ON THE CIRCLE CRITERION FOR BOUNDARY CONTROL SYSTEMS IN FACTOR FORM: LYAPUNOV STABILITY AND LUR'E EQUATIONS

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Abstract. A Lur'e feedback control system consisting of a nonlinear static sector type controller and a linear, infinite–dimensional system of boundary control in factor form is considered. A criterion of absolute strong asymptotic stability of the null equilibrium is obtained using a quadratic form Lyapunov functional. The construction of such functional is reduced to solving a Lur'e system of equations. For the solvability of the latter the main result is a sufficient condition using the strict circle inequality based on results by J.C. Oostveen and R.F. Curtain [27]. All results are illustrated in detail by electrical transmission line examples: 1) of the distortionless loaded $RLCG$–type and 2) of the unloaded RC–type.

Contents

1. Introduction

This paper uses some results of abstract linear systems in factor form, obtained by the authors in earlier papers [16], [17] and shortly recalled in Section 2; these systems are related but not identical to Salamon–Weiss abstract linear systems e.g. [36], see [16, Section 4.5], [17, Section 7]. The results of Section 2 combined with the input–output approach using *passivity* concepts lead in [18] to a circle criterion for the nonlinear Lur'e type feedback system described by Figure 3.1 below, consisting of a nonlinear static sector type controller followed by a linear infinite–dimensional system of boundary control in factor form in the

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loop. The present paper shows that Lyapunov state space theory together with the abstract results of Section 2 gives similar stability conditions.

An absolute stability criterion is derived in Section 3. It is obtained by using a quadratic form Lyapunov functional. An intricate procedure of evaluating the derivative of the quadratic form along the system trajectories is studied and successfully applied to get a novel so–called Lur'e system. The abstract results of Section 2 enable us to prove global strong asymptotic stability of the null equilibrium in Theorem 3.1. An important consequence is that the stability question depends on the solvability of the Lur'e system of equations (3.2) (or equivalently (3.3)).

This reduction is standard in finite–dimensional state space system theory and leads to a variety of solvability results commonly known as the Kalman–Popov lemma and the Yacubovič frequency–domain theorem. The main difficulty in getting a generalization of this lemma in the infinite–dimensional case is due to the fact that the open–loop linear system control and/or observation involve unbounded linear operators, which lead to some difficult mathematical questions. However it turns out that the proof of the Riccati results [27, Theorem 19 and Corollary 20] of Oostveen and Curtain is useful in our context. Its adaptation (with in particular Lemma 4.3) is essential in Section 4 on the solvability of the Lur'e system (3.2). It proceeds, modulo adaptation involving the transfer function mapping $\hat{g}(s) \longmapsto \hat{g}(s^{-1}) - \hat{g}(0)$, as in the spectral factorization method for solving the Riccati equation of Callier and Winkin [7]. One starts by giving appropriate spectral factorization results. Next one describes a state–feedback observation operator realization problem induced by a spectral factor, see (4.7). Finally the solvability of the Lur'e system is obtained in Theorem 4.1. Many other Lur'e system results are available such as [26, Theorem 4, p. 570], [22, Theorem 3, p. 902], [24], [2, Theorem 2.1, p. 179], [28, Theorem 3, p. 740], [4, Theorem 3.1], [29, Theorem 2] and [30, Theorem 3, p. 482]. However they do not fit our context.

Sections 5 and 6 present an exhaustive illustration of the results for the examples of

☛ a loaded distortionless electric $RLCG$–transmission line for which we prove the global strong asymptotic stability,

☛ an unloaded electric RC–transmission line for which we prove the global strong asymptotic stability too, despite the fact that here the factor control vector d is not admissible and only a weaker stability result follows from the proof of Theorem 3.1.

Related, although different absolute stability results have been proved in [3], [4] and [23]. A discussion and some prospects for further investigations are presented in the concluding Section 7.

2. Preliminary data

We start by recalling the notion of admissibility of output operators. To do this, in a Hilbert space H with a scalar product $\langle\cdot,\cdot\rangle_{\mathrm{H}}$, consider the homogeneous system with observation

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & Ax(t) \\ x(0) & = & x_0 \\ y(t) & = & Cx(t) \end{array}\right\}, \qquad t \geq 0 \ .$$

We assume that $A : (D(A) \subset \mathrm{H}) \longrightarrow \mathrm{H}$ generates a linear C_0–semigroup $\{S(t)\}_{t\geq 0}$ on H and $C \in \mathbf{L}(D_A, \mathrm{Y})$ is an observation (output) operator, where D_A stands for the space

$D(A)$ equipped with the graph norm and Y is an another Hilbert space with a scalar product $\langle\cdot,\cdot\rangle_{\mathrm{Y}}$.

Definition 2.1. The observation operator C is called (*infinite–time*) *admissible* if the *observability operator* $P : \mathrm{H} \longrightarrow \mathrm{L}^2(0,\infty;\mathrm{Y})$, $(Px)(t) := CS(t)x$ is defined and bounded on $D(A)$.

If the observation operator C is admissible then: P is *densely* defined, closable and for any $x \in D(A)$ the function $[0,\infty) \ni t \longmapsto (Px)(t) = CS(t)x \in \mathrm{Y}$ is continuous. By the standard operator theory $P^* \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{Y}),\mathrm{H})$ and $\overline{P} = P^{**} \in \mathbf{L}(\mathrm{H},\mathrm{L}^2(0,\infty;\mathrm{Y}))$.

Standard arguments involving continuity, the fact that $D(A)$ is dense in H, and the closed graph theorem lead to the following result also mentioned in [36] and [27].

Lemma 2.1. An operator $C \in \mathbf{L}(\mathrm{H},\mathrm{Y})$ is admissible iff $CS(\cdot)x \in \mathrm{L}^2(0,\infty;\mathrm{Y})$ for any $x \in \mathrm{H}$.

In addition to Lemma 2.1 observe that if $C \in \mathbf{L}(\mathrm{H},\mathrm{Y})$ is admissible then the adjoint of P is given by

$$P^*y = \int_0^\infty S^*(t)C^*y(t)dt, \qquad y \in \mathrm{L}^2(0,\infty;\mathrm{Y}) \ .$$

Consequently the admissibility for bounded control operators can be introduced by using adjoint operators. An operator $B \in \mathbf{L}(\mathrm{U},\mathrm{H})$, where U stands for a Hilbert space with scalar product $\langle\cdot,\cdot\rangle_{\mathrm{U}}$, is said to be an (*infinite–time*) *admissible* control operator if its adjoint $B^* \in \mathbf{L}(\mathrm{H},\mathrm{U})$ is an admissible observation operator with respect to the adjoint semigroup, i.e. if the observability map $x \longmapsto B^*S^*(\cdot)x$ is everywhere defined on H, now with output space U. The latter fact is necessary and sufficient for $Q \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U}),\mathrm{H})$ where Q is the *reachability operator* given by

$$Qu = \int_0^\infty S(t)Bu(t)dt \ .$$

In this paper we shall consider mainly SISO systems of boundary control in factor form [16],

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & A[x(t) + u(t)d] \\ y & = & c^{\#}x \end{array} \right\} . \tag{2.1}$$

assuming, if something else is not explicitly said, that $A : (D(A) \subset \mathrm{H}) \longrightarrow \mathrm{H}$ generates a linear exponentially stable (**EXS**), C_0–semigroup $\{S(t)\}_{t\geq 0}$ on H, $d \in \mathrm{H}$ is a factor control vector, $u \in \mathrm{L}^2(0,\infty)$ is a scalar control function, y is a scalar output defined by an A–bounded linear observation functional $c^{\#}$ (bounded on D_A). The restriction of $c^{\#}$ to $D(A)$ is representable as $c^{\#}\big|_{D(A)} = h^*A$ for some $h \in \mathrm{H}$.

Define two operators:

$$V \in \mathbf{L}(\mathrm{H},\mathrm{L}^2(0,\infty)), \qquad (Vx)(t) := h^*S(t)x$$

$$W \in \mathbf{L}(\mathrm{L}^2(0,\infty),\mathrm{H}), \qquad Wu := \int_0^\infty S(t)du(t)dt \ .$$

Recall that L and $R = L^*$,

$$Lf = f', \qquad D(L) = \mathrm{W}^{1,2}(0,\infty) \ ,$$

$$Rf = -f', \qquad D(R) = \{f \in \mathrm{W}^{1,2}(0,\infty) : \ f(0) = 0\}$$

are the generators of the semigroups of *left–* and *right–shifts* on $\mathrm{L}^2(0,\infty)$, respectively. With these notation and assumptions Definition 2.1 gets the following equivalent form.

Definition 2.2. The observation functional $c^{\#}$ is called *admissible* if the *observability operator*

$$P = VA, \qquad D(P) = D(A)$$

has a bounded continuous extension on H denoted by $\overline{P}$.

Definition 2.3. The factor control vector $d \in \mathrm{H}$ is called *admissible* if

$$\mathrm{Range}(W) \subset D(A) \ .$$

Remark 2.1. By Definition 2.3, $W \in \mathbf{L}(\mathrm{L}^2(0,\infty), \mathrm{H})$ because the semigroup $\{S(t)\}_{t\geq 0}$ is **EXS**. Moreover, the *reachability operator* Q satisfies $Q := AW \in \mathbf{L}(\mathrm{L}^2(0,\infty), \mathrm{H})$.

In the sequel $\Pi^+ := \{s \in \mathbb{C} : \mathrm{Re}\, s > 0\}$ denotes the open right–half complex plane, $\mathrm{H}^\infty(\Pi^+)$ is the Banach space of analytic functions f on Π^+, equipped with the norm $\|f\|_{\mathrm{H}^\infty(\Pi^+)} = \sup\limits_{s\in\Pi^+} |f(s)|$ and $\mathrm{H}^2(\Pi^+)$ is the Hardy space of functions f analytic on Π^+ such that $\sup\limits_{\sigma>0} \int_{-\infty}^{\infty} |f(\sigma + j\omega)|^2 \, d\omega < \infty$, where $f(j\omega) := \lim\limits_{\sigma\to 0+} f(\sigma + j\omega)$ exists for almost all $\omega \in \mathbb{R}$. The space $\mathrm{H}^2(\Pi^+)$ is unitarily isomorphic with $\mathrm{L}^2(0,\infty)$ through the normalized Laplace transform. To be more precise,

$$\langle f, g\rangle_{\mathrm{L}^2(0,\infty)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(j\omega)\overline{\hat{g}(j\omega)} d\omega$$

where $\hat{f}, \hat{g}$ are the Laplace transform of f and g, respectively.

Moreover [21, p. 134] we shall frequently use the unitary operator $U \in \mathbf{L}(\mathrm{H}^2(\Pi^+))$ given by

$$(U\varphi)(s) := (1/s)\varphi(1/s) \ , \tag{2.2}$$

which for the $j\omega$–axis $\mathrm{H}^2(\Pi^+)$–norm corresponds to the change of variable $\omega \longmapsto -\omega^{-1}$.

Finally we shall encounter Wiener and Callier–Desoer convolution algebras. Recall [5, pp. 652 - 653], [6, pp. 81 - 84], [8, pp. 337 - 338] that a scalar–valued Laplace transformable distribution f with support on $[0,\infty)$ is in the *Wiener class* $\mathcal{A}(\sigma)$ for some $\sigma \in \mathbb{R}$ if $f(t) = f_a(t) + f_{sa}(t)$ for $t \geq 0$ with $e^{-\sigma(\cdot)} f_a(\cdot) \in \mathrm{L}^1(0,\infty)$ and $f_{sa}(t) = \sum\limits_{i=0}^{\infty} f_i \delta_0(t - t_i)$, where δ_0 denotes the Dirac delta distribution and $t_0 = 0$ and $t_i > 0$ for $i > 0$ are such that $\sum\limits_{i=0}^{\infty} e^{-\sigma t_i} |f_i| < \infty$. Such distribution is in the *Callier–Desoer class* $\mathcal{A}_-(0)$ if it is in $\mathcal{A}(\sigma)$ for some $\sigma < 0$. $\widehat{\mathcal{A}}(\sigma)$ and $\widehat{\mathcal{A}}_-(0)$ denote the classes of Laplace transforms of such distributions. $\mathcal{A}(\sigma)$ is a *convolution Banach algebra* with norm

$$\|f\|_{\mathcal{A}(\sigma)} := \left\|e^{-\sigma(\cdot)} f_a(\cdot)\right\|_{\mathrm{L}^1(0,\infty)} + \sum_{i=0}^{\infty} e^{-\sigma t_i} |f_i| \ .$$

For more information see [6] or [8].

Definition 2.4. The operator $H \in \mathbf{L}(\mathrm{L}^2(0,\infty))$ is called *causal* or *nonanticipative* if

$$(Hu_T)_T = (Hu)_T \qquad \forall u \in \mathrm{L}^2(0,\infty)$$

where u_T denotes the *truncation* of u at time $T>0$, $u_T(t) = \left\{ \begin{array}{ll} u(t) & \text{if} \quad t<T \\ 0 & \text{otherwise} \end{array} \right\}$.

