+be^{-2sr}} \ . \tag{5.12}$$

For this spectral factor there holds $\phi(0) = \sqrt{\delta} = 4\sqrt{b}/(1+b)$.

To determine the vector $g \in \mathrm{H}$ we apply (4.14), which leads to the identity

$$\frac{\phi(s)-\sqrt{\delta}}{s} = \frac{\sqrt{b}}{1+b}\frac{b-3+(2+2b)e^{-sr}+(1-3b)e^{-2sr}}{s(1+be^{-2sr})} = g^*A(sI-A)^{-1}d \ , \tag{5.13}$$

valid for all $s \in \Pi^+$. A solution of (5.13) will be sought in the form

$$g = \begin{bmatrix} g_1\mathbf{1} \\ g_2\mathbf{1} \end{bmatrix}$$

where g_1 and g_2 are constants. On substituting g and (5.7) into (5.13) one obtains

$$g = \begin{bmatrix} g_1\mathbf{1} \\ g_2\mathbf{1} \end{bmatrix}, \qquad g_1 = \frac{\sqrt{b}(3b-1)}{1+b}, \quad g_2 = \frac{\sqrt{b}(b-3)}{1+b} \ . \tag{5.14}$$

The solution is unique by exact controllability. Using (5.14) we find with $x \in D(A)$

$$g^*Ax = g_1\int_{-r}^{0} x_1'(\theta)d\theta + g_2\int_{-r}^{0} x_2'(\theta)d\theta = \sqrt{b}(1-b)x_1(-r) + 2\sqrt{b}x_2(-r) \ . \tag{5.15}$$

Assuming a solution of (3.7) of the form

$$(\mathcal{H}x)(\theta) = Hx(\theta), \qquad H \in \mathbf{L}(\mathbb{R}^2), \quad H = H^*$$

and taking (5.1) with $q = (-b^2+6b-1)/a^2$ and (5.15) into account, we can reduce (3.7) to a *discrete matrix Lyapunov equation*

$$C_S^T H C_S - H = -R, \qquad R = \begin{bmatrix} b(1-b)^2 & 2b(1-b) \\ 2b(1-b) & (1-b)^2 \end{bmatrix} \ . \tag{5.16}$$

Here to calculate the matrix R we used the identity

$$(g^*Ax)^2 - q(h^*Ax)^2 = x^T(-r)Rx(-r) \qquad \forall x \in D(A) \ .$$

The solution of (5.16) is

$$H = \begin{bmatrix} b(1-b) & \dfrac{2b(1-b)}{1+b} \\ \dfrac{2b(1-b)}{1+b} & 1-b \end{bmatrix} > 0 \ .$$

Thus (by **AS**) we get a unique solution $\mathcal{H}$ of (3.7) which is coercive. Observe that $R > 0$ iff $q < 0$, and the latter holds only for $b \in (0, 3-2\sqrt{2})$.

Tedious but elementary calculations show that the second equation of the system (3.7) is valid for $e = (1+b)/a - 12b/[a(1+b)]$.

Since[4]

$$\left\|\frac{1}{1-\mu_0\hat{g}}\right\|_{\mathrm{H}^\infty(\Pi^+)} = \left(\frac{1+b}{1-b}\right)^2 ,$$

we get that assumption (**H6**) of Theorem 4.2 holds. Indeed, for $b>0$ we have

$$\mu_0 = \frac{k_1+k_2}{2} = \frac{1}{2}\left[\frac{1+b}{a} - \frac{8b}{a(1+b)} - \frac{1+b}{a}\right] = -\frac{4b}{a(1+b)} < 0$$

and therefore, by (5.3) we get

$$\frac{1}{1-\mu_0\hat{g}(s)} = \frac{(1+b)(1+be^{-2sr})}{(be^{-sr}+1)^2+b(1+e^{-sr})^2} \ .$$

A lower bound for the modulus of the denominator is obtained using

$$\begin{aligned}&\left|(1+be^{-sr})^2+b(1+e^{-sr})^2\right| \geq \left|1+be^{-sr}\right|^2 - b\left|1+e^{-sr}\right|^2 = \\ &= (1+be^{-\sigma r}\cos\omega r)^2 + e^{-2\sigma r}b^2\sin^2\omega r - b\left[(1+e^{-\sigma r}\cos\omega r)^2+e^{-2\sigma r}\sin^2\omega r\right] = \\ &= (1-b)(1-be^{-2\sigma r}) \geq (1-b)^2, \qquad s=\sigma+j\omega, \quad \sigma\geq 0 \ ,\end{aligned}$$

with inequality at $s = j\dfrac{\pi}{r}$. This means that

$$\left|\frac{1}{1-\mu_0\hat{g}(s)}\right| \leq \left(\frac{1+b}{1-b}\right)^2$$

and the upper bound is achieved at $s = j\dfrac{\pi}{r}$.

As in the previous case the realization problem is solved directly without using assumptions (**H8**), (**H9**) of Theorem 4.2. Below we confirm this solution by using spectral analysis.

For $b>0$ there are also two series of eigenvalues, corresponding to two roots $\xi^\pm = \dfrac{\pm j}{\sqrt{b}} = e^{-\lambda_n^\pm r}$ of the characteristic polynomial,

$$\lambda_n^+ = \frac{\ln\sqrt{b}}{r} + j\frac{2n\pi - \frac{\pi}{2}}{r}, \qquad \lambda_n^- = \frac{\ln\sqrt{b}}{r} + j\frac{2n\pi + \frac{\pi}{2}}{r}, \qquad n\in\mathbb{Z} \ .$$

The corresponding eigenvectors $\{x_n^+\}_{n\in\mathbb{Z}}$, $\{x_n^-\}_{n\in\mathbb{Z}}$ can be written as

$$\begin{bmatrix} x_n^+ & x_n^- \end{bmatrix} = e^{\frac{\ln\sqrt{b}}{r}\theta}\begin{bmatrix} x^+(0) & x^-(0)\end{bmatrix}\begin{bmatrix} y_n^+ & 0 \\ 0 & y_n^-\end{bmatrix}$$

where $\{y_n^+\}_{n\in\mathbb{Z}}$, $y_n^+(\theta) := \dfrac{1}{\sqrt{r}}e^{j\frac{2n\pi-\frac{\pi}{2}}{r}\theta}$ is the classical Fourier orthonormal basis of eigenvectors of a skew–adjoint operator B^+ acting in $\mathrm{L}^2(-r,0)$,

$$B^+y = y', \qquad D(B^+) = \{y\in \mathrm{W}^{1,2}(-r,0) : \ y(0) = -jy(-r)\} \ ,$$

[4]This formula remains valid for $b<0$.

while $\{y_n^-\}_{n\in\mathbb{Z}}$, $y_n^-(\theta) := \dfrac{1}{\sqrt{r}} e^{j\frac{2n\pi + \frac{\pi}{2}}{r}\theta}$ stands for the classical Fourier orthonormal basis of eigenvectors of a skew–adjoint operator B^- acting in $\mathrm{L}^2(-r,0)$,

$$B^- y = y', \qquad D(B^-) = \{y \in \mathrm{W}^{1,2}(-r,0) : \; y(0) = jy(-r)\} \; .$$

