PARAMETRIC OPTIMIZATION OF INFINITE – DIMENSIONAL SYSTEMS

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Abstract

The aim of this paper is to present an exact method of evaluating the quadratic performance index for infinite–dimensional systems. Our method relies on solving the Lyapunov operator equation with unbounded operators. For delay – differential systems of the neutral type an explicit construction of the solution to the Lyapunov operator equation is given. The results are illustrated by two examples of the determination of a classical controller setting. An approximation method of evaluating the quadratic performance index is also provided and compared with the exact method.

1 Introduction

Let us consider two examples motivating our investigations.

Example 1. The dynamics equations of the nuclear reactor temperature control system described in [5] are

$$\left\{ \begin{array}{rcl} T\dot{y}(t) + y(t) & = & p(t-r) \\ z_0 \mathbb{1}(t) + u(t) & = & p(t) \\ K_1\varepsilon(t) + K_2 \int_0^t \varepsilon(\tau)d\tau & = & u(t) \\ w - y(t) & = & \varepsilon(t) \end{array} \right\} \tag{1}$$

where K_1, K_2 are parameters, $\mathbb{1}$ denotes the Heaviside step function and r, T are fixed positive constants. If we assume that the system is asymptotically stable and until the moment of the appearance of a disturbance it remains in equilibrium, then for $t < 0$

$$\varepsilon = 0,\ u = K_2 \int_0^\infty \varepsilon(t)dt = w = p = y \ . \tag{2}$$

From (1) and (2) we get

$$\ddot{\varepsilon}(t) = -\frac{1}{T}\dot{\varepsilon}(t) - \frac{z_0}{T}\delta(t-r) - \frac{K_1}{T}\dot{\varepsilon}(t-r) - \frac{K_2}{T}\varepsilon(t-r)$$

where δ denotes Dirac's pseudofunction, together with the initial condition $\varepsilon(\theta) = 0,\ \dot{\varepsilon}(\theta) = 0$ for $\theta \in [-r, 0]$. Hence, introducing the state variables $x_1(t) = \varepsilon(t+r)$, $x_2(t) = \dot{\varepsilon}(t+r)$ and the notation $x_0 = z_0 a \neq 0$, $a = -\frac{1}{T} = -5$, $r = 0.5$, $b = -\frac{K_1}{T} = -5K_1$, $d = -\frac{K_2}{T} = -5K_2$ we obtain the final version of the dynamics equations

$$\left\{ \begin{array}{l} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = a x_2(t) + b x_2(t-r) + d x_1(t-r) \\ x_1(\theta) = 0, \quad -r \le \theta \le 0 \\ x_2(\theta) = 0, \quad -r \le \theta < 0, \quad x_2(0) = x_0 \end{array} \right\} . \tag{3}$$

The problem is to determine a pair (b, d) minimizing the integral performance index

$$J = \int_0^\infty \varepsilon^2(t)dt = \int_0^\infty x_1^2(t)dt \ . \tag{4}$$

The system (3) is a special case of the neutral system

$$\left\{ \begin{array}{ll} \dot{v}(t) = Av(t) + (AC+B)x(t-r), & t \ge 0 \\ v(t) = x(t) - Cx(t-r), & t \ge 0 \\ v(0) = v_0 & \\ x(\theta) = \phi(\theta) \text{ for almost every } \theta \in [-r, 0] & \end{array} \right\} \tag{5}$$

where $A,\ B,\ C \in \mathbf{L}(\mathbb{R}^n)$, $r > 0$, $v_0 \in \mathbb{R}^n$, ϕ is a function defined on $(-r, 0)$ with values in $\mathbb{R}^n$. This can be seen by taking $n = 2$, $v = x$, $\phi \equiv 0$, $C = 0 \in \mathbf{L}(\mathbb{R}^2)$ and

$$A = \begin{bmatrix} 0 & 1 \\ 0 & a \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ d & b \end{bmatrix}, \quad v_0 = \begin{bmatrix} 0 \\ x_0 \end{bmatrix} .$$

Example 2. The dynamics of the $RCLG$ transmission line without distortion is governed by the equations [5]

$$\left\{ \begin{array}{c} L\dfrac{\partial i(x,t)}{\partial t} = -\dfrac{\partial u(x,t)}{\partial x} - Ri(x,t) \\ C\dfrac{\partial u(x,t)}{\partial t} = -\dfrac{\partial i(x,t)}{\partial x} - Gu(x,t) \\ i(1,t)R_0 = u(1,t) \\ u(0,t) = w_0\mathbb{1}(t) - Ku(1,t) \end{array} \right\} , \tag{6}$$

$t \ge 0$, $0 \le x \le 1$, $\frac{R}{L} = \frac{G}{C} = \alpha$ with zero initial conditions.