Lemma 2.2. If $c^\#$ is admissible then $\overline{P}$, the closure of P has the form

$$\mathrm{Range}(V) \subset D(L), \qquad \overline{P} = LV$$

In particular for all $x_0 \in \mathrm{H}$, $(\overline{P}x_0)(t) = \dfrac{d}{dt}\left[h^* S(t)x_0\right] \in \mathrm{L}^2(0,\infty)$ with Laplace transform $\widehat{\left(\overline{P}x_0\right)}(s) = c^\#(sI-A)^{-1}x_0 \in \mathrm{H}^2(\Pi^+)$. Moreover if d is admissible then the *reachability operator* $Q = AW$ belongs to $\mathbf{L}(\mathrm{L}^2(0,\infty),\mathrm{H})$.

Lemma 2.3. If the *compatibility condition*

$$d \in D(c^\#) \tag{2.3}$$

holds then the function

$$\hat{g}(s) := sc^\#(sI-A)^{-1}d - c^\# d = sh^*A(sI-A)^{-1}d - c^\# d \tag{2.4}$$

is well–defined and analytic on the complex right half–plane Π^+.

If in addition to (2.3), $c^\#$ is admissible then:

- **(i)** $\hat{g}(s) = s\widehat{\left(\overline{P}d\right)}(s) - c^\# d$ with $\widehat{\overline{P}d} \in \mathrm{H}^\infty(\Pi^+) \cap \mathrm{H}^2(\Pi^+)$.
- **(ii)** The convolution operator K with kernel $\overline{P}d$, i.e., $Ku := \overline{P}d \star u$ belongs to $\mathbf{L}(\mathrm{L}^2(0,\infty))$ and it maps the domain of R into itself.

Lemma 2.3 leads to the following result [17, Theorem 4.1, Corollary 4.1, Theorem 4.2].

Lemma 2.4. If (2.3) holds, $c^\#$ is admissible and

$$\hat{g} \in \mathrm{H}^\infty(\Pi^+) \tag{2.5}$$

then the *input–output* operator F,

$$F = -KR - c^\# dI, \qquad D(F) = D(R)$$

is bounded and its closure $\overline{F}$ is causal and given by

$$\mathrm{Range}(K) \subset D(R), \qquad \overline{F} = -RK - c^\# dI \ .$$

Moreover $\hat{g}$ is then the *transfer function* of the system (2.1), and $\overline{F}$ is a convolution operator in the sense of distributions given by

$$\overline{F}u = g \star u, \qquad u \in \mathrm{L}^2(0,\infty) \ ,$$

with impulse response g given by

$$g := \mathcal{D}(\overline{P}d) - c^\# d\delta_0 \ ,$$

with Laplace transform $\hat{g}$ (here $\mathcal{D}$ denotes the distributional derivative, and δ_0 stands for the Dirac distribution at zero).

If in addition

$$c^\# \subset c^\#_L \ ,$$

where

$$c^\#_L x_0 = \lim_{h\to 0+} \frac{1}{h} c^\# \int_0^h S(\sigma)x_0 d\sigma, \qquad D(c^\#_L) = \left\{ x_0 \in \mathrm{H} : \ \exists \lim_{h\to 0+} \frac{1}{h} c^\# \int_0^h S(\sigma)x_0 d\sigma \right\}$$

is the Lebesgue extension of $c^\#$, then $c^\# d = \left(\overline{P}d\right)(0+)$ i.e. the Lebesgue value given by

$$\left(\overline{P}d\right)(0+) := \lim_{t\to 0+}\frac{1}{t}\int_0^t \left(\overline{P}d\right)(\tau)d\tau \ = \lim_{s\to\infty, s\in\mathbb{R}} s\widehat{\left(\overline{P}d\right)}(s),$$

and

$$\lim_{s\to\infty, s\in\mathbb{R}} \hat{g}(s) = 0 \ .$$

The following auxiliary result [17, Fact 3.2, p. 8] shall be needed.

Lemma 2.5. Let $c^\#$ be admissible. Let $\omega < 0$ be the growth constant of the **EXS** C_0–semigroup generated by A on H. Then for any $\sigma \in (\omega, 0]$

$$\left(t \longmapsto e^{-\sigma t}\left(\overline{P}x_0\right)(t)\right) \in \mathrm{L}^1(0,\infty)\cap \mathrm{L}^2(0,\infty) \qquad \forall x_0 \in \mathrm{H} \ .$$

As a consequence with $d \in \mathrm{H}$, $\widehat{\left(\overline{P}d\right)}(s) \in \widehat{\mathcal{A}}(\sigma)$ for any $\sigma \in (\omega, 0]$, and hence for such σ, is analytic and bounded in $\mathrm{Re}\, s > \sigma$ and thus also in a full neighborhood of $s = 0$.

In the sequel $\sigma(\cdot)$, $\sigma_P(\cdot)$, $\sigma_C(\cdot)$ will respectively denote the spectrum, the point (i.e. discrete) spectrum and the continuous spectrum of an operator. We shall need

Lemma 2.6. Let $A : (D(A) \subset \mathrm{H}) \longrightarrow \mathrm{H}$ be the generator an **EXS** C_0–semigroup $\{S(t)\}_{t\geq 0}$ on H. Then the semigroup $\{e^{tA^{-1}}\}_{t\geq 0}$, generated by $A^{-1} \in \mathbf{L}(\mathrm{H})$, is *strongly asymptotically stable* (**AS**), i.e. for every $x_0 \in \mathrm{H}$, $\lim\limits_{t\to\infty} e^{tA^{-1}}x_0 = 0$, if and only if $\{e^{tA^{-1}}\}_{t\geq 0}$ is *uniformly bounded*.

Proof. If the semigroup $\{S(t)\}_{t\geq 0}$ is **EXS**, then $\sigma_P(A^{-1}) \cap j\mathbb{R} = \emptyset$, $\sigma_P\big((A^*)^{-1}\big) \cap j\mathbb{R} = \emptyset$ and $0 \in \sigma_C(A^{-1})$ is the only possible point of the spectrum of A^{-1} on $j\mathbb{R}$. This together with the assumption that $\{e^{tA^{-1}}\}_{t\geq 0}$ is uniformly bounded gives that the semigroup $\{e^{tA^{-1}}\}_{t\geq 0}$ is **AS** by [1, Stability Theorem, p. 837], see also [25].

Conversely, if $\{e^{tA^{-1}}\}_{t\geq 0}$ is **AS**, then for all $x \in \mathrm{H}$, $\sup_{t\geq 0}\|e^{tA^{-1}}x\|_{\mathrm{H}} < \infty$, such that by the uniform boundedness principle $\sup_{t\geq 0}\|e^{tA^{-1}}\|_{\mathbf{L}(\mathrm{H})} < \infty$, whence $\{e^{tA^{-1}}\}_{t\geq 0}$ is uniformly bounded. □

Corollary 2.1. Let $A :\ (D(A) \subset \mathrm{H}) \longrightarrow \mathrm{H}$ be the generator an **EXS** C_0–semigroup $\{S(t)\}_{t\geq 0}$ on H. Then the semigroup $\{e^{tA^{-1}}\}_{t\geq 0}$ is **AS** if the operator inequality

$$\langle Ax, Xx\rangle_{\mathrm{H}} + \langle Xx, Ax\rangle_{\mathrm{H}} \leq 0 \qquad \forall x \in D(A) \ . \tag{2.6}$$

has a bounded self–adjoint solution $X = X^* \in \mathbf{L}(\mathrm{H})$ which is *coercive.*

Remark 2.2. This means that A is *similar* to a *dissipative operator*, [31, p.13] (to see this put $X = W^*W$, where $W \in \mathbf{L}(\mathrm{H})$ is a Banach isomorphism).

Proof. Put in (2.6) $X = W^*W$, where $W \in \mathbf{L}(\mathrm{H})$ is a Banach isomorphism, and set $z := WAx$. The latter defines a bijection of $D(A)$ onto H such that z is an arbitrary point of H. Hence (2.6) reduces to

$$\mathrm{Re}\langle (WA^{-1}W^{-1})z, z\rangle_{\mathrm{H}} \leq 0 \quad \forall z \in \mathrm{H} \ .$$

This means that the similar generator $WA^{-1}W^{-1}$ as well as its adjoint are dissipative. Hence by [31, Corollary 4.4, p.15], $\{e^{t(WA^{-1}W^{-1})} = We^{tA^{-1}}W^{-1}\}_{t\geq 0}$ is a *contraction* semi-group, i.e.

$$\left\|e^{t(WA^{-1}W^{-1})}\right\|_{\mathbf{L}(\mathrm{H})} \leq 1 \qquad \forall t \geq 0 \ .$$

Therefore

$$\left\|e^{tA^{-1}}\right\|_{\mathbf{L}(\mathrm{H})} \leq \left\|W^{-1}\right\|_{\mathbf{L}(\mathrm{H})} \|W\|_{\mathbf{L}(\mathrm{H})} < \infty \qquad \forall t \geq 0 \ ,$$

and the conclusion follows by Lemma 2.6. □

Corollary 2.2. Let $A : (D(A) \subset \mathrm{H}) \longrightarrow \mathrm{H}$ be the generator an **EXS** C_0–semigroup $\{S(t)\}_{t\geq 0}$ on H. Let $\sigma(A) = \overline{\sigma_P(A)}$ and let H have a Riesz basis of eigenvectors of A. Then the semigroup $\{e^{tA^{-1}}\}_{t\geq 0}$ is **AS**.

Proof. Here A is similar to the diagonal operator Λ of eigenvalues of A, which is dissipative as $\{S(t)\}_{t\geq 0}$ is **EXS**. Hence the conclusion follows by Corollary 2.1 and Remark 2.2. □

3. Asymptotic stability of the Lur'e feedback system

Consider the Lur'e feeback control system depicted in Figure 3.1, which consists of a linear part described by (2.1), and a scalar static controller nonlinearity $f : \mathbb{R} \longrightarrow \mathbb{R}$.

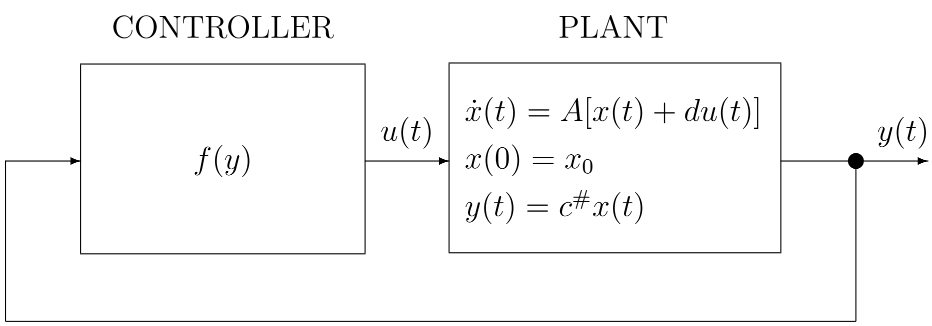

Figure 3.1. The Lur'e control system

Remark 3.1. For reasons of mathematical elegance the usual sign inversion is absent in the feedback loop of Figure 3.1. The standard set–up of the circle criterion as in e.g. [34, Section 5.6, Theorem 37 Case (iii), p. 227] is recovered by replacing $f(y)$ by $-f(y)$ and k_1 and k_2 below by respectively $-k_2$ and $-k_1$.

The aim here is to get sufficient conditions for global strong asymptotic stability for the Lur'e feedback system. For this purpose we assume:

(A1) The linear part of the feedback system from u to y is our boundary control system in factor form (2.1), where A generates an **EXS** semigroup $\{S(t)\}_{t\geq 0}$ on H, $c^{\#}$ is admissible and $d \in \mathrm{H}$; moreover conditions (2.3) and (2.5) hold. Hence, for any $x_0 \in \mathrm{H}$, its input–output equation in $\mathrm{L}^2(0,T)$ for any $T > 0$ is given by

$$y_T = \left(\overline{P}x_0\right)_T + \left(\overline{F}u\right)_T = \left(\overline{P}x_0\right)_T + \left(\overline{F}u_T\right)_T \ .$$

The last equality holds by the causality of $\overline{F}$.

(A2) There exist constants k_1 and $k_2 > k_1$ such that with

$$\delta := (1 + k_1 c^{\#} d)(1 + k_2 c^{\#} d) \geq 0, \qquad q := k_1 k_2, \qquad e := \frac{k_1 + k_2}{2} + k_1 k_2 c^{\#} d, \tag{3.1}$$

the *Lur'e system*

$$\left\{ \begin{array}{rcl} \left(A^{-1}\right)^* \mathcal{H} + \mathcal{H}A^{-1} - qhh^* & = & -gg^* \\ -\mathcal{H}d + eh & = & -\sqrt{\delta}g \end{array} \right\} \tag{3.2}$$

has a solution $(\mathcal{H}, g)$, $g \in \mathrm{H}$, $\mathcal{H} \in \mathbf{L}(\mathrm{H})$, $\mathcal{H} = \mathcal{H}^* \geq 0$, or so does the equivalent system

$$\left\{ \begin{array}{rcll} \langle Ax, \mathcal{H}x\rangle_{\mathrm{H}} + \langle x, \mathcal{H}Ax\rangle_{\mathrm{H}} & = & q\,(h^*Ax)^2 - (g^*Ax)^2 & \forall x \in D(A) \\ -\mathcal{H}d + eh & = & -\sqrt{\delta}g & \end{array} \right\} . \tag{3.3}$$

(A3) The factor control vector $d \in \mathrm{H}$ is admissible.