Finally, $x^{\pm}(0)$ is a solution of the equation $[I - \xi^{\pm} C_S]x(0) = 0$. In particular, for

$$x^+(0) = \begin{bmatrix} 1 \\ -j\sqrt{b} \end{bmatrix}, \qquad x^-(0) = \begin{bmatrix} 1 \\ j\sqrt{b} \end{bmatrix}$$

the system $\{x_n^+\}_{n\in\mathbb{Z}}$, $\{x_n^-\}_{n\in\mathbb{Z}}$ is the image of the system $\left\{\begin{bmatrix} y_n^+ \\ 0 \end{bmatrix}\right\}_{n\in\mathbb{Z}}$, $\left\{\begin{bmatrix} 0 \\ y_n^- \end{bmatrix}\right\}_{n\in\mathbb{Z}}$, being an orthonormal basis in $\mathrm{H} = \mathrm{L}^2(-r,0) \oplus \mathrm{L}^2(-r,0)$, under a bounded linear isomorphism. This implies that the system $\{x_n^+\}_{n\in\mathbb{Z}}$, $\{x_n^-\}_{n\in\mathbb{Z}}$,

$$x_n^+(\theta) = e^{\lambda_n^+\theta} \begin{bmatrix} \dfrac{1}{\sqrt{r}} \\ \dfrac{-j\sqrt{b}}{\sqrt{r}} \end{bmatrix}, \qquad x_n^-(\theta) = e^{\lambda_n^-\theta} \begin{bmatrix} \dfrac{1}{\sqrt{r}} \\ \dfrac{j\sqrt{b}}{\sqrt{r}} \end{bmatrix} \tag{5.17}$$

of eigenvectors of A forms a Riesz basis in H which shows that assumption (**vii**) of Theorem 4.2 holds. The biorthogonal Riesz basis of eigenvectors of the adjoint operator A^*, corresponding to the conjugate eigenvalues $\overline{\lambda_n^+} = \lambda_{-n}^-$, $\overline{\lambda_n^-} = \lambda_{-n}^+$ is given by

$$p_n^+(\theta) = e^{-\overline{\lambda_n^+}\theta} \begin{bmatrix} \dfrac{1}{2\sqrt{r}} \\ \dfrac{-j}{2\sqrt{br}} \end{bmatrix}, \qquad p_n^-(\theta) = e^{-\overline{\lambda_n^-}\theta} \begin{bmatrix} \dfrac{1}{2\sqrt{r}} \\ \dfrac{j}{2\sqrt{br}} \end{bmatrix} \; .$$

Now we have

$$\begin{aligned} \alpha_n^{\pm} := \; & \operatorname*{Res}_{s=\lambda_n^{\pm}} \left[\frac{\phi(s) - \sqrt{\delta}}{s}\right] = \frac{\sqrt{b}}{1+b} \operatorname*{Res}_{s=\lambda_n^{\pm}} \left[\frac{b - 3 + (2+2b)e^{-sr} + (1-3b)e^{-2sr}}{s(1+be^{-2sr})}\right] = \\ = \; & \frac{b - 1 \pm 2j\sqrt{b}}{2r\sqrt{b}\lambda_n^{\pm}} \; , \end{aligned}$$

and

$$\langle d, p_n^+\rangle_{\mathrm{H}} = \frac{j}{2\lambda_n^+\sqrt{br}}, \qquad \langle d, p_n^-\rangle_{\mathrm{H}} = \frac{-j}{2\lambda_n^-\sqrt{br}} \; .$$

Since $\langle d, p_n^+\rangle_{\mathrm{H}} \neq 0$ and $\langle d, p_n^-\rangle_{\mathrm{H}} \neq 0$ the system is approximately controllable, which confirms that assumption (**H7**) of Theorem 4.2 holds (moreover we know that the system is even exactly controllable). Therefore

$$g^* x_n^+ = \frac{\alpha_n^+}{\lambda_n^+ \langle d, p_n^+\rangle_{\mathrm{H}}} = \frac{(1-b)j + 2\sqrt{b}}{\sqrt{r}\lambda_n^+}, \qquad g^* x_n^- = \frac{\alpha_n^-}{\lambda_n^- \langle d, p_n^-\rangle_{\mathrm{H}}} = \frac{-(1-b)j + 2\sqrt{b}}{\sqrt{r}\lambda_n^-}$$

with $\{g^*x_n^+\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, $\{g^*x_n^-\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, whence the realizability problem has a solution $g \in \mathrm{H}$ such that

$$g^*Ax_n^+ = \frac{(1-b)j + 2\sqrt{b}}{\sqrt{r}}, \qquad g^*Ax_n^- = \frac{-(1-b)j + 2\sqrt{b}}{\sqrt{r}} \ .$$

The latter can be confirmed by inserting (5.17) into (5.15).

Since $g^*Ax_n^\pm$ and $\operatorname{Re}\lambda_n^\pm$ do not depend on $n \in \mathbb{Z}$, the spectral admissibility condition is satisfied iff

$$\sum_{n=-\infty}^{\infty} \left[\left|\operatorname{Re}\lambda_n^+ \hat{f}(-\overline{\lambda_n^+})\right|^2 + \left|\operatorname{Re}\lambda_n^- \hat{f}(-\overline{\lambda_n^-})\right|^2 \right] < \infty \qquad \forall \hat{f} \in \mathrm{H}^2(\Pi^+) \ ,$$

i.e., when the sequences $\{\lambda_n^\pm\}_{n\in\mathbb{Z}}$ are of the Carleson–Newman type, which is the case as they are uniformly separated and located on a vertical line parallel to the imaginary axis. Thus assumption (**H9**) of Theorem 4.2 holds.

By Theorem 3.1 the origin of the state space H is globally weakly asymptotically stable for all $f \in \mathcal{M} \cap \mathcal{S}$ with $k_1 = -(1+b)/a$ and $k_2 = -k_1$ if $b < 0$ and k_2 given by (5.11) if $b > 0$. This result is "ε–better" than that of [18], which means that now nonlinearities asymptotically close to the boundary of the sector $\mathcal{S}$ are allowed. This is contrary to [18] where the boundaries where cut away by taking straight lines with slopes $k_1+\varepsilon$ and $k_2-\varepsilon$, respectively. In [18] we have used the *input–output* approach. Moreover by Theorem 3.2 for the cases considered the origin of H is globally strongly asymptotically stable for all $f \in \mathcal{M} \cap \mathcal{S}_\epsilon$. This result agrees with that of [18].

For $b = 0$ the origin is strongly asymptotically stable for all $f \in \mathcal{M} \cap \mathcal{S}_\varepsilon$ with $k_1 = -(1+b)/a = -k_2$. This result agrees with that of [18].

6. Example 2: Unloaded RC–transmission line

Following [17, Subsection 5.2], the Hilbert space $\mathrm{H} = \mathrm{L}^2(0,1)$ with standard scalar product is used to model the dynamics of an unloaded RC transmission line according to (2.1) with:

- The state–space operator

  $$Ax = x'', \qquad D(A) = \{x \in \mathrm{H}^2(0,1) : \ x'(1) = 0, \ x(0) = 0\}$$

  which generates an **EXS** analytic self–adjoint semigroup on H. This is due to $A = A^* < 0$. Moreover, A has a system of eigenvectors $\{e_n\}_{n=0}^\infty$ (corresponding to its eigenvalues $\{\lambda_n\}_{n=0}^\infty$) that is an orthonormal basis of H (see [15, Formula (21)] or [16, Lemma 3.1 with K=0]),

  $$\left\{ \begin{array}{rcll} e_n(\theta) & = & \sqrt{2}\sin\left(\dfrac{\pi}{2} + n\pi\right)\theta \ , \qquad 0 \le \theta \le 1, & n \ge 0 \\ \lambda_n & = & -\left(\dfrac{\pi}{2} + n\pi\right)^2, & n \ge 0 \end{array} \right\} .$$

  Thus assumptions (**H1**) and (**H8**) of Theorem 4.2 are satisfied.
- The observation functional

  $$c^\# x = x(1), \qquad D(c^\#) = \{x \in \mathrm{L}^2(0,1) : \ x \text{ is left-continuous at } 1\} \supset \mathrm{C}[0,1] \ ,$$

  whose restriction to $D(A)$ reads as $c^\#\big|_{D(A)} = h^*A$ with $h(\theta) = -\theta$, $0 \le \theta \le 1$. It was proved in [14] that $c^\#$ is admissible and therefore assumption (**H3**) of Theorem 4.2 holds.

- The factor control vector d is given by
$$d = -\mathbf{1} \in \mathrm{L}^2(0,1), \qquad \mathbf{1}(\theta) = 1, \quad 0 \le \theta \le 1 \ ,$$
which is not admissible. For a proof see [16, Subsection 3.3] or for a shorter one [17, Appendix B]. Since
$$\langle d, e_n \rangle_{\mathrm{H}} = -\frac{\sqrt{2}}{\dfrac{\pi}{2} + n\pi} \neq 0 \qquad \forall n \in \mathbb{N} \cup \{0\} \ ,$$
the system is approximately controllable, i.e. assumption (**H7**) of Theorem 4.2 holds.