We pose the problem of the evaluation of K minimizing the integral performance index

$$J = \int_0^{\infty} \left[u(1,t) - \lim_{t\to\infty} u(1,t) \right]^2 dt \ . \qquad (7)$$

Using the *d'Alembert solutions* of a transmission line without distortion we can transform (6) into (5) with $n = 2$, $A = -I$, $B = C$, $v_0 = 0$ and

$$C = \begin{bmatrix} 0 & 1 \\ -\frac{\kappa}{\rho^2} & -k\left(1 + \frac{\kappa}{\rho^2}\right) \end{bmatrix}, \ \phi(\theta) = \begin{bmatrix} -\xi_\infty \\ -\xi_\infty \end{bmatrix}$$

for $\theta \in [-r, 0]$ where $\xi_\infty = \dfrac{w_0}{\left(1 + \frac{\kappa}{\rho^2}\right)(1+k)}$, $k = \dfrac{K(1+\kappa)}{\rho + \frac{\kappa}{\rho}}$, $\rho = e^{\alpha/v}$, $v = \dfrac{1}{\sqrt{LC}} = \dfrac{1}{r}$, $z = \sqrt{\dfrac{L}{C}}$, $\kappa = (R_0 - z)/(R_0 + z)$. The last three constants are called the *velocity of wave propagation*, the *wave impedance* of a line and the *reflection coefficient*, repsectively. The performance index (7) can be written as

$$J = \frac{(1+\kappa)^2}{\rho^2} \left[\|\phi\|^2_{\mathrm{L}^2(-r,0;\mathbb{R}^2)} + \int_0^{\infty} x_2^2(t) dt \right] \qquad (8)$$

2 Lyapunov operator equation

Let us consider an abstract dynamical system on a general Hilbert space H with scalar product $\langle \cdot, \cdot \rangle_{\mathrm{H}}$

$$\left\{ \begin{array}{ll} \dot{u}(t) = \mathcal{A} u(t), & t \geq 0 \\ u(0) = u_0 & \\ y = \mathcal{C} u & \end{array} \right\} \qquad (9)$$

with the operator $\mathcal{A} : (D(\mathcal{A}) \subset \mathrm{H}) \longrightarrow \mathrm{H}$ generating a linear C_0–semigroup $\{S(t)\}_{t\geq 0}$ on H and the operator $\mathcal{C} \in \mathbf{L}(D_{\mathcal{A}}, \mathrm{Y})$ where $D_{\mathcal{A}}$ denotes the domain $D(\mathcal{A})$ equipped with the norm induced by the scalar product $\langle u, v\rangle_{\mathcal{A}} := \langle u, v\rangle_{\mathrm{H}} + \langle \mathcal{A}u, \mathcal{A}v\rangle_{\mathrm{H}}$. Y is another Hilbert space with the scalar product $\langle \cdot, \cdot\rangle_{\mathrm{Y}}$. Observe also that if $\mathcal{A}^{-1} \in \mathbf{L}(\mathrm{H})$ then the norm induced by $\langle u, v\rangle_{\mathcal{A}}$ is equivalent to the norm induced by the scalar product $\langle \mathcal{A}u, \mathcal{A}v\rangle_{\mathrm{H}}$. In this case without loss of generality $\mathcal{C}$ can be represented as $\mathcal{C} = \mathcal{D}^*\mathcal{A}$ for some $\mathcal{D} \in \mathbf{L}(\mathrm{Y}, \mathrm{H})$.

Recall that the semigroup $\{S(t)\}_{t\geq 0}$ is *strongly asymptotically stable* (**AS**) if $s - \lim_{t\to\infty} S(t)u = 0$ for $u \in \mathrm{H}$. It is *exponentially stable* (**EXS**), if there exist $M \geq 1$, $\alpha > 0$ such that $\|S(t)\|_{\mathrm{L(H)}} \leq M e^{-\alpha t}$ for $t \geq 0$.