Next for the controller two sets describe restrictions to be imposed on the static nonlinearity $f : \mathbb{R} \longrightarrow \mathbb{R}$, namely

- For sufficiently small $\varepsilon > 0$, we define the sector:

$$\begin{aligned} \mathcal{S}_\varepsilon := \ & \Big\{ f \in \mathrm{C}(\mathbb{R}) : \ -\infty < k_1 < \frac{1}{2}\Big[k_1 + k_2 - \sqrt{(k_2 - k_1)^2 - 4\varepsilon}\Big] \leq \frac{f(y)}{y} \leq \\ \leq \ & \frac{1}{2}\Big[k_1 + k_2 + \sqrt{(k_2 - k_1)^2 - 4\varepsilon}\Big] < k_2 < \infty \qquad \forall y \in \mathbb{R} \setminus \{0\}\,,\ f(0) = 0 \Big\} \ . \end{aligned}$$

  In the sequel we shall also use $\mathcal{S}_0 := \lim_{\varepsilon \to 0+} \mathcal{S}_\varepsilon$.
- We denote by $\mathcal{M}$ the class of those functions $f \in \mathcal{S}_0 \cap \mathrm{C}(\mathbb{R})$ which are sufficiently smooth such that for the given Lur'e feedback system, for any $x_0 \in \mathrm{H}$, the truncated output y_T belongs to $\mathrm{L}^2(0,T)$ for any $T > 0$, i.e it solves the closed–loop fixed point output equation

$$y_T = \left(\overline{P}x_0\right)_T + \left(\overline{F}f(y_T)\right)_T \ .$$

Lemma 3.1. Let assumption **(A1)** hold. Let f belong to $\mathcal{M}$. Then:

(i) For any $T > 0$ the truncated output y_T and input u_T ($u = f(y)$) is in $\mathrm{L}^2(0,T)$.

(ii) If moreover **(A3)** holds, then the closed–loop state differential equation

$$\left\{ \begin{array}{rcl} \dot{x} & = & A[x + df(y)] \\ x(0) & = & x_0 \in \mathrm{H} \end{array} \right\} \tag{3.4}$$

has for any $T > 0$ the *weak* solution $x(\cdot) \in \mathrm{C}(0,T;\mathrm{H})$ given by

$$x(t) = S(t)x_0 + A\int_0^t S(t-\tau)df\big(y(\tau)\big)d\tau, \qquad t \geq 0 \ , \tag{3.5}$$

which satisfies *pointwise* in H almost everywhere

$$\left\{ \begin{array}{rcl} \dfrac{d}{dt}\big[A^{-1}x\big] = A^{-1}\dot{x} & = & x + df(y) \\ x(0) & = & x_0 \in \mathrm{H} \end{array} \right\} . \tag{3.6}$$

Proof. Ad **(i)**. As $f \in \mathcal{M}$ one gets that for any $T > 0$ the truncation y_T is in $\mathrm{L}^2(0,T)$ and moreover

$$\|u\|_{\mathrm{L}^2(0,T)} \leq \max\{|k_1|, |k_2|\}\, \|y\|_{\mathrm{L}^2(0,T)}$$

Ad **(ii)**. By **(i)** and **(A3)** formula (3.5) of the weak solution $x(\cdot) \in \mathrm{C}(0,T;\mathrm{H})$ of (3.4) holds by the analysis in [16, Section 4.2]. Next by Fubini's theorem with $u = f(y)$

$$\int_0^t [x(\tau) + du(\tau)]\, d\tau = A^{-1}[x(t) - x_0], \qquad t \geq 0 \ ,$$

such that, by Lebesgue's differentiation theorem for vector–valued functions, (3.6) holds pointwise in H almost everywhere. $\square$

Remark 3.2. For $w \in D(A^*)$ one has

$$\frac{d}{dt}\langle w, x\rangle_{\mathrm{H}} = \frac{d}{dt}\langle A^*w, A^{-1}x\rangle_{\mathrm{H}} = \langle A^*w, A^{-1}\dot{x}\rangle_{\mathrm{H}} = \langle A^*w, x + df(y)\rangle_{\mathrm{H}}$$

whence any pointwise a.e. solution of (3.6) is a weak solution of (3.4). The point is that the converse holds.

Theorem 3.1. Let assumptions **(A1)**$\div$**(A3)** hold. Let f belong to $\mathcal{S}_\varepsilon \cap \mathcal{M}$. Then the origin of the space H is *globally strongly asymptotically stable* (**GSAS**).

Proof. The objective is to get the quadratic form $V(x) = x^*\mathcal{H}x$ as a Lyapunov functional for the system (3.6). With $f \in \mathcal{M}$ and $u = f(y)$ its derivative along the solutions of (3.6) reads as

$$\begin{aligned}\dot{V} = \;& \dot{x}^*\mathcal{H}x + x^*\mathcal{H}\dot{x} = \dot{x}^*\mathcal{H}(A^{-1}\dot{x} - du) + (A^{-1}\dot{x} - du)^*\mathcal{H}\dot{x} = \\ = \;& \begin{bmatrix} \dot{x} \\ u \end{bmatrix}^* \begin{bmatrix} \mathcal{H}A^{-1} + (A^{-1})^*\mathcal{H} & -\mathcal{H}d \\ -d^*\mathcal{H} & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ u \end{bmatrix} .\end{aligned}$$

Moreover with $y = c^\# x = c^\# A^{-1}\dot{x} - c^\# df(y) = h^*\dot{x} - c^\# du$

$$[k_2 y - u][u - k_1 y] = [k_2 h^*\dot{x} - (k_2 c^\# d + 1)u][(k_1 c^\# d + 1)u - k_1 h^*\dot{x}] \;,$$

which by adding and subtracting gives

$$\dot{V} = -(k_2 y - u)(u - k_1 y) + \begin{bmatrix} \dot{x} \\ u \end{bmatrix}^* \begin{bmatrix} \mathcal{H}A^{-1} + (A^{-1})^*\mathcal{H} - qhh^* & -\mathcal{H}d + eh \\ -d^*\mathcal{H} + eh^* & -\delta \end{bmatrix} \begin{bmatrix} \dot{x} \\ u \end{bmatrix} .$$

Hence **(A2)** gives

$$\dot{V} = -\left[g^*\dot{x} + \sqrt{\delta}u\right]^2 - (k_2 y - u)(u - k_1 y) \le 0 \;.$$

Thus for $f \in \mathcal{M}$, V is a Lyapunov functional for the system (3.6). Now, due to $f \in \mathcal{S}_\varepsilon$, we have

$$\dot{V} = -\left[g^*\dot{x} + \sqrt{\delta}u\right]^2 - (k_2 y - u)(u - k_1 y) \le -\varepsilon y^2 \;. \tag{3.7}$$

This is because $\left\{\frac{1}{2}\left[k_1 + k_2 - \sqrt{(k_2 - k_1)^2 - 4\varepsilon}\right]y - u\right\}\left\{u - \frac{1}{2}\left[k_1 + k_2 + \sqrt{(k_2 - k_1)^2 - 4\varepsilon}\right]y\right\}$ $= (k_2 y - u)(u - k_1 y) - \varepsilon y^2$

Integrating both sides of (3.7) from 0 to t and using $\mathcal{H} \ge 0$ we obtain,

$$-V(x_0) \le V[x(t, x_0)] - V(x_0) \le -\varepsilon \int_0^t y^2(\tau)d\tau$$

whence

$$\|\mathcal{H}\|_{\mathbf{L}(\mathrm{H})} \|x_0\|_{\mathrm{H}}^2 \ge V(x_0) \ge \varepsilon \int_0^t y^2(\tau)d\tau \;.$$

This yields

$$\|y\|_{\mathrm{L}^2(0,\infty)} \le \sqrt{\frac{1}{\varepsilon}\|\mathcal{H}\|_{\mathbf{L}(\mathrm{H})}} \|x_0\|_{\mathrm{H}} \;.$$

Since $f \in \mathcal{S}_0$

$$\int_0^\infty u^2(t)dt = \int_0^\infty f^2[y(t)]dt = \int_0^\infty y^2(t)\frac{f^2[y(t)]}{y^2(t)}dt \le \max\left\{k_2^2, k_1^2\right\} \|y\|_{\mathrm{L}^2(0,\infty)}^2 \;,$$

whence

$$\|u\|_{\mathrm{L}^2(0,\infty)} \leq \max\{|k_2|,|k_1|\}\sqrt{\frac{1}{\varepsilon}\|\mathcal{H}\|_{\mathbf{L}(\mathrm{H})}}\,\|x_0\|_{\mathrm{H}} \quad . \tag{3.8}$$

Hence there holds that $y, u \in \mathrm{L}^2(0,\infty)$.

Since, by **(A3)**, $d \in \mathrm{H}$ is an admissible factor control vector, then

$$x(t) = S(t)x_0 + QR_t u \qquad t \geq 0 \tag{3.9}$$

where $Q \in \mathbf{L}(\mathrm{L}^2(0,\infty),\mathrm{H})$ is the reachability map of Remark 2.1 and $R_t \in \mathbf{L}(\mathrm{L}^2(0,\infty))$ denotes the reflection operator at $t > 0$,

$$(R_t u)(\tau) := \left\{ \begin{array}{cc} u(t-\tau), & \tau \in [0,t) \\ 0, & \tau \geq t \end{array} \right\}, \qquad \|R_t\|_{\mathbf{L}(\mathrm{L}^2(0,\infty))} \leq 1 \ .$$

There holds that $0 \leq t \longmapsto x(t) \in \mathrm{H}$ is strongly continuous. Using (3.8) and recalling that the exponential stability of the semigroup $\{S(t)\}_{t\geq 0}$ implies by the principle of uniform boundedness its stability (i.e the uniform boundedness for $t \geq 0$), we conclude that there exists a constant $\gamma > 0$, such that

$$\|x(t)\|_{\mathrm{H}} \leq \gamma\,\|x_0\|_{\mathrm{H}} \qquad \forall x_0 \in \mathrm{H}, \quad \forall t \geq 0 \ . \tag{3.10}$$

The stability of the null equilibrium easily follows from (3.10).

Considering state–attraction to zero, there holds that $\|S(t)x_0\|_{\mathrm{H}}$ tends to zero as $t \to \infty$ for any $x_0 \in \mathrm{H}$. Hence we may without loss of generality consider $x(t) = QR_t u$. For any fixed $u \in \mathrm{L}^2(0,\infty)$ define for $t_1 > 0$

$$u_{t_1}(t) := \left\{ \begin{array}{cc} 0, & t \in [0,t_1) \\ u(t), & t \geq t_1 \end{array} \right\} \ .$$

One gets then that for $t \geq t_1$

$$x(t) = QR_t u = S(t-t_1)QR_{t_1}u + QR_t u_{t_1} \ ,$$

where $\{S(t)\}_{t\geq 0}$ is **EXS**,

$$\|QR_t u_{t_1}\|_{\mathrm{H}} \leq \|Q\|_{\mathbf{L}(\mathrm{L}^2(0,\infty),\mathrm{H})}\,\|R_t\|_{\mathbf{L}(\mathrm{L}^2(0,\infty))}\,\|u_{t_1}\|_{\mathrm{L}^2(0,\infty)} \leq \|Q\|_{\mathbf{L}(\mathrm{L}^2(0,\infty),\mathrm{H})}\,\|u_{t_1}\|_{\mathrm{L}^2(0,\infty)} \ ,$$

and $\|u_{t_1}\|_{\mathrm{L}^2(0,\infty)}$ can be made arbitrarily small for t_1 sufficiently large. Therefore similarly as in the proof of [27, Lemma 12] one gets that $\lim_{t\to\infty}\|x(t)\|_{\mathrm{H}} = 0$. □

The final part of the proof of Theorem 3.1 relies on the observation that the admissibility of d implies two facts:

- ✎ $QR_{(\cdot)} \in \mathbf{L}(\mathrm{L}^2(0,\infty),\mathrm{L}^2(0,\infty;\mathrm{H}))$,
- ✎ $\lim_{t\to\infty}\|QR_t u\|_{\mathrm{H}} = 0 \quad \forall u \in \mathrm{L}^2(0,\infty)$ (or even $\mathrm{Range}(QR_{(\cdot)}) \subset \mathrm{BUC}_0[0,\infty;\mathrm{H})$ [18]).

Here we tacitly assumed the worst case where the controller can generate any control signal $u \in \mathrm{L}^2(0,\infty)$. In fact the control signal is being produced from the feedback loop as it is explained by the operator–theoretic diagram depicted in Figure 3.2.