It is easy to see that (2.3) holds with $c^{\#}d = -1$ and that by (2.4) the transfer function reads
$$\hat{g}(s) = \frac{1}{\cosh\sqrt{s}}, \qquad s \in \Pi^+ \ .$$
Moreover one has

$$\|\hat{g}\|_{\mathrm{H}^\infty(\Pi^+)} = 1 \ , \tag{6.1}$$

where the norm is attained at $s = 0$. For a more exhaustive discussion of these facts and many others see again [17]. In particular assumptions (**H2**) and (**H4**) of Theorem 4.2 hold.

It follows from (6.1) that (4.9) holds for $k_2 = -k_1 = 1$. More precisely one gets

$$\pi(j\omega) = 1 - \frac{1}{\left|\cosh\sqrt{j\omega}\right|^2} \ge \frac{\omega^2}{\omega^2 + 6} \ge 0 \qquad \forall \omega \in \mathbb{R} \ . \tag{6.2}$$

Indeed,
$$\frac{1}{\cosh\sqrt{j\omega}} = \left\{ \begin{array}{ll} \dfrac{1}{\cosh[(1+j)\Omega]}, & \omega \ge 0 \\ \dfrac{1}{\cosh[(1-j)\Omega]}, & \omega \le 0 \end{array} \right\}, \qquad \Omega := \sqrt{\frac{|\omega|}{2}} \ge 0$$
and
$$\begin{aligned} |\cosh[(1 \pm j)\Omega]|^2 = & \ |\cosh\Omega\cos\Omega \pm j\sinh\Omega\sin\Omega|^2 = \cosh^2\Omega\cos^2\Omega + \sinh^2\Omega\sin^2\Omega = \\ = & \ (1 + \sinh^2\Omega)\cos^2\Omega + \sinh^2\Omega\sin^2\Omega = \sinh^2\Omega + 1 - \sin^2\Omega \ . \end{aligned}$$
Hence
$$\pi(j\omega) = 1 - \frac{1}{\left|\cosh\sqrt{j\omega}\right|^2} = \frac{\sinh^2\Omega - \sin^2\Omega}{1 + \sinh^2\Omega - \sin^2\Omega} \ .$$
Now the Maclaurin expansions of sinh and sin give[5]:
$$\sinh\Omega + \sin\Omega \ge 2\Omega, \qquad \sinh\Omega - \sin\Omega \ge \frac{1}{3}\Omega^3 \qquad \forall \Omega \ge 0 \ ,$$
leading to
$$\sinh^2\Omega - \sin^2\Omega \ge \frac{2}{3}\Omega^4 \qquad \forall \Omega \ge 0 \ .$$
This jointly with the monotonicity of the function $x \longmapsto \dfrac{x}{1+x}$ on $x \ge 0$ yields
$$\pi(\omega) \ge \frac{2\Omega^4}{3 + 2\Omega^4} = \frac{\omega^2}{\omega^2 + 6} \ge 0 \qquad \forall \omega \in \mathbb{R} \ .$$

[5]By taking higher order terms one can improve the lower bounds arbitrarily.

The proof of (6.2) is now complete. Now by (6.2)

$$\int_{-\infty}^{\infty} \frac{\ln \pi(\omega)}{1+\omega^2} d\omega \geq 4 \int_0^{\infty} \frac{\ln \omega}{1+\omega^2} d\omega - 2\int_0^{\infty} \frac{\ln(\omega^2+6)}{1+\omega^2} d\omega = -2\pi \ln(1+\sqrt{6}) > -\infty \ .$$

Note that by Proposition 4.1 the second integral can be evaluated[6] by calculating $\phi(1)$ from the elementary rational spectral factorization problem

$$|\phi(j\omega)|^2 = \frac{\omega^2}{\omega^2+6} \ ,$$

the solution of which is

$$\phi(s) = \frac{s}{s+\sqrt{6}} \ .$$

Since assumption (**H6**) of Theorem 4.2 holds trivially, we shall have verified all the assumptions of Theorem 4.2 if (**H9**) holds.

The spectral factorization problem reads here as

$$\phi(s)\phi(-s) = 1 - \hat{g}(s)\hat{g}(-s) = 1 - \frac{1}{\cosh\sqrt{s}} \frac{1}{\cos\sqrt{s}} = \frac{\cosh\sqrt{s}\cos\sqrt{s} - 1}{\cosh\sqrt{s}} \frac{1}{\cos\sqrt{s}} \ .$$

where $\cosh\sqrt{s}$ and $\cos\sqrt{s}$ are entire functions of *exponential order* $\frac{1}{2}$ and of *exponential type* 0 [36, p. 63, p. 71] having the product representations

$$\cosh\sqrt{s} = \prod_{k=0}^{\infty} \left(1 - \frac{s}{\lambda_k}\right), \qquad \cos\sqrt{s} = \prod_{k=0}^{\infty} \left(1 + \frac{s}{\lambda_k}\right) \ .$$

Consider now the even entire function $\chi(s) := \cosh\sqrt{s}\cos\sqrt{s} - 1$. It is of exponential order $\frac{1}{2}$ and of exponential type 0 (with maximum modulus on $j\mathbb{R}$). Observe that

$$\chi(j\omega) = \cosh\sqrt{j\omega}\cos\sqrt{j\omega} - 1 = \sinh^2\Omega - \sin^2\Omega \geq \frac{2}{3}\Omega^4 = \frac{1}{6}\omega^2 \geq 0$$

and therefore

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\ln[\chi(j\omega)]}{1+\omega^2} d\omega \ &\geq \ \int_{-\infty}^{\infty} \frac{1}{1+\omega^2} \ln\left(\frac{\omega^2}{6}\right) d\omega = \\ &= \ 4\int_0^{\infty} \frac{\ln\omega}{1+\omega^2} d\omega - 2\ln 6 \int_0^{\infty} \frac{d\omega}{1+\omega^2} = -\pi \ln 6 > -\infty \ . \end{aligned}$$

As a consequence $\chi(s)$ is nonnegative on $j\mathbb{R}$ and an entire function of exponential type 0 of *class A* [21, p. 223]. Hence by Akhiezer's spectral factorization theorem for entire functions of exponential type [4, Theorem 7.5.1, p. 125], [21, Theorem 1, p. 437], [32, Theorem 3.6, p. 315] there exists an entire function $\vartheta(s)$ of *exponential type* 0 such that

$$\chi(s) = \cosh\sqrt{s}\cos\sqrt{s} - 1 = \vartheta(s)\vartheta(-s) \ . \tag{6.3}$$

It turns out that $\vartheta(s)$ has real zeros ζ_k for $k = 0, 1, 2, \ldots$ with $\zeta_0 = 0$, in the closed left complex half–plane. Indeed the ζ_k are roots of the equation

$$\cosh\sqrt{s} = \frac{1}{\cos\sqrt{s}} \ ,$$

which upon setting $s = -x^2$, $x \geq 0$, reads

$$\cos x = \frac{1}{\cosh x}, \qquad x \geq 0 \ . \tag{6.4}$$

[6] Alternatively one can apply [11, Formulae 864.54 and 864.62].