The observation operator $\mathcal{C}$ is called *admissible* if there exists $\gamma > 0$ such that

$$\int_0^{\infty} \|\mathcal{C}S(t)u_0\|^2_{\mathrm{Y}}\, dt \leq \gamma \, \|u_0\|^2_{\mathrm{H}} \qquad \forall u_0 \in D(\mathcal{A}) \qquad (10)$$

i.e., the densely defined *observability map* $P : \mathrm{H} \longrightarrow \mathrm{L}^2(0,\infty;\mathrm{Y})$, $(Pu_0)(t) := \mathcal{C}S(t)u_0$ is bounded.

Theorem 2.1. $\mathcal{C}$ is admissible iff there exists $\mathcal{H} = \mathcal{H}^* \in \mathbf{L}(\mathrm{H})$, $\mathcal{H} \geq 0$, and $\mathcal{H}$ satisfies the *Lyapunov operator equation*

$$\langle \mathcal{A}u, \mathcal{H}v\rangle_{\mathrm{H}} + \langle u, \mathcal{H}\mathcal{A}v\rangle_{\mathrm{H}} = -\langle \mathcal{C}u, \mathcal{C}v\rangle_{\mathrm{Y}} \qquad (11)$$

for all $u, v \in D(\mathcal{A})$.

Theorem 2.2. If $\dim \mathrm{Y} < \infty$ and there exists $k \in \mathrm{L}^2(0,\infty)$ such that for almost all $t \geq 0$:

$$\|\mathcal{C}S(t)u_0\|_{\mathrm{Y}} \leq k(t)\, \|u_0\|_{\mathrm{H}} \qquad \forall u_0 \in D(\mathcal{A})$$

then (11) has a solution $\mathcal{H} \in \mathbf{L}(\mathrm{H})$ with $\mathcal{H} = \mathcal{H}^*$ and $\mathcal{H} \geq 0$, and $\mathcal{H}$ is a Hilbert–Schmidt (**HS**) operator.

Theorem 2.3. If $\{S(t)\}_{t\geq 0}$ is **AS**, then (11) has at most one self–adjoint, bounded and nonnegative solution.

The proofs of the above results can be found in [6]. If the assumptions of Theorem 2.3 and Theorem 2.1 or 2.2 hold then the unique solution of (11) takes the form

$$\langle u, \mathcal{H}v\rangle_{\mathrm{H}} = \langle \overline{P}u, \overline{P}v\rangle_{\mathrm{L}^2(0,\infty;\mathrm{Y})} \ . \qquad (12)$$

3 Evaluation of the quadratic integral performance index for neutral systems

Motivated by examples of Section 1, we pose the problem of evaluating the quadratic integral performance index

$$J = \int_0^{\infty} [v^T(t), x^T(t-r)] G \begin{bmatrix} v(t) \\ x(t-r) \end{bmatrix} dt \qquad (13)$$

with $G := \begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix} = G^T \geq 0$ over trajectories of the neutral system (5). We shall give a solution to this problem employing the results of Section 2.

In the state space $\mathrm{H} = \mathbb{M}^2 = \mathbb{R}^n \oplus \mathrm{L}^2(-r, 0; \mathbb{R}^n)$ we can write (5) as (9) with

$$\mathcal{A}u = \begin{bmatrix} Av + (AC+B)\psi(-r) \\ \psi' \end{bmatrix}, \qquad D(\mathcal{A}) =$$

$$= \{u \in \mathbb{R}^n \oplus \mathrm{W}^{1,2}(-r, 0; \mathbb{R}^n) : \ v = \psi(0) - C\psi(-r)\}$$

where $u = \begin{bmatrix} v \\ \psi \end{bmatrix}$ and $u_0 = \begin{bmatrix} v_0 \\ \phi \end{bmatrix}$ is the current and initial state, respectively. It can be proved (see [4]) that $\mathcal{A}$ generates a linear C_0–semigroup $\{S(t)\}_{t\geq 0}$ on H,

$$S(t) \begin{bmatrix} v_0 \\ \phi \end{bmatrix} = \begin{bmatrix} v(t) \\ x_t \end{bmatrix}, \ t \geq 0$$

where $x_t : \ [-r, 0] \ni \theta \longmapsto x_t(\theta) = x(t+\theta) \in \mathbb{R}^n$. This semigroup is **EXS** iff

$$|\sigma(C)| < 1 \ , \qquad (14)$$

i.e., the spectrum of C is in an open unit circle and all roots of the *characteristic quasipolynomial*

$$\det[\lambda I - \lambda e^{-r\lambda}C - A - e^{-r\lambda}B] \tag{15}$$

have negative real parts [4, Lemma 6.2.11, p. 151]. In what follows, we assume that (14) and (15) hold.