Predicting some facts which will be rigorously stated in Section 6 we announce here that in an abstract parabolic case where A generates an **EXS** analytic semigroup all system operators: S, $QR_{(\cdot)}$, $\overline{P}$ and $\overline{F}$ feature some balanced smoothing properties. Thanks to this even if d is not admissible one can still have $QR_{(\cdot)} \in \mathbf{L}(\mathrm{L}^2(0,\infty),\mathrm{L}^2(0,\infty;\mathrm{H}))$ and $\lim_{t\to\infty}\|QR_t u\|_{\mathrm{H}} = 0$ now only for controls truly generated by the feedback. These last two

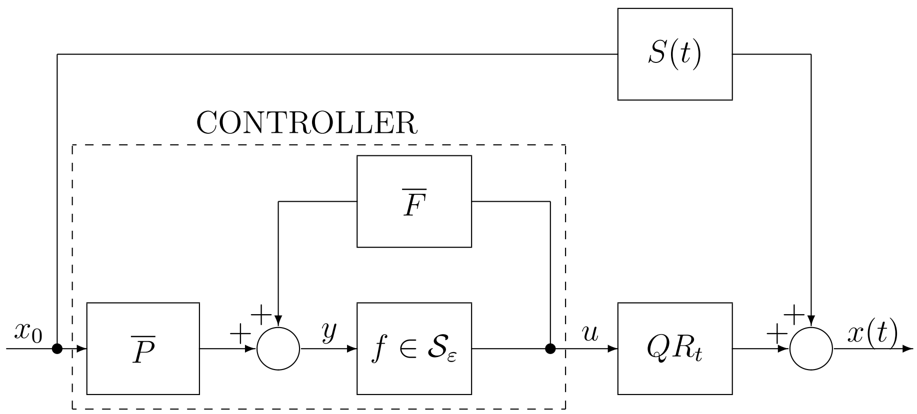

FIGURE 3.2. The operator–theoretic diagram of the Lur'e control system

facts turn out to be sufficient for proving that the origin is **GSAS** under some assumptions imposed on d which are essentially weaker than **(A3)**.

4. SUFFICIENT CRITERION FOR SOLVABILITY OF THE LUR'E SYSTEM OF EQUATIONS

In this section we shall get sufficient conditions for checking **(A2)**, i.e. for the solvability of the Lur'e system of equations (3.2) or equivalently (3.3) with respect to the pair $(\mathcal{H}, g)$. Our inspiration stems from the Oostveen and Curtain Riccati results in [27], modulo adaptation to our context, where d is not supposed to be admissible i.e. **(A3)** does not hold as an intellectual challenge motivated by the "parabolic regularity" mentioned above and examined in detail in Section 6 where d is not admissible.

The method for getting our main result Theorem 4.1 is as in the spectral factorization method for solving the Riccati equation of Callier and Winkin [7], modulo the transfer function mapping $\hat{g}(s) \longmapsto \hat{g}(s^{-1}) - \hat{g}(0)$. Spectral factorization is handled first. Some other preliminary results follow next, and finally we get our result.

4.1. Spectral factorization.

The following result is important in our context.

Lemma 4.1. Let $\omega \longmapsto \pi(j\omega)$ be a real–valued, nonnegative function on the $j\omega$–axis such that π belongs to $\mathrm{L}^\infty(\mathbb{R})$ and $\pi(j\omega) = \pi(-j\omega)$. Let in addition π be *coercive*, i.e. there exists an $\varepsilon > 0$ such that $\pi(j\omega) \geq \varepsilon$ for all $\omega \in \mathbb{R}$. Then:

(i) There exists a function $\phi \in \mathrm{H}^\infty(\Pi^+)$ such that

$$\pi(j\omega) = \phi(j\omega)\phi(-j\omega) = |\phi(j\omega)|^2 \quad , \tag{4.1}$$

and $1/\phi$ is as well in $\mathrm{H}^\infty(\Pi^+)$. Moreover $\phi(s)$ can be chosen to be *real*, i.e. it satisfies $\phi(s) = \overline{\phi(\overline{s})}$, meaning that a Taylor expansion in the open right–half plane has real coefficients or that $\phi(s)$ takes real values for real arguments; furthermore such $\phi(s)$ is unique modulo a ± 1 factor.

(ii) If moreover $\pi(j\omega)$ has an analytic extension in a domain containing a full neighborhood of $s = 0$ which is para–Hermitian self–adjoint (i.e. $\pi(s) = \pi(-s)$), then

$$\left(s \longmapsto \frac{\phi(s) - \phi(0)}{s}\right) \in \mathrm{H}^\infty(\Pi^+) \cap \mathrm{H}^2(\Pi^+) \quad . \tag{4.2}$$

and the factor $\phi(s)$ of assertion **(i)** is unique by the normalization condition $\phi(0) = \sqrt{\pi(0)}$.

Remark 4.1. π is called a *spectral density function* and ϕ is called a *spectral factor*. Moreover equation (4.1) is called a *spectral factorization equation*, and the problem of finding a spectral factor is the *spectral factorization problem*.

Proof of Lemma 4.1. Part **(i)** is well–known and thus its proof is only roughly sketched. It is traditionally first obtained on the unit circle of the complex z–plane and then solved on the imaginary axis of the complex s–plane by using a linear fractional transformation $z = (s-1)^{-1}(s+1)$ which maps bijectively the closed right–half plane onto the closed unit disc. Results are associated with G.Szegö, see especially [21, two Theorems, p. 53; Chapter 8], [20, Subsection 1.14], and [32, Section 6.1][1]. Accordingly one gets a spectral factor $\phi \in \mathrm{H}^\infty(\Pi^+)$ satisfying (4.1) and $1/\phi \in \mathrm{H}^\infty(\Pi^+)$ is an *invertible outer function* given by Szegö's formula

$$\phi(s) = c\exp\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{sj\omega-1}{j\omega-s}\frac{\ln\pi(j\omega)}{1+\omega^2}d\omega\right], \qquad s\in\Pi^+$$

where $c \in \mathbb{C}$ is a constant of modulus 1. Call $\psi(s)$ the above exponential function. Then it follows readily that ψ is *real*, i.e. $\psi(s) = \overline{\psi(\overline{s})}$ and by restricting $c \in \mathbb{R}$, it follows that ϕ is a *real* spectral factor.

Concerning **(ii)** note that, since the spectral density function has a para–Hermitian self–adjoint analytic extension in a domain containing a full neighbourhood of $s = 0$, then we have there the factorization

$$\pi(s) = \phi(s)\phi(-s)\ , \tag{4.3}$$

with $\phi(s)$ regular at $s = 0$ (this can be seen by considering the successive self–adjoint polynomial approximations and their factorizations of the Taylor expansion π near zero). This jointly with $\phi \in \mathrm{H}^\infty(\Pi^+)$ leads to the fact that the function $s \longmapsto \dfrac{\phi(s)-\phi(0)}{s}$ is analytic and bounded in a full neighborhood of $s = 0$ and finally is in $\mathrm{H}^\infty(\Pi^+)\cap\mathrm{H}^2(\Pi^+)$. Due to the analyticity of $\phi(s)$ at $s = 0$ an outer spectral factorization of the statement **(i)** can be made unique by the normalization condition $\phi(0) = \sqrt{\pi(0)} > 0$. □

4.2. **State–feedback realization problem.** The following assumptions hold, where the first four ones are equivalent to **(A1)**:

(H1) The operator $A : (D(A) \subset \mathrm{H}) \longrightarrow \mathrm{H}$ generates an **EXS** linear C_0–semigroup on H;
(H2) The compatibility condition (2.3) holds;
(H3) The observation functional $c^\#$ is admissible, $c^\#\big|_{D(A)} = h^*A$;
(H4) The transfer function $\hat{g}$, defined by (2.4), satisfies (2.5);
(H5) There exist k_1, k_2, $k_1 < k_2$ such that the *Popov function*

$$\begin{aligned}\pi(j\omega) := \;& 1-(k_1+k_2)\operatorname{Re}[\hat{g}(j\omega)] + k_1k_2\left|\hat{g}(j\omega)\right|^2 = \\ =\;& \delta - 2e\operatorname{Re}[\hat{g}(j\omega)+c^\# d] + q\left|\hat{g}(j\omega)+c^\# d\right|^2, \qquad \omega\in\mathbb{R}\ ,\end{aligned} \tag{4.4}$$

satisfies coercivity condition[2]

$$\pi(j\omega) \geq \varepsilon > 0 \qquad \forall\omega\in\mathbb{R}\ . \tag{4.5}$$

[1]For operator–valued results see e.g. [11, Theorem 5] or [32, Theorem 3.7, Theorem 6.14].

[2]If $k_1k_2 < 0$ then the frequency–domain inequality (4.5) means geometrically that the plot of the transfer function $\hat{g}(j\omega)$ is located inside the circle with centre at $(k_1^{-1}+k_2^{-1})/2$ and radius $(k_2^{-1}-k_1^{-1})/2$. In particular, this yields $\hat{g} \in \mathrm{H}^\infty(\Pi^+)$.

Note that as $\hat{g} \in \mathrm{H}^\infty(\Pi^+)$ and $\hat{g}(s) = \overline{\hat{g}(\overline{s})}$ one gets $\pi \in \mathrm{L}^\infty(\mathbb{R})$, $\pi(j\omega) = \pi(-j\omega)$. It follows from Lemma 4.1 that the spectral factorization problem (4.1) with the Popov spectral density function π has a solution ϕ such that $1/\phi$ in $\mathrm{H}^\infty(\Pi^+)$. Furthermore, as $\hat{g}(s) + c^\# d = s\big(\widehat{Pd}\big)(s)$, it follows by Lemma 2.5 that the Popov function has a para–Hermitian self–adjoint analytic extension in a domain containing a full neighborhood of $s = 0$ which reads

$$\begin{aligned} \pi(s) := \;& 1 - \frac{(k_1+k_2)}{2}[\hat{g}(s) + \hat{g}(-s)] + k_1k_2\hat{g}(s)\hat{g}(-s) = \\ =\;& \delta - es\left[\big(\widehat{Pd}\big)(s) - \big(\widehat{Pd}\big)(-s)\right] - qs^2\big(\widehat{Pd}\big)(s)\big(\widehat{Pd}\big)(-s) \ . \end{aligned} \tag{4.6}$$

Hence $\pi(0) = \delta > 0$ and again by Lemma 4.1 the spectral factorization problem is uniquely solvable by adding the requirement $\phi(0) = \sqrt{\pi(0)} = \sqrt{\delta}$ and

$$\left(s \longmapsto \frac{\phi(s) - \phi(0)}{s} = \frac{\phi(s) - \sqrt{\delta}}{s}\right) \in \mathrm{H}^\infty(\Pi^+) \cap \mathrm{H}^2(\Pi^+) \ .$$

Henceforth given **(H1)÷(H5)**, we call *realization problem* that of finding a $g \in \mathrm{H}$ satisfying the identity:

$$\frac{\phi(s) - \sqrt{\delta}}{s} = g^* A(sI - A)^{-1}d \qquad \forall s \in \Pi^+ \ , \tag{4.7}$$

where $\phi \in \mathrm{H}^\infty(\Pi^+)$ is that spectral factor of the Popov density function π which satisfies $1/\phi \in \mathrm{H}^\infty(\Pi^+)$ and ϕ is analytic at $s = 0$ with $\phi(0) = \sqrt{\delta}$ (the outer normalized spectral factor). The realization equation (4.7) is equivalent to

$$\phi(s^{-1}) = \sqrt{\delta} - g^*(sI - A^{-1})^{-1}d \qquad \forall s \in \Pi^+ \setminus \{0\} \ . \tag{4.8}$$

This will turn out to be a realization of the spectral factor of the Popov function in the proof of Theorem 4.1 due to the Oostveen and Curtain Lemma 4.3: in that proof it is seen that g^* is proportional to a state–feedback operator dictated by a solution of a Riccati equation.

Lemma 4.2. If the pair (A^{-1}, d) is *approximately reachable* i.e. $\overline{\mathrm{Span}\,\{A^{-n}d\}_{n=0}^\infty} = \mathrm{H}$ then the realization problem (4.7), or its equivalent form (4.8) has at most one solution.

Proof. Indeed, if there were two solutions g_1 and g_2 then we would have

$$[g_1 - g_2]^*(sI - A^{-1})^{-1}d = 0 \qquad \forall s \in \Pi^+ \setminus \{0\}$$

and by approximate reachability: $g_1 = g_2$. □

4.3. **Sufficient criterion using a strict circle inequality.** The proof of the Riccati results [27, Theorem 19 and Corollary 20] of Oostveen and Curtain contains the lemma below, where the admissibility of the *bounded* observation and control operators C and B is as in the beginning of Section 2[3]. Other infinite–dimensional Riccati results exist, e.g. [33], [37], [38], but their application in the proof of Theorem 4.1 is not obvious.

[3]For the case that the Popov function is nonnegative but not coercive, see [9] as a complement of information.