The graphs of the functions on the left–hand– and right–hand side intersect in simple roots $x_0 = 0$, $x_k > 0$ for $k \in \mathbb{N}$, such that with

$$\epsilon_k := x_k - \left[\frac{\pi}{2} + k\pi\right], \qquad k \in \mathbb{N} \cup \{0\} \ ,$$

one gets the *exponentially fast converging root–error bound*

$$|\epsilon_k| \le e^{0.01\pi}\pi e^{-\sqrt{-\lambda_k}} \le e^{-1.49\pi}\pi := \epsilon\pi \approx 0.00927\pi \qquad \forall k \in \mathbb{N} \ , \tag{6.5}$$

with $\epsilon_0 = -\dfrac{\pi}{2}$ and $-\lambda_k = \left(\dfrac{\pi}{2}+k\pi\right)^2$. For more information on this formula as well as (6.10) and (6.11) below, see Appendix B. Then with $\eta_k := \lambda_k - \zeta_k = \lambda_k + x_k^2 = \lambda_k + (\epsilon_k + \sqrt{-\lambda_k})^2 = \epsilon_k\left(2\sqrt{-\lambda_k} + \epsilon_k\right)$ and $\epsilon := e^{-1.49\pi}$, one gets *the exponentially fast converging absolute error bound*

$$|\eta_k| = |\lambda_k - \zeta_k| \le 2\pi(1+\epsilon)e^{0.01\pi}\sqrt{-\lambda_k}e^{-\sqrt{-\lambda_k}} \le 3\epsilon(1+\epsilon)\pi^2 \qquad \forall k \in \mathbb{N} \ , \tag{6.6}$$

i.e.

$$|\eta_k| \le 2.083\pi\sqrt{-\lambda_k}e^{-\sqrt{-\lambda_k}} \le 0.02807\pi^2 \qquad \forall k \in \mathbb{N} \ ,$$

where the last expressions are the next to last ones at $k = 1$.

The relative error bound reads

$$\left|\frac{\eta_k}{\lambda_k}\right| \le 2\pi(1+\epsilon)e^{0.01\pi}\frac{e^{-\sqrt{-\lambda_k}}}{\sqrt{-\lambda_k}} \le \frac{4\epsilon(1+\epsilon)}{3} =: \rho \qquad \forall k \in \mathbb{N} \ , \tag{6.7}$$

i.e.

$$\left|\frac{\eta_k}{\lambda_k}\right| \le 2.083\pi\frac{e^{-\sqrt{-\lambda_k}}}{\sqrt{-\lambda_k}} \le 0.01247 \qquad \forall k \in \mathbb{N} \ ,$$

with a similar comment concerning the last expressions as above, whence

$$0 < 1 - \rho \le \left|\frac{\zeta_k}{\lambda_k}\right| \le 1 + \rho \approx 1.01247 \qquad \forall k \in \mathbb{N} \ . \tag{6.8}$$

From (6.8) and the fact that

$$\sum_{k=1}^{\infty} \frac{1}{|\lambda_k|^{\frac{1}{2}+\alpha}} < \infty, \qquad \alpha > 0 \ .$$

we get

$$\sum_{k=1}^{\infty} \frac{1}{|\zeta_k|^{\frac{1}{2}+\alpha}} < \infty, \qquad \alpha > 0 \ . \tag{6.9}$$

Hence upon applying to $\chi(s) = \vartheta(s)\vartheta(-s)$ Weierstrass's factorization theorem and subsequently Hadamard's factorization theorem as indicated in Appendix B, there holds the product representation:

$$\vartheta(s) = Ks\prod_{k=1}^{\infty}\Big(1 - \frac{s}{\zeta_k}\Big), \qquad K = \frac{1}{\sqrt{6}} \ . \tag{6.10}$$

There holds that $\vartheta(s)$ is an entire function of *order* $\frac{1}{2}$ and of *exponential type* 0. The former is by Borel's theorem, while the latter follows from [36, Theorem 7, p. 71]).

The error analysis above suggests that

$$\prod_{l=1}^{\infty}\Big(1 + \frac{\lambda_k}{\zeta_l}\Big) \approx \prod_{l=1}^{\infty}\Big(1 + \frac{\lambda_k}{\lambda_l}\Big) = \Big(1 + \frac{\lambda_k}{\lambda_0}\Big)^{-1}\cosh\sqrt{-\lambda_k} \ .$$

Indeed upon defining the product–ratio

$$R(\lambda_k) := \frac{\prod_{l=1}^{\infty}\Big(1+\dfrac{\lambda_k}{\zeta_l}\Big)}{\prod_{l=1}^{\infty}\Big(1+\dfrac{\lambda_k}{\lambda_l}\Big)} \ ,$$

one finds (with ρ defined in (6.7)) a constant $R = \left[\dfrac{1}{1-\rho(1+\rho)}\right]^{1/(1-e^{-\pi})} \approx 1.0134$ such that

$$R^{-1} \leq R(\lambda_k) \leq R \qquad \forall k \in \mathbb{N}\cup\{0\}\ . \tag{6.11}$$

As

$$\vartheta(\lambda_k)\vartheta(-\lambda_k) = \cos\sqrt{-\lambda_k}\cosh\sqrt{-\lambda_k} - 1 = -1\ ,$$

with

$$\vartheta(-\lambda_k) = K(-\lambda_k)\prod_{l=1}^{\infty}\Big(1+\frac{\lambda_k}{\zeta_l}\Big) = K(-\lambda_k)R(\lambda_k)\Big(1+\frac{\lambda_k}{\lambda_0}\Big)^{-1}\cosh\sqrt{-\lambda_k}\ ,$$

one finds then a constant $L = \dfrac{4R}{-K\lambda_0} = 4.024$ such that

$$|\vartheta(\lambda_k)| \leq Le^{-\sqrt{-\lambda_k}} \qquad \forall k \in \mathbb{N}\cup\{0\}\ , \tag{6.12}$$

i.e. the sequence $\{\vartheta(\lambda_k)\}_{k=0}^{\infty}$ *tends to zero exponentially fast.*

By the considerations above the general form of the spectral factor is

$$\phi(s) = \frac{\vartheta(s)}{\cosh\sqrt{s}}, \qquad s \neq \lambda_k, \quad k = 0,1,2,\dots\ .$$

Since $\phi(0) = \sqrt{\delta} = 0$, from (4.23) we find for $k \in \mathbb{N}\cup\{0\}$

$$\alpha_k = \operatorname*{Res}_{s=\lambda_k}\Big[\frac{\phi(s)}{s}\Big] = \lim_{s\to\lambda_k}(s-\lambda_k)\frac{\vartheta(s)}{s(\cosh\sqrt{s}-\cosh\sqrt{\lambda_k})} = \frac{2\vartheta(\lambda_k)}{\sqrt{\lambda_k}\sinh\sqrt{\lambda_k}} = \frac{2(-1)^{k+1}\vartheta(\lambda_k)}{\sqrt{-\lambda_k}}$$

and therefore with the orthonormal A–eigenvector basis $\{e_k\}_{k=0}^{\infty}$ of H

$$\frac{\alpha_k}{\lambda_k\langle d, e_k\rangle_{\mathrm{H}}} = \frac{\sqrt{2}(-1)^{k+1}\vartheta(\lambda_k)}{(-\lambda_k)}\ ,$$

where (6.12) holds. This shows that (4.21) holds. Using the standard estimate (with $\operatorname{Re} s > 0$)

$$\begin{aligned} |\hat{f}(s)|\sqrt{2\operatorname{Re} s} = &\ \big|\langle f, e^{-\overline{s}(\cdot)}\rangle_{\mathrm{L}^2(0,\infty)}\big|\sqrt{2\operatorname{Re} s} \leq \\ \leq &\ \|f\|_{\mathrm{L}^2(0,\infty)}\|e^{-s(\cdot)}\|_{\mathrm{L}^2(0,\infty)}\sqrt{2\operatorname{Re} s} = \|f\|_{\mathrm{L}^2(0,\infty)}\ , \end{aligned}$$

we get (with real negative eigenvalues λ_k and using (6.12))

$$\begin{aligned}\sum_{k=0}^{\infty}\left|\frac{\alpha_k}{\langle d, e_k\rangle_{\mathrm{H}}}\right|^2\left|\hat{f}(-\lambda_k)\right|^2 = & \sum_{k=0}^{\infty}\left|\frac{\alpha_k}{\sqrt{-2\lambda_k}\langle d, e_k\rangle_{\mathrm{H}}}\right|^2\left|\hat{f}(-\lambda_k)\sqrt{-2\lambda_k}\right|^2 \leq \\ \leq & L^2\|f\|^2_{\mathrm{L}^2(0,\infty)}\sum_{k=0}^{\infty}\frac{e^{-2\sqrt{-\lambda_k}}}{(-\lambda_k)} \leq L^2\|f\|^2_{\mathrm{L}^2(0,\infty)}\frac{e^{-\pi}}{(-\lambda_0)}\sum_{k=0}^{\infty}e^{-2k\pi} = \\ = & L^2\|f\|^2_{\mathrm{L}^2(0,\infty)}\frac{1}{(-\lambda_0)}\frac{e^{-\pi}}{1-e^{-2\pi}} < \infty \qquad \forall f \in \mathrm{L}^2(0,\infty)\ .\end{aligned}$$

Hence (4.22) holds and so does assumption (**H9**) of Theorem 4.2.