An $\mathcal{A}$ – bounded linear observation operator

$$\mathcal{C}\begin{bmatrix} v \\ \psi \end{bmatrix} = G^{\frac{1}{2}}\begin{bmatrix} v \\ \psi(-r) \end{bmatrix}, \qquad \mathrm{Y} = \mathbb{R}^{2n}$$

corresponds to the integrand in (13). Since the semigroup $\{S(t)\}_{t\geq 0}$ is **EXS** we have for $u_0 \in \mathrm{H}$:

$$\int_0^\infty \|x(t-r)\|^2_{\mathbb{R}^n}\,dt = \sum_{k=0}^{\infty}\int_{kr}^{(k+1)r}\|x(t-r)\|^2_{\mathbb{R}^n}\,dt =$$

$$= \sum_{k=0}^{\infty}\int_{-r}^{0}\|x_{kr}(\theta)\|^2_{\mathbb{R}^n}\,d\theta \leq \|u_0\|^2_{\mathrm{H}}\,\frac{M^2}{1-e^{-2\mu r}} \quad .$$

Employing the Rayleigh inequality we get

$$\|\mathcal{C}S(\cdot)u_0\|^2_{\mathrm{L}^2(0,\infty;\mathbb{R}^{2n})} \leq \lambda_{\max}(G)\left[\frac{1}{2\mu}+\right.$$

$$\left.+\frac{1}{1-e^{-2\mu r}}\right]M^2\,\|u_0\|^2_{\mathrm{H}} \qquad \forall u_0 \in D(\mathcal{A})$$

and thus (10) holds. It follows from Theorem 2.1 and Theorem 2.3, and (12) that $J(u_0) = \langle u_0, \mathcal{H}u_0\rangle$ for all $u_0 \in \mathrm{H}$ where $\mathcal{H}$ is a unique bounded self–adjoint nonnegative solution to the Lyapunov operator equation (11) which reduces now to

$$\langle \mathcal{A}u, \mathcal{H}u\rangle_{\mathrm{H}} + \langle u, \mathcal{H}\mathcal{A}u\rangle_{\mathrm{H}} =$$

$$-[v^T, \psi^T(-r)]G\begin{bmatrix} v \\ \psi(-r)\end{bmatrix} \qquad \forall u \in D(\mathcal{A}) \quad . \tag{16}$$

A solution of (16) will be sought in the form

$$\mathcal{H}u = \begin{bmatrix} \alpha v + \int_{-r}^{0}\beta(\theta)\psi(\theta)d\theta \\ \beta^T(\cdot)v + \int_{-r}^{0}\delta(\cdot,\sigma)\psi(\sigma)d\sigma + \gamma\psi(\cdot)\end{bmatrix} \tag{17}$$

where $\alpha = \alpha^T$, $\gamma = \gamma^T$, $\delta(\theta,\sigma) = \delta^T(\sigma,\theta)$, $(\theta,\sigma) \in [-r,0]\times[-r,0]$, $\beta \in \mathrm{C}^\infty[-r,0]$, and δ is C^∞ on triangles with vertices at $(-r,-r)$, $(0,-r)$, $(0,0)$, and $(-r,-r)$, $(-r,0)$, $(0,0)$, respectively.

Taking (17) into account in (16), after some manipulations we come to a system of equation determining α, β, γ and δ,

$$\left.\begin{array}{l} A^T\alpha + \alpha A + \beta^T(0) + \beta(0) + \gamma = -P \\ \gamma C + \alpha(AC+B) + \beta(0)C - \beta(-r) = -Q \\ C^T\gamma C - \gamma = -R \\ \dfrac{\partial\delta(\theta,\sigma)}{\partial\sigma} + \dfrac{\partial\delta(\theta,\sigma)}{\partial\theta} = 0 \\ A^T\beta(\theta) - \dfrac{d\beta(\theta)}{d\theta} + \delta(0,\theta) = 0 \\ (B^T + C^TA^T)\beta(\theta) - \delta(-r,\theta) + C^T\delta(0,\theta) = 0 \end{array}\right\} \tag{18}$$

The general solution of the fourth equation of (18) is

$$\delta(\theta,\sigma) = \left\{\begin{array}{ll} \Phi(\theta-\sigma), & \theta < \sigma \\ \Phi^T(\sigma-\theta), & \theta > \sigma \end{array}\right\}\ , \tag{19}$$