Lemma 4.3. Let $A : (D(A) \subset \mathrm{H}) \longrightarrow \mathrm{H}$ generate an **AS** linear C_0–semigroup on H, let $B \in \mathbf{L}(\mathrm{U},\mathrm{H})$, let $C \in \mathbf{L}(\mathrm{H},\mathrm{Y})$ be an admissible observation operator, let the transfer function

$$G(s) := C(sI - A)^{-1}B \tag{4.9}$$

belong to $\mathrm{H}^\infty(\Pi^+, \mathbf{L}(\mathrm{U},\mathrm{Y}))$ and $Q \in \mathbf{L}(\mathrm{Y})$, $Q = Q^*$, $N \in \mathbf{L}(\mathrm{Y},\mathrm{U})$, $R \in \mathbf{L}(\mathrm{U})$, $R = R^* \geq \eta I > 0$ such that the *Popov function*

$$\Pi(j\omega) := R + NG(j\omega) + [NG(j\omega)]^* + [G(j\omega)]^* QG(j\omega), \qquad \omega \in \mathbb{R} \tag{4.10}$$

is in $\mathrm{L}^\infty(\mathbb{R}, \mathbf{L}(\mathrm{U}))$. Assume moreover that the Popov function is *coercive* i.e.

$$\Pi(j\omega) \geq \varepsilon I > 0 \qquad \forall \omega \in \mathbb{R} \tag{4.11}$$

Then the *operator Riccati equation*:

$$A^*Xx + XAx - (B^*X + NC)^* R^{-1}(B^*X + NC)x + C^*QCx = 0 \qquad \forall x \in D(A) \tag{4.12}$$

has a self-adjoint bounded solution $X = X^*$, $X \in \mathbf{L}(\mathrm{H})$,

$$X = \Psi^* \mathcal{T} \Psi, \qquad \mathcal{T} := Q - (Q\mathbb{F} + N^*)\mathcal{R}^{-1}(Q\mathbb{F} + N^*)^* \ , \tag{4.13}$$

where Ψ and $\mathbb{F}$ are, respectively, the extended observability map and the extended input–output map associated with the system triple (A, B, C) and

$$\mathcal{R} := R + N\mathbb{F} + \mathbb{F}^*N + \mathbb{F}Q\mathbb{F}^*$$

is the *Toeplitz operator* with the Popov function Π as its *symbol*, and such that with

$$F_X := -R^{-1}(B^*X + NC) \in \mathbf{L}(\mathrm{H},\mathrm{U}) \tag{4.14}$$

there holds: W, $W^{-1} \in \mathrm{H}^\infty(\Pi^+, \mathbf{L}(\mathrm{U}))$, where

$$W(s) := I - F_X(sI - A)^{-1}B \ , \tag{4.15}$$

and

$$\Pi(j\omega) = W(j\omega)^* RW(j\omega) \qquad \forall \omega \in \mathbb{R} \ . \tag{4.16}$$

Remark 4.2. Here F_X is a state–feedback operator and $W(s)$ is the control loop return difference induced by $u = F_X x$. To prove that $W \in \mathrm{H}^\infty(\Pi^+, \mathbf{L}(\mathrm{U}))$, one needs according to the proof of [27, Theorem 19, pp. 961 - 962], to revisit the proof of [27, Lemma 18]. The arguments in the latter proof use only the fact that B is *finite–time admissible* (which is the case as B is bounded) whence one can guarantee that the spectral factorization W in (4.16) has a realization (A, B, C_W, D_W) with bounded operators B, C_W, D_W resulting in a well–defined extended output equation

$$y = \Psi_W x_0 + \mathbb{F}_W u$$

where $\Psi_W \in \mathbf{L}(\mathrm{H}, \mathrm{L}^2(0,\infty))$ and $\mathbb{F}_W \in \mathbf{L}(\mathrm{L}^2(0,\infty))$ with

$$\Psi_W x_0 = (\mathbb{F}_W^*)^{-1}(\mathbb{F}^*Q + N)\Psi x_0$$

and

$$\left(\widehat{\mathbb{F}_W u}\right)(j\omega) = W(j\omega)\hat{u}(j\omega) \ .$$

Using this result in the proof of [27, Theorem 19] it turns out that $C_W = -F_X$ and $D_W = I$ and that $W \in \mathrm{H}^\infty(\Pi^+, \mathbf{L}(\mathrm{U}))$. Thus here the solution of the Riccati equation is stabilizing in the sense that the latter property holds, i.e. *the control loop return difference stabilizing property*. If the pair (A, B) is reachable then such solution is unique.

We have not assumed that B is admissible, because in the context of Theorem 4.1 this would require that d is admissible, which we do not assume. If B is admissible, then X is a *unique strongly stabilizing solution* [27], where in particular $A+BF_X$ is the generator of an **AS** semigroup obtained by the state–feedback $u = F_X x$.

Theorem 4.1. Let assumptions **(H1)÷(H5)** hold. Moreover assume that:

(H6) The operator $A : (D(A) \subset \mathrm{H}) \longrightarrow \mathrm{H}$ is such that the semigroup generated by A^{-1} is **AS**;

Then:

(i) The system (3.2) has a solution $(\mathcal{H}, g)$, $\mathcal{H} \in \mathbf{L}(\mathrm{H})$, $\mathcal{H} = \mathcal{H}^* \geq 0$, $g \in \mathrm{H}$, where in particular: g is the solution of the realization equation (4.7), where ϕ is the spectral factor of the Popov function π (given by (4.4)) such that $\phi(0) = \sqrt{\delta}$, and both ϕ and $1/\phi$ are in $\mathrm{H}^\infty(\Pi^+)$;

(ii) Assume that the pair (A^{-1}, d) is approximately reachable. Then this g can be obtained by solving the realization problem (4.7) or its equivalent form (4.8), while $\mathcal{H}$ can then be determined by solving the first (i.e. Lyapunov) equation of (3.2).

Proof. Ad **(i)**. Consider Lemma 4.3. Set U and Y equal to $\mathbb{R}$ and replace the triples (A, B, C) and (Q, N, R) respectively by $(A^{-1}, d, -h^*)$ and $(q, -e, \delta)$, where by (4.5) $\pi(0) = \delta > 0$. Then:

❶ The semigroup $\{e^{tA^{-1}}\}_{t\geq 0}$ is **AS** by assumption **(H6)**;

❷ The admissibility of h^* with respect to the semigroup $\{e^{tA^{-1}}\}_{t\geq 0}$ follows by the admissibility of $c^{\#}|_{D(A)} = h^*A$ with respect to the semigroup generated by A valid by **(H3)**. Indeed, by the unitary operator $U \in \mathbf{L}(\mathrm{H}^2(\Pi^+))$ defined in (2.2), $\big(\widehat{\overline{P}x_0}\big)(s) = h^*A(sI-A)^{-1}x_0$ is mapped into $\big(\widehat{U\overline{P}x_0}\big)(s) = -h^*(sI-A^{-1})^{-1}x_0$. Since $c^{\#}$ is admissible then by the Paley–Wiener theory $h^*e^{(\cdot)A^{-1}}x_0 \in \mathrm{L}^2(0,\infty)$ for any $x_0 \in \mathrm{H}$ and the claim follows from Lemma 2.1;

❸ The transfer function (4.9) gives

$$G(s) = -h^*(sI-A^{-1})^{-1}d = s^{-1}h^*A(s^{-1}I-A)^{-1}d \ , \tag{4.17}$$

whence

$$G(s) = \hat{g}(s^{-1}) + c^{\#}d = \hat{g}(s^{-1}) - \hat{g}(0) \ , \tag{4.18}$$

where $\hat{g}$ is the transfer function in (2.4). The transfer function described in (4.17) and (4.18) is in $\mathrm{H}^\infty(\Pi^+)$ due to **(H4)**;

❹ The Popov function (4.10) reads

$$\Pi(j\omega) = \delta - 2e\,\mathrm{Re}[G(j\omega)] + q\,|G(j\omega)|^2 \qquad \forall\omega \in \mathbb{R} \ ,$$

such that by (4.18) and (4.4)

$$\Pi(j\omega) = \pi\big((j\omega)^{-1}\big) \qquad \forall\omega \in \mathbb{R}\setminus\{0\} \ ; \tag{4.19}$$

The Popov function Π satisfies the coercivity condition (4.11) by (4.19) and **(H5)**. All assumptions of Lemma 4.3 are valid and applying the latter gives that the Riccati operator equation (4.12), which reads here as

$$(A^{-1})^*X + XA^{-1} - \frac{1}{\delta}(Xd+eh)(Xd+eh)^* + qhh^* = 0 \ , \tag{4.20}$$

has a solution $X = X^* \in \mathbf{L}(\mathrm{H})$. The symbol of the Toeplitz operator $\mathcal{T}$, defined in (4.13), reads with $U = Y = \mathbb{R}$ as

$$\begin{aligned}\mathcal{T}(j\omega) = &\ \ Q - [QG(j\omega) + N^*]\,\Pi^{-1}(j\omega)\,[QG(j\omega) + N^*]^* = -(N^2 - QR)\Pi^{-1}(j\omega) = \\ = &\ \ -(e^2 - q\delta)\Pi^{-1}(j\omega) = -\frac{1}{4}(k_2 - k_1)^2\,\Pi^{-1}(j\omega) \qquad \forall \omega \in \mathbb{R}\ ,\end{aligned}$$

whence by (4.13) $X \le 0$. This solution is such that with

$$F_X = -\frac{1}{\delta}(d^*X + eh^*) \tag{4.21}$$

there holds: $W, 1/W \in \mathrm{H}^\infty(\Pi^+)$, $\Pi(j\omega) = \dfrac{1}{\delta}|W(j\omega)|^2$ for all $\omega \in \mathbb{R}$, where

$$W(s)\sqrt{\delta} = 1 - F_X(sI - A^{-1})^{-1}d\ . \tag{4.22}$$

Hence by (4.20), (4.21) the pair $(\mathcal{H}, g)$, $\mathcal{H} := -X \ge 0$, $g := \sqrt{\delta}F_X^*$ is a solution of (3.2)[4]. Next the function $\phi(s) := \sqrt{\delta}W(s^{-1})$ is in $\mathrm{H}^\infty(\Pi^+)$ jointly with $1/\phi$ and by (4.19) ϕ satisfies (4.1). As $A^{-1} \in \mathbf{L}(\mathrm{H})$, W is analytic at $\{\infty\}$ and takes the value 1 at $\{\infty\}$, i.e. $\lim\limits_{|s|\to\infty} W(s) = 1$, whence ϕ is analytic at 0 and $\lim\limits_{s\to 0}\phi(s) =: \phi(0) = \sqrt{\delta}$. Finally it follows from (4.22) and (4.21) that g satisfies the realization equation (4.8).

Ad **(ii)**. By **(i)** and Lemma 4.2 the realization equation (4.8) has a unique solution (uniquely determined by the spectral factor ϕ),

$$g := -\frac{1}{\sqrt{\delta}}(-\mathcal{H}d + eh)\ ,$$

where $\mathcal{H}$ is a solution of the Riccati operator equation

$$(A^{-1})^*\mathcal{H} + \mathcal{H}A^{-1} + \frac{1}{\delta}(-\mathcal{H}d + eh)(-\mathcal{H}d + eh)^* - qhh^* = 0\ .$$

Hence we conclude that the second element in the pair $(\mathcal{H}, g)$ being a solution of (3.2) can be determined by solving the realization problem, while the first element can then be determined by solving the first (i.e. Lyapunov) equation of (3.2). □

Remark 4.3. Assertion **(ii)** of Theorem 4.1 is important in that it facilitates finding a solution $(\mathcal{H}, g)$ of the Lur'e system (3.2). Indeed as stated, g found from the realization equation can be inserted into the right–hand side of the first (i.e. Lyapunov) equation of the Lur'e system (3.2) to get $\mathcal{H}$, avoiding solving the open–loop Riccati operator equation.

The following example shows that if the pair (A^{-1}, d) is not approximately reachable, then one still can compute an appropiate solution of the Lur'e system (3.2) by using the realization equation and reachable restrictions. We are inspired here by [7, Section 3]. Consider $\mathrm{H} = \mathbb{R}^2$, $\mathrm{U} = \mathrm{Y} = \mathbb{R}$. Let

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix},\quad d = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix},\quad c^\# = \begin{bmatrix} 1 \\ 1 \end{bmatrix}^T \rightsquigarrow h = \begin{bmatrix} -1 \\ -\frac{1}{2} \end{bmatrix},\quad k_1 = -1,\quad k_2 = 0\ .$$

---

[4]Let $\Pi(j\omega) = |M(j\omega)|^2$, where both M and $1/M$ are in $\mathrm{H}^\infty(\Pi^+)$. Then

$$\langle x, \mathcal{H}x\rangle_{\mathrm{H}} = -\langle \Psi x, \mathcal{T}\Psi x\rangle_{\mathrm{L}^2(0,\infty)} = \left\|\frac{1}{2}(k_2 - k_1)M^{-1}(j\omega)\big(\widehat{\Psi x}\big)(j\omega)\right\|^2_{\mathrm{H}^2(\Pi^+)} \qquad \forall x \in \mathrm{H}$$

displays how spectral factorization defines $\mathcal{H}$.

Here $c^{\#}d = \frac{1}{2} = \delta = -e$, $\hat{g}(s) = \frac{-1}{s+2}$ and in (4.5) we can take $\varepsilon = \frac{1}{2}$. The pair

$$\mathcal{H} = \frac{3-2\sqrt{2}}{8}\begin{bmatrix} 9 & 4\sqrt{2} \\ 4\sqrt{2} & 4 \end{bmatrix}, \qquad g = \frac{\sqrt{2}-1}{2}\begin{bmatrix} -3 \\ -\sqrt{2} \end{bmatrix}$$

solves (3.2) and via (4.21) and (4.22) is associated with the spectral factor $\phi(s) = \frac{s+\sqrt{2}}{s+2}$ satisfying $\phi, 1/\phi \in \mathrm{H}^\infty(\Pi^+)$ and normalized by $\phi(0) = 1/\sqrt{2}$. However since the pair (A^{-1}, d) is not approximately reachable only the second component of g, i.e g_2, can be recovered from the realization equation. However this component defines the reachable part $\begin{bmatrix} 0 & g_2 \end{bmatrix}^T$ of g and this enables us by solving the reachable restriction (i.e. here the (2,2)–entry equation) of the first Lyapunov equation of (3.2) to recover the reachable part of $\mathcal{H}$, viz. element h_{22}. Backsubstitution of this explicit element into the Riccati equation allows then (by solving linear equations) to find the remaining elements of $\mathcal{H}$ and finally of g.