Contrary to Theorem 4.2, the assumptions of Theorem 4.3 are easily checked. In (6.2) the equalities are attained at $\omega = 0$ and therefore (4.27) cannot be satisfied until $k_2 = -k_1 = 1$. Nevertheless, if we decrease $k_2 = -k_1$ from 1 to $\sqrt{1-\varepsilon}$, where $\varepsilon > 0$ is small then (4.27) will hold. Indeed, by (6.1) we have then

$$1 - (1-\varepsilon)\left|\hat{g}(j\omega)\right|^2 \geq 1 - (1-\varepsilon) = \varepsilon > 0 \qquad \forall \omega \in \mathbb{R}\ .$$

Furthermore, for $k_2 = -k_1 = \sqrt{1-\varepsilon}$ all the assumptions of Theorem 4.3 (except for (**H10**) and (**H12**)) are satisfied, because their verification is similar as in the case of the assumptions of Theorem 4.2. As for (**H10**) and (**H12**) one must check that the semigroup generated by A_0, which here equals A^{-1}, is uniformly bounded and hence **AS**. This can be done in many ways. In particular, we can repeat the Riesz basis property method, used in the proof of Theorem 4.2[7], and conclude that the semigroup $\{e^{tA_0}\}_{t\geq 0}$ is uniformly bounded.

By Theorem 4.3 the Lur'e system (3.2) has a solution $(\mathcal{H}, g)$ such that $\mathcal{H} \in \mathbf{L}(\mathrm{H})$, $\mathcal{H} = \mathcal{H}^* \geq 0$, and $g \in \mathrm{H}$ with g^*A admissible with respect to the semigroup generated by A. Since one does not know whether $\mathcal{H}$ is coercive, one cannot use Theorem 3.1. Moreover as d is not admissible, Theorem 3.2 cannot be applied. However, by the proof of Theorem 3.2, (3.10) is valid without the admissiblity of d, so one can conclude that $y, u \in \mathrm{L}^2(0,\infty)$ provided that f satisfies the sector condition

$$\left|\frac{f(y)}{y}\right| \leq \sqrt{1-\varepsilon}, \qquad \forall y \neq 0, \quad f(0) = 0\ .$$

This confirms the result of Grabowski and Callier [18, p. 10], where the same has been derived using the input–output approach.

Theorem 4.2 provides an positive answer to the question of solvability of the Lur'e system of equations (3.2) for $k_2 = -k_1 = 1$. Consequently $y, u \in \mathrm{L}^2(0,\infty)$ even if f satisfies the weaker sector condition

$$\left|\frac{f(y)}{y}\right| < 1, \qquad \forall y \neq 0, \quad f(0) = 0\ . \tag{6.13}$$

This result can be sharpened using the output equation

$$y = \overline{P}x_0 + g \star u\ ,$$

[7]See the text just after formula (4.25) with some simplifications due to the fact that A has a system of eigenvectors being an orthonormal basis, so the similarity transformation T is trivial, i.e., $T = I$.

where the observability map $\overline{P}$ is given by

$$
(6.14)\qquad
\begin{aligned}
(\overline{P}x_0)(t) = &\sum_{k=0}^{\infty} e^{\lambda_k t} c^{\#} e_k \langle x_0, e_k\rangle_{\mathrm{H}} = \\
= &\sqrt{2}\sum_{k=0}^{\infty}(-1)^k e^{-\left(\frac{\pi}{2}+k\pi\right)^2 t}\langle x_0, e_k\rangle_{\mathrm{H}} \qquad \forall t>0, \quad \forall x_0\in \mathrm{H}
\end{aligned}
$$

and therefore $\overline{P}x_0$ is continuous function of $t>0$ and decays exponentially as t tends to infinity. The impulse response g is the derivative of $\overline{P}d$ and thus g is continuous for $t\ge 0$, $(\overline{P}d)(0)=-1=c^{\#}d$[8], $g(0)=0$ and g decays exponentially as t tends to infinity, see [17, Subsection 5.2] for details. Since $u\in \mathrm{L}^2(0,\infty)$ then by the standard properties of convolution $g\star u$ is a continuous function of $t\ge 0$ and $\lim\limits_{t\to\infty}(g\star u)(t)=0$. Hence y is continuous for $t>0$ and $\lim\limits_{t\to\infty} y(t)=0$. This jointly with continuity of f and the sector conditions (6.13) implies that u is continuous for $t>0$ and $\lim\limits_{t\to\infty} u(t)=0$.

The closed–loop linear semigroup generator corresponding to the linear feedback law $f(y)=\mu y$ takes the form

$$A_\mu x = x'', \qquad D(A_\mu)=\left\{x\in \mathrm{H}^2(0,1):\ x'(1)=0,\ x(0)=\mu x(1)\right\} .$$

It is proved in [13] that A_μ generates an analytic semigroup on $\mathrm{L}^2(0,1)$ which is **EXS** for $\mu\in(-\cosh\pi,1)$ with $\cosh\pi\approx 11.592$. Hence, the Hurwitz sector is essentially bigger than the sector (k_1,k_2) obtained above.

7. Discussion

7.1. **A survey of known results on the solvability of Lur'e equations.**

7.1.1. *Nudel'man and Schwartzman* [26, Theorem 4, p. 570].

Theorem 7.1. Let $A\in \mathbf{L}(\mathrm{H})$ be the generator of an **EXS** linear C_0–semigroup on H and let the vector $B\in\mathrm{H}$ be *cyclic* for A, i.e., $\{A^kB\}_{k=0}^{\infty}$ is a *complete* system in H. Then the existence of a solution $R\in\mathbf{L}(\mathrm{H})$, $R=R^*\ge 0$ of

$$
(7.1)\qquad \left\{\begin{array}{c} A^*R+RA\le 0\\ -RB+C=0\end{array}\right\} ,
$$

is equivalent to

$$
(7.2)\qquad \operatorname{Re} C^*(j\omega I-A)^{-1}B\ge 0 \qquad \forall\omega\in\mathbb{R} .
$$

This theorem cannot be applied to decide the solvability of (3.2) as the semigroup generated by A^{-1} cannot be **EXS** unless $\dim\mathrm{H}<\infty$, because 0 belongs to the continuous spectrum of A^{-1} (clearly A^{-1} has an unbounded inverse). One may think to use the possible *exponential stabilizability* of the pair (A^{-1},d). However this does not hold, as otherwise the semigroup $\{e^{t(A^{-1}+dg^*)}\}_{t\ge0}$ would be **EXS** for some $g\in\mathrm{H}$. Since the operator dg^* has rank one then by a result of Triggiani [33] the semigroup $\{e^{tA^{-1}}\}_{t\ge0}$ would also be **EXS** which is not the case if $\dim\mathrm{H}=\infty$. Hence the pair (A^{-1},d) is not exponentially stabilizable.

[8]For $x_0=d$ the series in (6.14) converges for $t\ge0$, and its convergence at $t=0$ is conditional which follows from Leibniz's criterion.

7.1.2. *Likhtarnikov and Yacubovič* [23, Theorem 3, p. 902].