$(\theta,\sigma) \in [-r,0]\times[-r,0]$, $\theta \neq \sigma$, $\Phi \in \mathrm{C}^\infty[-r,0]$. Substituting (19) into the subsystem consisting of the fifth and sixth equation of (18), and eliminating Φ from the resulting system, we obtain

$$\delta(\theta,\sigma) = \left\{\begin{array}{ll} \dfrac{d\beta^T(\kappa)}{d\kappa} - \beta^T(\kappa)A\mid_{\kappa=\theta-\delta}, & \theta < \delta \\ \dfrac{d\beta(\kappa)}{d\kappa} - A^T\beta(\kappa)\mid_{\kappa=\delta-\theta}, & \theta > \delta \end{array}\right\} \tag{20}$$

Now, (18) reduces to the Lyapunov matrix equation

$$C^T\gamma C - \gamma = -R \tag{21}$$

and the boundary–value problem

$$\left\{\begin{array}{l} \dfrac{d}{d\theta}[\beta(\theta) + \vartheta(\theta)C] = A^T\beta(\theta) + \vartheta(\theta)B \\ \dfrac{d}{d\theta}[C^T\beta(\theta) + \vartheta(\theta)] = -B^T\beta(\theta) - \vartheta(\theta)A \\ A^T\alpha + \alpha A + \beta^T(0) + \beta(0) + \gamma = -P \\ \gamma C + \alpha(AC+B) + \beta(0)C - \beta(-r) = -Q \end{array}\right\} \tag{22}$$

where

$$\vartheta(\theta) = \beta^T(-r-\theta), \qquad -r \leq \theta \leq 0 \tag{23}$$

Castelan and Infante [1], [2] and [3] have derived (22) in the case $C = 0$, i.e., for retarded systems and a much more complicated version of (22) for neutral systems provided that $\mathrm{W}^{1,2}(-r,0;\mathbb{R}^n)$ was chosen as a state space. Employing the *Kronecker* product of matrices ([8, Section 8.4]) we get

$$\frac{d}{d\theta}\begin{bmatrix}\mathrm{col}\ \beta \\ \mathrm{col}\ \vartheta\end{bmatrix} = N_1^{-1}N_2\begin{bmatrix}\mathrm{col}\ \beta \\ \mathrm{col}\ \vartheta\end{bmatrix}\ , \tag{24}$$

$$N_1 := \begin{bmatrix} I\otimes I & I\otimes C^T \\ C^T\otimes I & I\otimes I\end{bmatrix},$$

$$N_2 := \begin{bmatrix} A^T\otimes I & I\otimes B^T \\ -B^T\otimes I & -I\otimes A^T\end{bmatrix}$$

and col β, col ϑ denote n^2–dimensional vectors having rows composed of the rows of matrices β and ϑ, respectively. By the *Schur lemma* and (14) we have $\det N_1 = \det(I\otimes I - C^T\otimes C^T) \neq 0$. An eigensolution of (24) is $e^{\lambda\theta}\begin{bmatrix} L \\ M\end{bmatrix}$ where λ is a root of the characteristic polynomial

$$\det\left[(\lambda I - A^T)\otimes(\lambda I + A^T)+\right.$$

$$\left.+(B^T + \lambda C^T)\otimes(B^T - \lambda C^T)\right]$$

and matrices L, $M \in \mathbf{L}(\mathbb{C}^{n^2})$ satisfy the system

$$\left\{\begin{array}{l} \lambda L + \lambda MC = A^TL + MB \\ \lambda C^TL + \lambda M = -B^TL - MA \end{array}\right\}\ . \tag{25}$$

By multiplying the equations of (25) by (-1), transposing and reordering them, one can see that $e^{-\lambda\theta}\begin{bmatrix} M^T \\ L^T \end{bmatrix}$ is also the eigensolution. Assume from now that all eigenvalues of (24) have *linear elementary divisors.* Then the corresponding eigenvectors form a basis in $\mathbb{C}^{n^2}$ and the general solution of (24) is

$$\sum_{i=1}^{n^2}\left\{\kappa_i e^{\lambda_i\theta}\begin{bmatrix} L_i \\ M_i \end{bmatrix}+\mu_i e^{-\lambda_i\theta}\begin{bmatrix} M_i^T \\ L_i^T \end{bmatrix}\right\}\ .$$

This solution satisfies the functional equation (23) iff $\mu_i = \kappa_i e^{-\lambda_i r}$, and finally

$$\beta(\theta)=\sum_{i=1}^{n^2}\kappa_i\left[e^{\lambda_i\theta}L_i+e^{-\lambda_i(r+\theta)}M_i^T\right]\ . \tag{26}$$