We are now ready for two examples in which the function π, given by (4.4), will first be tested for the condition

$$\pi(j\omega) \geq 0 \qquad \forall \omega \in \mathbb{R} \ , \tag{4.23}$$

which is weaker than the coercivity condition (4.5).

5. Example 1: Distortionless loaded $RLCG$–transmission line

In this section we discuss an electrical transmission line as a plant in Figure 3.1 illustrating hereby the results of the previous sections.

The *distortionless* transmission line is a $RLCG$ line for which $\alpha := R/L = G/C$. Following [17, Subsection 5.1] consider such line loaded by a resistance $R_0 > 0$. Recall that the system dynamics can be described by

$$\left\{ \begin{array}{rcl} w(t) & = & C_S w(t-r) + u(t) b_0 \\ y(t) & = & c_0^T w(t-r) \end{array} \right\} \tag{5.1}$$

where

$$C_S = \begin{bmatrix} 0 & 1 \\ -b & 0 \end{bmatrix}, \qquad b = \frac{\kappa}{\rho^2}, \qquad \kappa = \frac{R_0 - z}{R_0 + z}, \qquad z = \sqrt{\frac{L}{C}}, \qquad \rho = e^{\alpha r} \ ,$$

$$b_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad c_0 = \begin{bmatrix} 0 \\ a \end{bmatrix}, \qquad a = \frac{1+\kappa}{\rho} > 0 \ .$$

By using the Hilbert space $\mathrm{H} = \mathrm{L}^2(-r,0) \oplus \mathrm{L}^2(-r,0)$ with $r = \sqrt{LC}$ equipped with the standard scalar product, one can convert its dynamics into an abstract model in factor form as in (2.1). More precisely:

- The state space operator A takes the form

$$Ax = x', \qquad D(A) = \Big\{ x \in \mathrm{W}^{1,2}(-r,0) \oplus \mathrm{W}^{1,2}(-r,0) : \ x(0) = C_S x(-r) \Big\} \ .$$

and generates a C_0–semigroup $\{S(t)\}_{t\geq 0}$ on H (or even a C_0–group if $\det C_S \neq 0$). This semigroup is **EXS** iff $|\lambda(C_S)| < 1$ or equivalently $|b| < 1$ [12, pp. 148 - 154], which is the case[5]. Thus assumption **(H1)** holds.

- The observation functional $c^{\#}$ is given by

$$c^{\#}x = c_0^T x(-r), \qquad D(c^{\#}) = \left\{x \in \mathrm{H}: \ c_0^T x \text{ is right–continuous at } -r\right\} ,$$

and is representable on $D(A)$ as

$$c^{\#}\big|_{D(A)} = h^* A, \qquad h = \vartheta \begin{bmatrix} b\mathbf{1} \\ -\mathbf{1} \end{bmatrix} \in \mathrm{H}, \qquad \vartheta := \frac{a}{1+b} ,$$

where $\mathbf{1}$ denotes the constant function taking the value 1 on $[-r, 0]$. The admissibility of $c^{\#}$ was implicitly discussed in [14, p. 363]. The Lyapunov proof of this fact is presented in [17]. Thus assumption **(H3)** holds.

- The factor control vector is identified as

$$d = \frac{-1}{1+b} d_0, \qquad d_0 = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix} \in \mathrm{H} ,$$

where d is admissible [17, p. 20], whence assumption **(A3)** holds. By the proof presented therein the pair (A^{-1}, d) is exactly (hence approximately) reachable.

The compatibility condition (2.3) holds with $c^{\#}d = -\vartheta$ and by (2.4) the transfer function reads

$$\hat{g}(s) = \frac{ae^{-sr}}{1 + be^{-2sr}} .$$

This is confirmed by applying the Laplace transform directly to (5.1). Moreover,

$$\|\hat{g}\|_{\mathrm{H}^\infty(\Pi^+)} = \frac{a}{1 - |b|} .$$

and thus (2.5) is satisfied. The situation is even better, namely we have that g is in the Callier–Desoer algebra $\mathcal{A}_-(0)$. All these results and many others can be found in [17]. In particular assumptions **(A1)** and **(H1)**$\div$**(H4)** hold.

The closed–loop semigroup generator corresponding to the linear feedback $f(y) = \mu y$ takes the form

$$A_\mu x = x', \qquad D(A_\mu) = \left\{x \in \mathrm{W}^{1,2}(-r, 0) \oplus \mathrm{W}^{1,2}(-r, 0): \ x(0) = \left[C_S + \mu b_0 c_0^T\right] x(-r)\right\} .$$

Indeed, $D(A_\mu)$ consists of these x for which $x + \mu dc^{\#}x \in D(A)$. The latter holds if $x \in \mathrm{W}^{1,2}(-r, 0) \oplus \mathrm{W}^{1,2}(-r, 0)$ and $x(0) + \mu dc^{\#}x = C_S\left[x(-r) + \mu dc^{\#}x\right]$, or equivalently, if $x(0) = \left[C_S + \mu b_0 c_0^T\right] x(-r)$. The semigroup generated on $\mathrm{H} = \mathrm{L}^2(-r, 0) \oplus \mathrm{L}^2(-r, 0)$ by A_μ is **EXS** iff all eigenvalues of the matrix $C_S + \mu b_0 c_0^T = \begin{bmatrix} 0 & 1 \\ -b & a\mu \end{bmatrix}$, are in the open unit disk [12]. This is the case if

$$|\mu| < \frac{1+b}{a} . \tag{5.2}$$

[5]An alternative proof follows by applying Datko's theorem on **EXS** see e.g. [8, Theorem 5.1.3, p. 217] upon noting that the operator $(\mathcal{G}x)(\theta) := [rD + \theta I]x(\theta)$, $\theta \in [-r, 0]$, where D denotes a unique solution to the discrete matrix Lyapunov equation $C_S^T D C_S - D = -I$, belongs to $\mathbf{L}(\mathrm{H})$ is self–adjoint and nonnegative, and solves the Lyapunov operator equation

$$\langle x, \mathcal{G}Ax\rangle_{\mathrm{H}} + \langle Ax, \mathcal{G}x\rangle_{\mathrm{H}} = -\|x\|^2_{\mathrm{H}} \qquad \forall x \in D(A) .$$

Stability condition (5.2) yields the *Hurwitz sector* which has to be compared with a sector (k_1, k_2) generated by the frequency–domain inequality (4.5). It is clear that by (5.2) the upper limit for k_2 is $\dfrac{1+b}{a}$ and the lower limit for k_1 is $-\dfrac{1+b}{a}$. Now we can verify assumption **(A2)**. This will be done separately for $b \leq 0$ and for $b > 0$.

5.1. **The case of nonpositive** b**.** Substituting $k_2 = -k_1 = \dfrac{1+b}{a}$ into (4.4) gives

$$\pi(j\omega) = 1 - \Big(\frac{1+b}{a}\Big)^2 |\hat{g}(j\omega)|^2 = \frac{-4b\sin^2\omega r}{(1-b)^2 + 4b\cos^2\omega r} \geq 0 \qquad \forall\omega \in \mathbb{R}$$

and therefore the Hurwitz sector (5.2) agrees with the sector implied by (4.23).

To have (4.5) satisfied we replace $k_2 = -k_1 = \dfrac{1+b}{a}$ by $k_2 = -k_1 = \sqrt{\Big(\dfrac{1+b}{a}\Big)^2 - \nu}$ with sufficiently small $\nu > 0$ getting

$$\pi(j\omega) = \frac{-4b\sin^2\omega r}{(1-b)^2 + 4b\cos^2\omega r} + \nu\big|\hat{g}(j\omega)\big|^2 \geq \nu \inf_{\omega\in\mathbb{R}} \big|\hat{g}(j\omega)\big|^2 = \frac{\nu a^2}{(1+|b|)^2} := \eta > 0 \;\; \forall\omega \in \mathbb{R} \ ,$$

whence **(H5)** holds. Finally **(H6)** is valid by Remark 2.2 and Corollary 2.1 because A is dissipative. To see this note that, as $|b| < 1$ ($\iff$ $|\lambda(C_S)| < 1$), $C_S^T C_S - I = \operatorname{diag}\{b^2 - 1, 0\} \leq 0$, whence

$$\langle Ax, x\rangle_{\mathrm{H}} + \langle x, Ax\rangle_{\mathrm{H}} = x^T(-r)\left[C_S^T C_S - I\right] x(-r) \leq 0 \ .$$

Now all assumptions of Theorem 4.1 are met and by the latter the Lur'e system (3.2) with

$$k_2 = -k_1 = \sqrt{\Big(\frac{1+b}{a}\Big)^2 - \nu} \quad \rightsquigarrow \quad q = -\Big(\frac{1+b}{a}\Big)^2 + \nu, \; e := \vartheta\Big[\Big(\frac{1+b}{a}\Big)^2 - \nu\Big], \; \delta = \nu\vartheta^2$$

has a solution $(\mathcal{H}, g)$, $\mathcal{H} \in \mathbf{L}(\mathrm{H})$, $\mathcal{H} = \mathcal{H}^* \geq 0$, whence **(A2)** is met. By Theorem 3.1 the origin of H is **GSAS** for any $f \in \mathcal{S}_{2\nu} \cap \mathcal{M}$. This agrees with the result in [18, Subsection 4.1] modulo $\varepsilon = 2\nu$.

Moreover here an explicit solution $(\mathcal{H}, g)$ of the equivalent Lur'e system (3.3) is obtained by the method of Theorem 4.1/**(ii)** as follows. For finding a real spectral factor ϕ such that $\phi(0) = \sqrt{\delta} = \vartheta\sqrt{\nu} = \dfrac{a\sqrt{\nu}}{1+b}$ and ϕ, $1/\phi \in \mathrm{H}^\infty(\Pi^+)$, $\phi(0) = \sqrt{\delta}$ and equation (4.3) suggest

$$\phi(s) = \frac{\alpha + \beta e^{-sr} + \gamma e^{-2sr}}{1 + be^{-2sr}} \ , \tag{5.3}$$

where the real triple (α, β, γ) satisfies

$$\alpha + \beta + \gamma = a\sqrt{\nu}, \qquad \beta(\alpha+\gamma) = 0, \qquad \alpha\gamma = b, \qquad \alpha^2 + \beta^2 + \gamma^2 = a^2\nu - 2b \ . \tag{5.4}$$

Here the fourth equation results from the former ones, which are hence essential. The condition $1/\phi \in \mathrm{H}^\infty(\Pi^+)$ is equivalent to the condition that $\gamma z^2 + \beta z + \alpha = 0$ has two roots of modulus larger than one. This implies that $|\alpha| > |\gamma|$, whence $\alpha + \gamma \neq 0$ and by the second equation $\beta = 0$. Hence, by the first and third one, an appropriate well defined spectral factor is given by

$$\phi(s) = \frac{\alpha + \gamma e^{-2sr}}{1 + be^{-2sr}}, \qquad \alpha = \frac{a\sqrt{\nu} + \sqrt{a^2\nu - 4b}}{2}, \quad \gamma = \frac{a\sqrt{\nu} - \sqrt{a^2\nu - 4b}}{2} \ .$$

Now
$$\frac{\phi(s)-\phi(0)}{s}=\frac{(b\alpha-\gamma)(1-e^{-2sr})}{(1+b)s(1+be^{-2sr})}\ ,$$
whence, with $A(sI-A)^{-1}d=\dfrac{e^{s\theta}}{1+be^{-2sr}}\begin{bmatrix} e^{-sr}\\ 1\end{bmatrix}$ for $\theta\in[-r,0]$, the solution of the realization equation (4.7) is unique by the approximate reachability of the pair (A^{-1},d) and reads
$$g=g_0\begin{bmatrix}\mathbf{1}\\ \mathbf{1}\end{bmatrix},\qquad g_0=(1+b)^{-1}\ (b\alpha-\gamma)\ ,$$
and thus $g^*Ax=(\gamma-b\alpha)x_1(-r)$. Assuming $(\mathcal{H}x)(\theta)=Hx(\theta)$ where $H=H^T\in \mathrm{L}(\mathbb{R}^2)$, the Lyapunov operator equation in (3.3) is reduced to
$$x^T(-r)[C_S^THC_S-H]x(-r)=q(h^*Ax)^2-(g^*Ax)^2\qquad \forall x\in D(A)$$
with a unique solution
$$H=\mathrm{diag}\{\gamma^2-b^2,1-\alpha^2\}=H^T>0\ .$$
To see this, put
$$H=\begin{bmatrix} h_1 & h_{12}\\ h_{12} & h_2\end{bmatrix}\ ,$$
and get using (5.4) with $\beta=0$,

(5.5) $\quad (1+b)h_{12}=0,\quad h_1-b^2h_2=\gamma^2+b^2\alpha^2-2b^2,\quad h_2-h_1=(b^2+1)-(\alpha^2+\gamma^2)\ .$