Theorem 7.2. Let $A \in \mathbf{L}(\mathrm{H})$ be the generator of a linear C_0–semigroup on H, let $B \in \mathbf{L}(\mathrm{U}, \mathrm{H})$, let the pair (A, B) be exponentially stabilizable[9] and let there exist a $\delta > 0$ such that

$$F(x,u) \geq \delta \left(\|x\|_{\mathrm{H}}^2 + \|u\|_{\mathrm{U}}^2\right) \qquad \forall (\omega, x, u) \in \mathbb{R} \times D(A) \times \mathrm{U}, \quad j\omega x = Ax + Bu \ ,$$

where

$$F(x,u) = \langle x, Qx\rangle_{\mathrm{H}} + \langle x, Su\rangle_{\mathrm{H}} + \langle Su, x\rangle_{\mathrm{H}} + \langle u, Ru\rangle_{\mathrm{H}} \tag{7.3}$$

with $Q \in \mathbf{L}(\mathrm{H})$, $Q = Q^*$, $S \in \mathbf{L}(\mathrm{U}, \mathrm{H})$, $R \in \mathbf{L}(\mathrm{U})$ and $R = R^*$. Then there exists an operator $\mathcal{H} \in \mathbf{L}(\mathrm{H})$, $\mathcal{H} = \mathcal{H}^* \geq 0$ which satisfies for some $\eta > 0$:

$$\begin{aligned} &\langle Ax + Bu, \mathcal{H}x\rangle_{\mathrm{H}} + \langle x, \mathcal{H}(Ax + Bu)\rangle_{\mathrm{H}} + \\ &+ F(x,u) \geq \eta \left(\|x\|_{\mathrm{H}}^2 + \|u\|_{\mathrm{U}}^2\right) \qquad \forall (x,u) \in D(A) \times \mathrm{U} \ . \end{aligned} \tag{7.4}$$

Theorem 7.2 does not apply to (3.2), because the left-hand side of (3.5) is not representable as the left-hand side of (7.4). Furthermore, then the quadratic form F is not continuous. Theorem 7.2 does not apply also to (3.7), as then the control operator B is not bounded and still the quadratic form F is not continuous.

7.1.3. *Balakrishnan* [2, Theorem 2.1, p. 179].

Theorem 7.3. Let $A : (D(A) \subset \mathrm{H}) \longrightarrow \mathrm{H}$ generate an **EXS** linear C_0–semigroup on H, let A^{-1} be compact, let A have a system of eigenvectors which forms a Riesz basis of H, and let the following restrictions hold:

$$B \in D(A), \quad C \in D(A^*), \quad \langle C, B\rangle_{\mathrm{H}} \neq 0, \quad \langle C, AB\rangle_{\mathrm{H}} \neq 0 \ ,$$

the pair (A, B) is *approximately controllable*[10] and (7.2) holds. Then there exists $R \in \mathbf{L}(\mathrm{H})$, $R = R^* \geq 0$ satisying (7.1).

The main difference between Theorem 4.2 and the result above lies in the assumptions concerning d and h. To be more precise, comparing the Lur'e system (7.1) with (3.7) one can see that (7.1) corresponds to (3.7) with $d = B$, $eh = C$, $\mathcal{H} = R$, and[11] $\delta = 0$, $q = 0$. The domain assumption $B \in D(A)$ implies that Ad is meaningful and belongs to H, while the domain assumption $C \in D(A^*)$ implies that A^*h is meaningful and

$$c^{\#}x = \langle Ax, h\rangle_{\mathrm{H}} = \langle x, A^*h\rangle_{\mathrm{H}}, \qquad x \in D(A) \ .$$

Hence by the Riesz representation theorem the observation functional $c^{\#}$ extends to a bounded, linear everywhere defined functional and we conclude that the Lur'e system considered by Balakrishnan corresponds to a system with bounded control and observation operators.

[9] Originally Likhtarnikov and Yacubovič have assumed the so-called L^2–*controllability* of the system, however Louis and Wexler [24] proved that this concept is equivalent to exponential stabilizability. In fact Louis and Wexler rediscovered the theorem by generalizing an earlier result also due to Likhtarnikov and Yacubovič [22]

[10] Equivalently, B is a cyclic vector for A^{-1}, i.e., $\{A^{-k}B\}_{k=0}^{\infty}$ is a complete system in H.

[11] In the context of (3.2) the two last equalities mean that $k_1 = 0$, $k_2 = -\dfrac{1}{c^{\#}d} = \dfrac{1}{\hat{g}(0)} > 0$.

7.1.4. *Oostveen and Curtain* [27, Theorem 19 and Corollary 20].

Theorem 7.4. Let $A : (D(A) \subset \mathrm{H}) \longrightarrow \mathrm{H}$ generate an **AS** linear C_0–semigroup on H, let $B \in \mathbf{L}(\mathrm{U}, \mathrm{H})$ be an admissible control operator[12], let $C \in \mathbf{L}(\mathrm{H}, \mathrm{Y})$ be an admissible observation operator[13], let the transfer function

$$G(s) := C(sI - A)^{-1}B \tag{7.5}$$

belong to $\mathrm{H}^\infty(\Pi^+, \mathbf{L}(\mathrm{U}, \mathrm{Y}))$ and $N \in \mathbf{L}(\mathrm{Y}, \mathrm{U})$, $Q \in \mathbf{L}(\mathrm{Y})$, $Q = Q^*$, $R \in \mathbf{L}(\mathrm{U})$, $R = R^* \geq \delta I > 0$. There holds: if the *Popov function* satisfies

$$R + NG(j\omega) + [NG(j\omega)]^* + [G(j\omega)]^* QG(j\omega) \geq \varepsilon I > 0 \tag{7.6}$$

then the *Riccati operator equation*:

$$A^*Xx + XAx - (B^*X + NC)^* R^{-1}(B^*X + NC)x + C^*QCx = 0 \qquad \forall x \in D(A) \tag{7.7}$$

has a unique self-adjoint bounded solution such that the operator $[A - BR^{-1}(B^*X + NC)]$ generates an **AS** linear C_0–semigroup on H.

Theorem 7.4 inspired Theorem 4.3 of Section 4. For an interesting complement of information for the case that the Popov function is nonnegative but not coercive, see [9].

7.1.5. *Pandolfi* [29, Theorem 3, p. 740].

Theorem 7.5. Let $A \in \mathbf{L}(\mathrm{H})$ generate a linear C_0–group on H, $B \in \mathbf{L}(\mathrm{U}, \mathrm{H})$ and let the pair (A, B) be exactly controllable. There holds: if

$$F(x, u) \geq 0 \qquad \forall (\omega, x, u) \in \mathbb{R} \times D(A) \times \mathrm{U}, \quad j\omega x = Ax + Bu \ ,$$

where the form F is given by (7.3) with $Q \in \mathbf{L}(\mathrm{H})$, $Q = Q^*$, $S \in \mathbf{L}(\mathrm{U}, \mathrm{H})$, $R \in \mathbf{L}(\mathrm{U})$, and $R = R^*$, then there exists an $\mathcal{H} \in \mathbf{L}(\mathrm{H})$, $\mathcal{H} = \mathcal{H}^*$ satisfying

$$\langle Ax + Bu, \mathcal{H}x\rangle_{\mathrm{H}} + \langle x, \mathcal{H}(Ax + Bu)\rangle_{\mathrm{H}} + F(x, u) \geq 0 \qquad \forall (x, u) \in D(A) \times \mathrm{U} \ .$$

Observe that here the coercivity assumptions of Theorem 7.2 are eliminated by replacing the L^2–controllability by the stronger requirement of exact controllability; simultaneously A is now the generator of a group.

7.1.6. *Bucci* [5, Theorem 3.1].

Theorem 7.6. Let $A \in \mathbf{L}(\mathrm{H})$ generate an **EXS** linear analytic C_0–semigroup on H, let there exists an $\alpha \in (0, 1)$ such that $D \in \mathbf{L}(\mathrm{U}, D[D(-A)^\alpha]$. There holds: if there exists a $\delta > 0$ such that

$$F(A(j\omega I - A)^{-1}Du, u) \geq \delta \|u\|^2_{\mathrm{U}} \qquad \forall (\omega, u) \in \mathbb{R} \times \mathrm{U} \ ,$$

where F is given by (7.3) with $Q \in \mathbf{L}(\mathrm{H})$, $Q = Q^*$, $S \in \mathbf{L}(\mathrm{U}, \mathrm{H})$, $R \in \mathbf{L}(\mathrm{U})$, and $R = R^* \geq 0$, then there exists an $\mathcal{H} \in \mathbf{L}(\mathrm{H})$, $\mathcal{H} = \mathcal{H}^*$ satisfying

$$\langle \mathcal{H}(x - Du), Ax\rangle_{\mathrm{H}} + \langle Ax, \mathcal{H}(x - Du)\rangle_{\mathrm{H}} + F(x - Du, u) \geq 0 \quad \forall (x, u) \in D(A) \times \mathrm{U} \tag{7.8}$$

This result can be applied to decide the solvability of system (3.7) but only when the observational fuctional $c^{\#}$ extends to a linear bounded, everywhere defined functional.

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[12]This holds iff $B^*(sI - A)^{-1}x \in \mathrm{H}^2(\Pi^+, \mathrm{U})$ for all $x \in \mathrm{H}$.

[13]This holds iff $C^*(sI - A)^{-1}x \in \mathrm{H}^2(\Pi^+, \mathrm{Y})$ for all $x \in \mathrm{H}$.

7.1.7. *Pandolfi* [30, Theorem 2].

Theorem 7.7. Let $A \in \mathbf{L}(\mathrm{H})$ generate a linear analytic C_0–semigroup on H, let $D \in \mathrm{H}$, A^{-1} be compact, let A have a system of eigenvectors which forms a Riesz basis of H, let $\sigma_P(A) = \{z_n\}_{n\in\mathbb{N}}$ where the z_n are simple eigenvalues, let there exist a $\mu > 0$ such that $|\operatorname{Im} z_n / \operatorname{Re} z_n| < \mu$ uniformly with respect to $n \in \mathbb{N}$. There holds: if

$$\pi(\omega) := F(-A(j\omega - A)^{-1}Du, u) \geq 0 \qquad \forall(\omega, u) \in \mathbb{R} \times \mathrm{U} \ ,$$

where F is given by (7.3) with $Q \in \mathbf{L}(\mathrm{H})$, $Q = Q^*$, $S \in \mathrm{H}$, $R \in \mathbb{R}$ and there exists an exponent $\delta < 1$ such that

$$|\omega|^{\delta}\pi(\omega) > H,$$

for some positive H and large enough $|\omega|$, then there exists an $\mathcal{H} \in \mathbf{L}(\mathrm{H})$, $\mathcal{H} = \mathcal{H}^*$ satisfying (7.8).

This result can be applied to decide the solvability of system (3.7) but only when the observation fuctional $c^{\#}$ extends to a linear bounded, everywhere defined functional.

7.1.8. *Pandolfi* [31, Theorem 3, p. 482].

Theorem 7.8. Let $A \in \mathbf{L}(\mathrm{H})$ generate a linear C_0–group on H, let $D \in \mathbf{L}(\mathrm{U}, \mathrm{H})$ be an admissible factor control operator and the pair (A^{-1}, D) be exactly controllable. There holds: if

$$F(x - Du, u) \geq 0 \qquad \forall(\omega, x, u) \in \mathbb{R} \times D(A) \times \mathrm{U}, \quad j\omega(x - Du) = Ax$$

where the form F is given by (7.3) with $Q \in \mathbf{L}(\mathrm{H})$, $Q = Q^*$, $S \in \mathbf{L}(\mathrm{U}, \mathrm{H})$, $R \in \mathbf{L}(\mathrm{U})$ and $R = R^*$, then there exists an $\mathcal{H} \in \mathbf{L}(\mathrm{H})$, $\mathcal{H} = \mathcal{H}^*$ satisfying (7.8).

Again the quadratic form F is continuous, so the result is related to the solvability of (3.7) with bounded observation.

7.2. **Conclusions.** The most important results of this paper are:

- Some circle criterion type absolute stability criteria presented in Section 3 based on quadratic Lyapunov functionals (Theorems 3.1, 3.2). These criteria give results similar to those of the input–output approach, however to verify here the Theorem assumptions one has to examine the solvability problem of the Lur'e system (3.2).
- Solvability results for Lur'e systems in Section 4, where in particular, appropriate versions of the Kalman–Yacubovič lemma are presented (Theorems 4.2 and 4.3).
- A detailed presentation of two examples of electrical transmission–lines, illustrating the results of previous sections, in Sections 5 and 6. The discussion shows that this paper's stability criteria are checkable.

Some considerations for generalizing the results are now discussed. Some further investigations should be made to replace the *Riesz basis assumption* in Theorem 4.2 by weaker assumptions. This probably will require the derivation of a new admissibility criterion for observation functionals in terms of the cyclic system $\{A^{-k}d\}_{k=0}^{\infty}$ rather than in terms of a Riesz basis of eigenvectors of A. Note that up to now there is no satisfactory characterization of admissibility of the non–spectral type.

Futhermore, more involved numerical analysis is in order to complete the results of the example of Section 6 for finding *interesting approximations of* $(\mathcal{H}, g)$ – the solution of the Lur'e system (4.14). In view of the exponential decay of $\vartheta(\lambda_k)$, it is a good bet that some spectral approximations could converge very fast to the solution $(\mathcal{H}, g)$. This involves numerical analysis which falls outside the scope of the present paper.

Finally observe that, except for the case $b \leq 0$, all examples above show that the absolute stability conditions generated by the circle criterion are significantly more conservative than the Hurwitz sector condition. It is known that for finite–dimensional autonomous *continuous* Lur'e systems *Popov's method* leads to considerably better stability conditions than the circle criterion. It is less known that a generalization of Popov's method to finite–dimensional autonomous *discrete* Lur'e systems is possible only by *further restricting the class of admissible nonlinearities*. This causes one to expect some difficulties to get an appropriate Popov type stability criterion for the system described by (3.4), which is sufficiently general to handle discrete–time systems, as can be seen by noting that (5.2) is an equivalent model giving the essentially discrete–time dynamics of the electrical distortionless loaded *RLCG*–transmission line. An additional observation is that the input–output approach for finite–dimensional feedback systems is usually based on some smoothness assumptions imposed on the system output. Thus an other difficulty for obtaining a generalization of Popov's method will be that one has to examine some *differentiability properties of the system output*.

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Appendix A: Derivation of Equation (3.9)

Let $a := k_2 - k_1 > 0$, $b := k_2 + k_1$ and $\Delta := a^2 - 4\varepsilon$. Then

$$\left[k_2 - \frac{f(y)}{y}\right]\left[\frac{f(y)}{y} - k_1\right] =$$
$$= \left\{k_2 - \frac{b+\sqrt{\Delta}}{2} + \left[\frac{b+\sqrt{\Delta}}{2} - \frac{f(y)}{y}\right]\right\}\left\{\left[\frac{f(y)}{y} - \frac{b-\sqrt{\Delta}}{2}\right] + \frac{b-\sqrt{\Delta}}{2} - k_1\right\} =$$
$$= \left\{\frac{a-\sqrt{\Delta}}{2} + \left[\frac{b+\sqrt{\Delta}}{2} - \frac{f(y)}{y}\right]\right\}\left\{\left[\frac{f(y)}{y} - \frac{b-\sqrt{\Delta}}{2}\right] + \frac{a-\sqrt{\Delta}}{2}\right\} =$$
$$= \frac{a-\sqrt{\Delta}}{2}\left[\frac{f(y)}{y} - \frac{b-\sqrt{\Delta}}{2}\right] + \left[\frac{b+\sqrt{\Delta}}{2} - \frac{f(y)}{y}\right]\left[\frac{f(y)}{y} - \frac{b-\sqrt{\Delta}}{2}\right] +$$
$$+ \frac{\left(a-\sqrt{\Delta}\right)^2}{4} + \left[\frac{b+\sqrt{\Delta}}{2} - \frac{f(y)}{y}\right]\frac{a-\sqrt{\Delta}}{2} =$$
$$= \left[\frac{b+\sqrt{\Delta}}{2} - \frac{f(y)}{y}\right]\left[\frac{f(y)}{y} - \frac{b-\sqrt{\Delta}}{2}\right] + \varepsilon \ .$$

Now since $f \in \mathcal{S}_\varepsilon$ we have

$$y^2\left[k_2 - \frac{f(y)}{y}\right]\left[\frac{f(y)}{y} - k_1\right] = y^2\left[\frac{b+\sqrt{\Delta}}{2} - \frac{f(y)}{y}\right]\left[\frac{f(y)}{y} - \frac{b-\sqrt{\Delta}}{2}\right] + \varepsilon y^2 \geq \varepsilon y^2 \ .$$