5.2. **The case of positive b.** The Hurwitz sector (5.2) is essentially larger than the sector implied by (4.23), because for $k_1=-\dfrac{1+b}{a}$ we cannot take $k_2=\dfrac{1+b}{a}$ to have the latter satisfied. An another choice of k_1, k_2 has to be proposed. Assuming $k_1=-\dfrac{1+b}{a}$ we search for the maximal admissible value of k_2 for which (4.23) holds. Since
$$\pi(j\omega)=1-(k_1+k_2)\,\mathrm{Re}[\hat{g}(j\omega)]+k_1k_2\,|\hat{g}(j\omega)|^2=$$
$$=\frac{(1+b)^2\cos^2\omega r+(1-b)^2\sin^2\omega r+(1+b)^2\cos\omega r-k_2a(1+b)\cos\omega r-k_2a(1+b)}{(1-b)^2+4b\cos^2\omega r}$$
then, treating the numerator as a polynomial in $\cos\omega r$, we give k_2 its maximal admissible value for which the frequency domain inequality (4.23) holds, viz.
$$k_2=\frac{1+b}{a}-\frac{8b}{a(1+b)}=\frac{b^2-6b+1}{a(1+b)}\ ,$$
whence
$$\pi(j\omega)=\frac{4b(1+\cos\omega r)^2}{(1-b)^2+4b\cos^2\omega r}\geq 0\ .$$
For meeting (4.5), we replace $k_1=-\dfrac{1+b}{a}$ and $k_2=\dfrac{1+b}{a}-\dfrac{8b}{a(1+b)}$ successively by $k_{1,2}=-\dfrac{4b}{a(1+b)}\mp\sqrt{\dfrac{(1-b)^4}{a^2(1+b)^2}-\nu}$, with $\nu>0$ sufficiently small giving
$$\pi(j\omega)=\frac{4b(1+\cos\omega r)^2}{(1-b)^2+4b\cos^2\omega r}+\nu\big|\hat{g}(j\omega)\big|^2\geq\nu\inf_{\omega\in\mathbb{R}}\big|\hat{g}(j\omega)\big|^2=\frac{\nu a^2}{(1+|b|)^2}:=\eta>0\ \ \forall\omega\in\mathbb{R}\ .$$

Thus **(H5)** holds. **(H6)** holds as the method of Subsection 5.1 for checking **(H6)** does not depend on the sign of b. Thus all assumptions of Theorem 4.1 are met and by its assertion the Lur'e system (3.2) with

$$k_{1,2} = -\frac{4b}{a(1+b)} \mp \sqrt{\frac{(1-b)^4}{a^2(1+b)^2} - \nu} \quad \rightsquigarrow$$

$$\rightsquigarrow \quad q = \frac{-b^2+6b-1}{a^2} + \nu, \qquad e := \frac{b^2-10b+1}{a(1+b)} + \nu\vartheta, \qquad \delta = \frac{16b}{(1+b)^2} + \nu\vartheta^2$$

has a solution $(\mathcal{H}, g)$, $\mathcal{H} \in \mathbf{L}(\mathrm{H})$, $\mathcal{H} = \mathcal{H}^* \geq 0$, whence **(A2)** holds. By Theorem 3.1 the origin of H is **GSAS** for any $f \in \mathcal{S}_{2\nu} \cap \mathcal{M}$. This agrees with the result in [18, Subsection 4.1] modulo $\varepsilon = 2\nu$.

Moreover, as in the case $b \leq 0$, an explicit solution $(\mathcal{H}, g)$ of the equivalent Lur'e system (3.3) is possible by the method of Theorem 4.1/**(ii)** as follows. For finding a real spectral factor ϕ such that $\phi(0) = \sqrt{\delta} = \dfrac{\sqrt{16b + a^2\nu}}{1+b}$ and ϕ, $1/\phi \in \mathrm{H}^\infty(\Pi^+)$, $\phi(0) = \sqrt{\delta}$ and equation (4.3) suggest, as in the case $b \leq 0$, that $\phi(s)$ is given by (5.3), where the real triple (α, β, γ) satisfies

$$\alpha + \beta + \gamma = \sqrt{16b + a^2\nu}, \quad \beta(\alpha+\gamma) = 4b, \quad \alpha\gamma = b, \quad \alpha^2 + \beta^2 + \gamma^2 = 6b + a^2\nu \ ,$$

and the first three equations are essential. By the second equation one has that β is nonzero, such that the first and second equation deliver $\beta^2 - \beta\sqrt{16b + a^2\nu} + 4b = 0$, and the second and third one give that α and γ must be the roots of $\beta x^2 - 4bx + \beta b = 0$. This delivers four possible solutions (β, α, γ). The condition $1/\phi \in \mathrm{H}^\infty(\Pi^+)$ implies, as in the case $b \leq 0$, that $|\alpha| > |\gamma|$, whence a unique appropriate solution (β, α, γ) is given by

$$\beta = \frac{1}{2}\left[\sqrt{16b + a^2\nu} - \sqrt{a^2\nu}\right], \qquad \alpha = \frac{2b}{\beta} + \sqrt{\left(\frac{2b}{\beta}\right)^2 - b}, \qquad \gamma = \frac{2b}{\beta} - \sqrt{\left(\frac{2b}{\beta}\right)^2 - b} \ .$$

Thus one gets a well defined spectral factor and

$$\frac{\phi(s) - \phi(0)}{s} = \frac{[\gamma - b(\alpha+\beta)]e^{-2sr} + \beta(1+b)e^{-sr} + [b\alpha - (\beta+\gamma)]}{(1+b)s(1+be^{-2sr})} \ .$$

The solution of the realization equation (4.7) is unique by the approximate reachability of the pair (A^{-1}, d) and reads then

$$g = \begin{bmatrix} g_1\mathbf{1} \\ g_2\mathbf{1} \end{bmatrix}, \qquad g_1 = \frac{b(\alpha+\beta) - \gamma}{1+b}, \qquad g_2 = \frac{b\alpha - (\beta+\gamma)}{1+b} \ ,$$

whence $g^*Ax = (\gamma - b\alpha)x_1(-r) + \beta x_2(-r)$. Assuming $(\mathcal{H}x)(\theta) = Hx(\theta)$ where $H = H^T \in \mathrm{L}(\mathbb{R}^2)$, the Lyapunov operator equation in (3.3) is reduced to

$$x^T(-r)[C_S^T H C_S - H]x(-r) = q(h^*Ax)^2 - (g^*Ax)^2 \qquad \forall x \in D(A) \ .$$

The unique solution

$$H = \begin{bmatrix} h_1 & h_{12} \\ h_{12} & h_2 \end{bmatrix} = H^T > 0 \ ,$$

results from equations (5.5), where the right–hand side of the first equation has to be replaced by $\beta(\gamma - b\alpha)$, whence

$$h_{12} = (1+b)^{-1}\beta(\gamma - b\alpha), \qquad h_1 = \gamma^2 - b^2, \qquad h_2 = 1 - \alpha^2 \ .$$

6. Example 2: Unloaded RC–transmission line

Following [17, Subsection 5.2], the Hilbert space $\mathrm{H} = \mathrm{L}^2(0,1)$ with standard scalar product is used to model the dynamics of an unloaded RC transmission line according to (2.1) with:

- The state–space operator

$$Ax = x'', \qquad D(A) = \{x \in \mathrm{H}^2(0,1) : \ x'(1) = 0, \ x(0) = 0\}$$

  which generates an **EXS** analytic self–adjoint semigroup on H. This is due to $A = A^* < 0$. Moreover, A has a system of eigenvectors $\{e_n\}_{n=0}^{\infty}$ (corresponding to its eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$) that is an orthonormal basis of H (see [15, Formula (21)] or [16, Lemma 3.1 with K=0]),

$$\left\{ \begin{array}{lcll} e_n(\theta) & = & \sqrt{2}\sin\left(\frac{\pi}{2} + n\pi\right)\theta \ , \qquad 0 \le \theta \le 1, & n \ge 0 \\ \lambda_n & = & -\left(\frac{\pi}{2} + n\pi\right)^2, & n \ge 0 \end{array} \right\} .$$

  Thus assumption **(H1)** is satisfied.
- The observation functional

$$c^{\#}x = x(1), \qquad D(c^{\#}) = \{x \in \mathrm{L}^2(0,1) : \ x \text{ is left–continuous at } 1\} \supset \mathrm{C}[0,1] \ ,$$

  whose restriction to $D(A)$ reads as $c^{\#}\big|_{D(A)} = h^*A$ with $h(\theta) = -\theta$, $0 \le \theta \le 1$. It was proved in [14] that $c^{\#}$ is admissible and therefore assumption **(H3)** holds.
- The factor control vector d is given by

$$d = -\mathbf{1} \in \mathrm{L}^2(0,1), \qquad \mathbf{1}(\theta) = 1, \quad 0 \le \theta \le 1 \ ,$$

  and is not admissible. For a proof see [16, Subsection 3.3] or for a shorter one [17, Appendix B].

It is easy to see that (2.3) holds with $c^{\#}d = -1$ and that by (2.4) the transfer function reads

$$\hat{g}(s) = \frac{1}{\cosh\sqrt{s}}, \qquad s \in \Pi^+ \ .$$

Moreover one has

$$\|\hat{g}\|_{\mathrm{H}^\infty(\Pi^+)} = 1 \ , \tag{6.1}$$

where the norm is attained at $s = 0$. For a more exhaustive discussion of these facts and many others see [17]. In particular **(H2)** and **(H4)** hold, as well as **(A1)**.

The closed–loop semigroup generator corresponding to the linear feedback $f(y) = \mu y$ takes the form

$$A_\mu x = x'', \qquad D(A_\mu) = \left\{x \in \mathrm{H}^2(0,1) : \ x'(1) = 0, \ x(0) = \mu x(1)\right\} \ .$$

It is proved in [13] that A_μ generates an analytic semigroup on $\mathrm{L}^2(0,1)$ which is **EXS** for $\mu \in (-\cosh\pi, 1)$ with $\cosh\pi \approx 11.592$.

It follows from (6.1) that (4.23) holds for $k_2 = -k_1 = 1$. The Hurwitz sector is essentially larger than the sector (k_1, k_2) for which (4.23) is satisfied. The assumptions of Theorem 4.1 are easily checked, provided that we decrease $k_2 = -k_1$ from 1 to $\sqrt{1-\nu}$, where $\nu > 0$ is small. Then **(H5)** holds as by (6.1)

$$\pi(j\omega) = 1 - (1-\nu)\,|\hat{g}(j\omega)|^2 \ge 1 - (1-\nu) = \nu > 0 \qquad \forall\omega \in \mathbb{R} \ .$$

Finally **(H6)** is valid by Corollary 2.2 as H has an orthonormal basis of eigenvectors of A.

By Theorem 4.1 the Lur'e system (3.2) with $e = 1-\nu = -q$, $\delta = \nu$ has a solution $(\mathcal{H}, g)$, $\mathcal{H} \in \mathbf{L}(\mathrm{H})$, $\mathcal{H} = \mathcal{H}^* \geq 0$, $g \in \mathrm{H}$. Thus **(A1)** and **(A2)** hold but so does not **(A3)** as d is not admissible. Hence Theorem 3.1 cannot be applied, but some special analysis below shows that the first part of its proof is valid. The results of the analysis are what we have previously announced as the "parabolic regularity".

First with $f \in \mathcal{M}$, conclusion **(i)** of Lemma 3.1 holds. Next as the semigroup $\{S(t)\}_{t\geq 0}$ is analytic and **EXS**, $AS(t) \in \mathbf{L}(\mathrm{H})$ for $t > 0$ and $\lambda \longmapsto A(\lambda I - A)^{-1} \in \mathrm{H}^\infty(\Pi^+, \mathbf{L}(\mathrm{H}))$. Moreover, by the vector version of the Paley–Wiener theorem

$$\left\|QR_{(\cdot)}u\right\|_{\mathrm{L}^2(0,\infty;\mathrm{H})} = \|AS(\cdot) \star du\|_{\mathrm{L}^2(0,\infty;\mathrm{H})} \leq \left\|A\left[(\cdot)I - A\right]^{-1}\right\|_{\mathrm{H}^\infty(\Pi^+,\mathbf{L}(\mathrm{H}))} \|d\|_{\mathrm{H}} \|u\|_{\mathrm{L}^2(0,\infty)}$$

for any $u \in \mathrm{L}^2(0,\infty)$ and any $d \in \mathrm{H}$. Thus for any $T > 0$ with $u \in \mathrm{L}^2(0,T)$ one gets $\big(AS(\cdot) \star du(\cdot)\big)_T \in \mathrm{L}^2(0,T;\mathrm{H})$. Therefore with $x_0 \in \mathrm{H}$

$$x(t) := S(t)x_0 + \int_0^t AS(t-\tau)du(\tau)d\tau, \qquad t \geq 0 \tag{6.2}$$

satisfies pointwise in H almost everywhere the initial value problem (3.6) since by Fubini's theorem there holds that

$$\int_0^t [x(\tau) + du(\tau)]\, d\tau = A^{-1}[x(t) - x_0], \qquad t \geq 0\ .$$

Hence the first part of the proof of Theorem 3.1 is valid, whence $y, u \in \mathrm{L}^2(0,\infty)$ for any $f \in \mathcal{S}_{2\nu} \cap \mathcal{M}$. This agrees with the result in [18, Subsection 4.2] modulo $\varepsilon = 2\nu$.

Finally it turns out that the null equilibrium of (3.6) is **GSAS** as will be shown next. We start by observing that for $AS(t) \in \mathbf{L}(\mathrm{H})$ for $t > 0$, one gets most importantly

Lemma 6.1. There holds[6] for $t > 0$

$$\|AS(t)h\|_{\mathrm{H}} \leq \sqrt{2}\sqrt{\frac{\sqrt{t}+1}{\sqrt{t}}}\, e^{-\pi^2 t/4}\ , \tag{6.3}$$

$$\|AS(t)d\|_{\mathrm{H}} \leq \frac{\pi}{\sqrt{2}}\sqrt{\frac{t\sqrt{t}+1}{t\sqrt{t}}}\, e^{-\pi^2 t/4}\ , \tag{6.4}$$

and

$$\int_0^t \big\|AS(t-\tau)d\big\|_{\mathrm{H}} \big\|AS(\tau)h\big\|_{\mathrm{H}} d\tau \leq \pi^2\sqrt{3}(1+t)e^{-\pi^2 t/4}\ . \tag{6.5}$$

Proof. Because of similarity for proving (6.3) and (6.4) we handle only (6.4) in detail. To see this one uses successively

$$\lambda_n\langle d, e_n\rangle_{\mathrm{H}} = d^*Ae_n = \sqrt{-2\lambda_n}, \qquad \lambda_n - \lambda_0 \leq -\pi^2 n^2 \ \text{ and } \ \frac{\lambda_n}{\lambda_0} \leq 9n^2 \ \text{ for } \ n \in \mathbb{N},$$

$$xe^{-x} \leq e^{-1} \text{ for } x \geq 0 \ \text{ and } \ \int_0^\infty e^{-y^2}dy = \frac{\sqrt{\pi}}{2}\ .$$

[6]The estimates (6.3) and (6.4) are known as the Balakrishnan–Washburn estimates [35].

There results

$$
\begin{aligned}
&\|AS(t)d\|_{\mathrm{H}}^2 = \sum_{n=0}^{\infty}\left|\langle AS(t)d, e_n\rangle_{\mathrm{H}}\right|^2 = -\sum_{n=0}^{\infty} 2\lambda_n e^{2\lambda_n t} = \\
&= -2\lambda_0 e^{2\lambda_0 t}\left[1+\sum_{n=1}^{\infty}\frac{\lambda_n}{\lambda_0}\, e^{2(\lambda_n-\lambda_0)t}\right] \le -2\lambda_0 e^{2\lambda_0 t}\left[1+\frac{9}{\pi^2 t}\sum_{n=1}^{\infty}\pi^2 n^2 t\, e^{-2\pi^2 n^2 t}\right] \le \\
&\le -2\lambda_0 e^{2\lambda_0 t}\left[1+\frac{9}{\pi^2 et}\sum_{n=1}^{\infty} e^{-\pi^2 n^2 t}\right] \le -2\lambda_0 e^{2\lambda_0 t}\left[1+\frac{9}{\pi^2 et}\int_0^{\infty} e^{-\pi^2 n^2 t} dn\right] = \\
&= e^{2\lambda_0 t}\ \frac{2\pi^2 et\sqrt{\pi t}+9}{4et\sqrt{\pi t}} \le \frac{\pi^2}{2}\ \frac{t\sqrt{t}+1}{t\sqrt{t}}\ e^{2\lambda_0 t}\ .
\end{aligned}
$$

For (6.3) one uses $\lambda_n\langle h, e_n\rangle_{\mathrm{H}} = h^* A e_n = c^{\#} e_n = (-1)^n\sqrt{2}$ instead of $\lambda_n\langle d, e_n\rangle_{\mathrm{H}}$.

For (6.5) one has by (6.3) and (6.4) with

$$
\eta(t) := \sqrt{2}\sqrt{\sqrt{t}+1} \ \text{ and } \ \zeta(t) := \frac{\pi}{\sqrt{2}}\sqrt{t\sqrt{t}+1}\ ,
$$

$$
\begin{aligned}
&\int_0^t \left\|AS(t-\tau)d\right\|_{\mathrm{H}}\left\|AS(\tau)h\right\|_{\mathrm{H}} d\tau \le \int_0^t \zeta(t-\tau)\ \frac{e^{\lambda_0(t-\tau)}}{(t-\tau)^{3/4}}\ \eta(\tau)\ \frac{e^{\lambda_0\tau}}{\tau^{1/4}}\ d\tau \le \\
&\le \zeta(t)\eta(t)e^{\lambda_0 t}\int_0^t \frac{d\tau}{(t-\tau)^{3/4}\tau^{1/4}} = \zeta(t)\eta(t)e^{\lambda_0 t}\int_0^1 \frac{d\xi}{(1-\xi)^{3/4}\xi^{1/4}} = \\
&= \pi^2\sqrt{(t\sqrt{t}+1)(\sqrt{t}+1)}\ e^{\lambda_0 t}\sqrt{2} \le \pi^2\sqrt{3}(1+t)\ e^{-\pi^2 t/4}\ ,
\end{aligned}
$$

where the last integral is the Beta–function $B(\frac{1}{4},\frac{3}{4}) = \Gamma(\frac{1}{4})\Gamma(\frac{3}{4}) = \pi\csc(\frac{\pi}{4}) = \pi\sqrt{2}$. □

Now recall that (6.2) holds with $u \in \mathrm{L}^2(0,\infty)$. Therefore, since $\{S(t)\}_{t\ge 0}$ is **EXS**, the null equilibrium will be **GSAS**, if for any initial state of the feedback system of Figure 3.1, $\displaystyle\int_0^t \left\|AS(t-\tau)du(\tau)\right\|_{\mathrm{H}} d\tau$ is bounded and tends to zero as $t\to\infty$. To see this, we start by observing that

$$
\begin{aligned}
&\int_0^t \|AS(t-\tau)du(\tau)\|_{\mathrm{H}}\, d\tau = \int_0^t \|AS(t-\tau)df\,(y(\tau))\|_{\mathrm{H}}\, d\tau \le \\
&\le \max\{|k_1|,|k_2|\}\int_0^t \|AS(t-\tau)d\|_{\mathrm{H}}\, |y(\tau)| d\tau \le \qquad (6.6)\\
&\le \max\{|k_1|,|k_2|\}\left\{\int_0^t \|AS(t-\tau)d\|_{\mathrm{H}}\left|\left(\overline{P}x_0\right)(\tau)\right| d\tau + \int_0^t \|AS(t-\tau)d\|_{\mathrm{H}}\left|\left(\overline{F}u\right)(\tau)\right| d\tau\right\}.
\end{aligned}
$$

Now using the self–adjointness of A as well as the analyticity of $\{S(t)\}_{t\ge 0}$ there holds by Lemma 2.2

$$
\left|\left(\overline{P}x_0\right)(\tau)\right| = |\langle AS(t)h, x_0\rangle_{\mathrm{H}}| \le \|AS(t)h\|_{\mathrm{H}}\, \|x_0\|_{\mathrm{H}}\ .
$$

Hence we get

$$
\int_0^t \|AS(t-\tau)d\|_{\mathrm{H}}\left|\left(\overline{P}x_0\right)(\tau)\right| d\tau \le \|x_0\|_{\mathrm{H}}\int_0^t \|AS(t-\tau)d\|_{\mathrm{H}}\, \|AS(\tau)h\|_{\mathrm{H}}\, d\tau
$$

such that by (6.5)

$$\int_0^t \|AS(t-\tau)d\|_{\mathrm{H}} \left|\left(\overline{P}x_0\right)(\tau)\right| d\tau \leq \|x_0\|_{\mathrm{H}}\pi^2\sqrt{3}(1+t)e^{-\pi^2 t/4} \ . \tag{6.7}$$

We study next the convolution term

$$r(t) := \int_0^t \|AS(t-\tau)d\|_{\mathrm{H}} \left|\left(\overline{F}u\right)(\tau)\right| d\tau \ . \tag{6.8}$$

First recall Lemma 2.4 and [17, end of Subsection 5.2]. Here the impulse response g of the open–loop linear system is a continuous function, $g(0) = 0$ and g decays exponentially as t tends to infinity, such that g is in $\mathrm{L}^2(0,\infty)$ and the input–output map is a convolution $\overline{F}u = g \star u$ with $u \in \mathrm{L}^2[0,\infty)$. Thus, by [10, Exercise 4, p.242], $\overline{F}u \in \mathrm{BUC}_0[0,\infty)$. Use now $p(t) := \|AS(t)d\|_{\mathrm{H}}$ and $q(t) := \left|\left(\overline{F}u\right)(t)\right|$. Thus $q \in \mathrm{BUC}_0[0,\infty) \subset \mathrm{L}^\infty[0,\infty)$, and by (6.4) for $t > 0$

$$p(t) \leq \pi e^{\lambda_0 t}\left\{t^{-3/4}\chi_{[0,1]}(t) + \chi_{(1,\infty)}(t)\right\} \ , \tag{6.9}$$

whence $p \in \mathrm{L}^1[0,\infty)$. Thus by (6.8) and [10, Theorem 14, p.241; Exercise 3, p.242], $r = p \star q \in \mathrm{BUC}[0,\infty)$. Moreover by (6.8), (6.9) and standard manipulations, for $t \geq 1$

$$r(t) = (p \star q)(t) \leq 4\pi \sup_{\tau\in[t-1,t]} q(\tau) + \pi e^{\lambda_0}\left(e^{\lambda_0(\cdot)} \star q\right)(t-1) \ .$$

Hence as $q \in \mathrm{BUC}_0[0,\infty)$, $\lim\limits_{t\to\infty} r(t) = 0$. Thus $r \in \mathrm{BUC}_0[0,\infty)$. Finally we are done by (6.6), and the fact that r and the right–hand side of (6.7) are in $\mathrm{BUC}_0[0,\infty)$.

Remark 6.1. As we already know $\langle d, e_n\rangle_{\mathrm{H}} \neq 0$ for any $n \in \mathbb{N}$, whence the pair (A^{-1}, d) is approximately reachable and the method of Theorem 4.1/**(ii)** applies to determine an explicit solution $(\mathcal{H}, g)$ of the Lur'e system (3.3) but it is more involved than in previous example. For some detail on solving by symmetric extraction the spectral factorization problem for $k_2 = -k_1 = 1$, see [19, Example 2, pp. 31-34].

7. Discussion and conclusions

The most important results of this paper are:

- ✍ A criterion for the absolute global strong asymptotic stability presented in Section 3 based on quadratic Lyapunov functionals viz. Theorem 3.1, whose assumptions however require to check the solvability of the Lur'e system (3.2).
- ✍ Solvability results for this Lur'e system in Section 4, leading to Theorem 4.1. The criterion of Section 3 jointly with those of Section 4 lead to results similar to those of the input–output approach [18],
- ✍ A detailed presentation of two examples of electrical transmission–lines, illustrating the results of previous sections, in Sections 5 and 6. The discussion shows that this paper's stability criteria are checkable.

In [23] a circle criterion has been derived for a nonlinear feedback system having in its feedback loop, an integrator and a sector nonlinearity in front of an infinite-dimensional Weiss-Salamon linear plant. Due to the smoothing action of the integrator, the results of [23] are not comparable with those of the present paper.

Moreover observe that, except for the case $b \leq 0$ of Example 1, all examples above show that the absolute stability conditions generated by the circle criterion are significantly more conservative than the Hurwitz sector condition. It is known that for finite–dimensional autonomous *continuous* Lur'e systems *Popov's method* leads to considerably better stability conditions than the circle criterion. It is less known that a generalization of Popov's method to finite–dimensional autonomous *discrete* Lur'e systems is possible only by *further restricting the class of admissible nonlinearities*. This causes one to expect some difficulties to get an appropriate Popov type stability criterion for the system described by

$$\left\{ \begin{array}{rcl} \dfrac{d}{dt}\left[A^{-1}x\right] & = & A^{-1}\dot{x} = x + df(y) \quad x_0 \in \mathrm{H} \\ y & = & c^{\#}x \end{array} \right\} \tag{7.1}$$

which is sufficiently general to handle discrete–time systems, as can be seen by noting that (5.1) is an equivalent model giving the essentially discrete–time dynamics of the electrical distortionless loaded $RLCG$–transmission line (see [14, p. 365] for details). An additional observation is that the input–output approach for finite–dimensional feedback systems is usually based on some smoothness assumptions imposed on the system output. Thus an other difficulty for obtaining a generalization of Popov's method will be that one has to examine some *differentiability properties of the system output*. This is mainly why in [4] a version of Popov's criterion has been successfully derived using the Lyapunov method (and improved in [3] with the aid of former Popov's approach combined with regularity results for the solution to the closed loop) for the infinite–dimensional Lur'e system of *indirect control*,

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & A\left\{x(t) + df[\sigma(t)]\right\} \\ \dot{\sigma}(t) & = & \langle q, x(t)\rangle_{\mathrm{H}} - \rho f[\sigma(t)] \end{array} \right\} .$$

Regarding the variable σ as the system output one can readily notice that here the output is differentiable. This is in contrast to (7.1) where the output y is generally not differentiable.

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