Appendix B: Error analysis concerning the Example of Section 6

Ad inequality (6.5). Observe that for $k \in \mathbb{N} \cup \{0\}$, $(-1)^{k+1}\epsilon_k > 0$. Moreover for $k \in \mathbb{N}$, $|\epsilon_k| \leq |\epsilon_{k-1}| \leq 0.5\pi$ and for $|\epsilon| \leq 0.5\pi$, $(2/\pi)|\epsilon| \leq \sin|\epsilon|$. Put $x = \epsilon + (0.5+k)\pi$ in equation (6.4). The latter reads then

$$(-1)^{k+1}\sin\epsilon = \frac{1}{\cosh[\epsilon + (0.5+k)\pi]}, \qquad k \in \mathbb{N}, \quad |\epsilon| \leq 0.5\pi \ .$$

Thus for $k \in \mathbb{N}$ with k odd there holds

$$\frac{2}{\pi}|\epsilon_k| \leq \sin|\epsilon_k| = \frac{1}{\cosh[|\epsilon_k| + (0.5+k)\pi]} \leq \frac{1}{\cosh[(0.5+k)\pi]} \leq 2e^{-(0.5+k)\pi} = 2e^{-\sqrt{-\lambda_k}} \ .$$

Hence for $k \in \mathbb{N}$ with k odd

$$|\epsilon_k| \leq \pi e^{-\sqrt{-\lambda_k}} \leq \pi e^{-1.5\pi} < 0.01\pi \ .$$

Then for $k \in \mathbb{N}$ with k even, $-\epsilon_k = |\epsilon_k| \leq |\epsilon_{k-1}| < 0.01\pi$, such that

$$\begin{aligned}\frac{2}{\pi}|\epsilon_k| \leq & \ \sin|\epsilon_k| = \frac{1}{\cosh[-|\epsilon_k| + (0.5+k)\pi]} \leq \frac{1}{\cosh[(0.49+k)\pi]} \leq \\ \leq & \ 2e^{-(0.49+k)\pi} = 2e^{0.01\pi}e^{-(0.5+k)\pi} = 2e^{0.01\pi}e^{-\sqrt{-\lambda_k}} \ .\end{aligned}$$

Thus for $k \in \mathbb{N}$ with k even

$$|\epsilon_k| \leq e^{0.01\pi}\pi e^{-\sqrt{-\lambda_k}} \ .$$

Hence upon comparing with the odd case, inequality (6.5) holds. □

Ad equation (6.10). Observe that $\chi(s) = \vartheta(s)\vartheta(-s)$ has a double root at $\zeta_0 = 0$ and simple real roots at $\pm\zeta_k$ with $\zeta_k \approx \lambda_k < 0$ for $k \in \mathbb{N}$. The latter satisfy inequality (6.9). Hence $\chi(s)$ is of *genus* 0 and *order* $\frac{1}{2}$ (the latter can be confirmed by Borel's theorem [36, Theorem 6, p. 69]). Consequently by Weierstrass's factorization theorem, [36, pp. 54 - 57],

$$\chi(s) = e^{p(s)}(-s^2)\prod_{k=1}^{\infty}\Big(1-\frac{s}{\zeta_k}\Big)\prod_{k=1}^{\infty}\Big(1+\frac{s}{\zeta_k}\Big) \ ,$$

where $p(s)$ is a real polynomial. Now by Hadamard's factorization Theorem, [36, pp. 74 - 75], the degree of $p(s)$ cannot exceed $\frac{1}{2}$. Hence $p(s)$ is a constant such that

$$\chi(s) = \cosh\sqrt{s}\cos\sqrt{s} - 1 = \vartheta(s)\vartheta(-s) = K^2(-s^2)\prod_{k=1}^{\infty}\Big(1-\frac{s}{\zeta_k}\Big)\prod_{k=1}^{\infty}\Big(1+\frac{s}{\zeta_k}\Big) \ ,$$

where K is a constant. Observe that near zero, for s real and small

$$1 + \frac{(-s^2)}{6} + o\left(-s^2\right) - 1 = K^2(-s^2) \ ,$$

whence $K = \dfrac{1}{\sqrt{6}}$. From these considerations one gets equation (6.10). □

Ad inequality (6.11). By definition of η_k we have:

$$R(\lambda_k) = \prod_{l=1}^{\infty}\big[1+\mu_l(\lambda_k)\big], \qquad \mu_l(\lambda_k) := \frac{\eta_l}{\zeta_l}\ \frac{\lambda_k}{\lambda_l+\lambda_k} \ .$$

Note that in definition of $\mu_l(\lambda_k)$ the second factor is positive and less than 1 and the first factor is related to the relative error, i.e. small. Recalling that $\rho = (4/3)(1+\epsilon)\epsilon \approx 0.01247$, $\epsilon = e^{-1.49\pi}$, (6.8) and (6.7) yield

$$|\mu_l(\lambda_k)| \le \left|\frac{\lambda_l}{\zeta_l}\right|\ \left|\frac{\eta_l}{\lambda_l}\right| \le (1+\rho)2\pi(1+\epsilon)e^{0.01\pi}\ \frac{e^{-\sqrt{-\lambda_l}}}{\sqrt{-\lambda_l}} \le \rho(1+\rho) \approx 0.01263 \qquad \forall l \in \mathbb{N} \ ,$$

where the last expression is the third one evaluated at $l = 1$.

Therefore, since $|\mu_l(\lambda_k)| \le \rho(1+\rho) \approx 0.01263$ for all $l \in \mathbb{N}$, the graph of $(-1,1) \ni x \longmapsto \ln(1+x)$ yields that for $\beta := -\dfrac{\ln\big[1-\rho(1+\rho)\big]}{\rho(1+\rho)} \approx 1.0064$, there holds

$$-\beta|\mu_l(\lambda_k)| \le \ln[1+\mu_l(\lambda_k)] \le \beta|\mu_l(\lambda_k)| \qquad \forall l \in \mathbb{N} \ .$$

Combining we get then

$$\exp\Big[-\beta\sum_{l=1}^{\infty}|\mu_l(\lambda_k)|\Big] \le R(\lambda_k) \le \exp\Big[\beta\sum_{l=1}^{\infty}|\mu_l(\lambda_k)|\Big] \ , \tag{7.9}$$

where for all $k \in \mathbb{N}\cup\{0\}$

$$\begin{aligned}\sum_{l=1}^{\infty}|\mu_l(\lambda_k)| \le\ & 2\pi(1+\epsilon)e^{0.01\pi}(1+\rho)\sum_{l=1}^{\infty}\frac{e^{-\sqrt{-\lambda_l}}}{\sqrt{-\lambda_l}} \le \frac{2\pi(1+\epsilon)e^{0.01\pi}(1+\rho)}{\sqrt{-\lambda_1}}\sum_{l=1}^{\infty}e^{-\sqrt{-\lambda_l}} = \\ =\ & \frac{4(1+\rho)(1+\epsilon)}{3}\ e^{0.01\pi}e^{-0.5\pi}\sum_{l=1}^{\infty}e^{-l\pi} = \frac{4(1+\rho)(1+\epsilon)}{3}\frac{e^{0.01\pi}e^{-1.5\pi}}{1-e^{-\pi}} = \\ =\ & \frac{4(1+\rho)\epsilon(1+\epsilon)}{3(1-e^{-\pi})} = \frac{\rho(1+\rho)}{1-e^{-\pi}} \ .\end{aligned}$$

Thus for all $k \in \mathbb{N} \cup \{0\}$,

$$\beta \sum_{l=1}^{\infty} |\mu_l(\lambda_k)| \leq \frac{-\ln\big[1-\rho(1+\rho)\big]}{1-e^{-\pi}} \ .$$

This and (7.9) imply inequality (6.11) with $R = \left[\dfrac{1}{1-\rho(1+\rho)}\right]^{1/(1-e^{-\pi})} \approx 1.0134.$ □

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