Putting (26) into the third and fourth equations of (22) we get

$$\left\{\begin{array}{l} \gamma + A^T\alpha+\alpha A+\sum_{i=1}^{n^2}\kappa_i W_i=-P \\ \gamma C+\alpha(AC+B)+\sum_{i=1}^{n^2}\kappa_i V_i=-Q \\ W_i := L_i+L_i^T+e^{-\lambda_i r}(M_i+M_i^T) \\ V_i := e^{-\lambda_i r}(M_i^TC-L_i)+(L_iC-M_i^T) \end{array}\right\}\ .$$

Applying the Kronecker product once more yields

$$\underbrace{\begin{bmatrix} A^T\otimes I+I\otimes A^T & \text{col } W_i \\ -I\otimes(AC+B)^T & \text{col } V_i \end{bmatrix}}_{2n^2 \text{ vectors } (i=1,2,\ldots,n^2)}\begin{bmatrix} \text{col } \alpha \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \\ \cdot \\ \cdot \\ \kappa_{n^2} \end{bmatrix}=$$

$$=\begin{bmatrix} -\text{col } \gamma-\text{col } P \\ \text{col } Q+\text{col } (\gamma C) \end{bmatrix}\ . \tag{27}$$

The matrix of the system (27) is nonsingular. Indeed, if this is not the case, then taking $P=Q=R=0$ (in virtue of (21) and (14) we also have $\gamma=0$) and making use of formulae (27), (26), (20), (21) and (17) we can generate a nonzero operator $\mathcal{H}$ being a solution to the Lyapunov operator equation (16). However, this contradicts the uniqeness of the null solution for $\mathcal{C}=0$ which is a consequence of (14), (15) and Theorem 2.3. Finally, (27) has a unique solution which means that formulae: (27), (26), (20), (21) and (17) determine an operator $\mathcal{H}$ being the unique solution of the Lyapunov operator equation (16).

Let us indicate two possible simplifications of the performance index evaluation which can arrise in practical applications. The first is symmetry of matrices α, γ, P which causes that (27) contains $\frac{1}{2}n(n-1)$ redundant equations. The second is that for a large variety of initial conditions the evaluation of the performance index does not require the knowledge of all entries of α, $\beta(\theta)$, $\delta(\theta,\sigma)$, γ (e.g. for $u_0=\begin{bmatrix} v_0 \\ \mathbf{0} \end{bmatrix}$ it sufficies to determine only the matrix α).

4 Examples

Example 1: Nuclear reactor temperature control problem – continued. Treating a and r as fixed constants, we seek for a pair (b,d) belonging to the domain of stability which minimizes the performance index. The stability domain of the characteristic quasipolynomial $(s^2-as)+(-bs-d)e^{-sr}$ in the plane $0bd$ is an open bounded set with boundary $\Gamma_1\cup\Gamma_2$ where the curve Γ_1 is given parametrically for $\omega\in[0,\omega_0]$

$$\left\{\begin{array}{l} b(\omega)=-a\cos\omega r-\omega\sin\omega r \\ d(\omega)=-\omega^2\cos\omega r+a\omega\sin\omega r \end{array}\right\}\ .$$

Here ω_0 denotes the smallest positive root of the equation $\omega=a\tan\omega r$ and Γ_2 is an interval of $0b$ axis with ends coinciding with the ends of Γ_1.

The performance index (4) has a form (13) with $P=cc^T$, $c^T=[1,0]$, $Q=R=0\in\mathrm{L}(\mathbb{R}^2)$. Taking into account the nature of initial conditions we conclude that $J(u_0)=x_0^2\alpha_{22}$ where α_{22} is the lower right entrie of the matrix α. α_{22} can be determined from (27) which yields

$$\frac{J(u_0)}{x_0^2}=\frac{G_3F_3-F_2G_5}{2d^2\{aF_3G_1-F_1G_3+G_1F_2-aG_5F_1\}} \tag{28}$$

with

$$\begin{aligned} F_1 &= d-b\lambda_1-e^{-\lambda_1 r}\lambda_1(\lambda_1+a) \\ F_2 &= e^{-\lambda_1 r}(d^2-b^2\lambda_1^2)-\lambda_1(\lambda_1+a)(d-b\lambda_1) \\ F_3 &= e^{-\lambda_1 r}\lambda_1^2(\lambda_1+a)+\lambda_1(d-b\lambda_1) \\ \lambda_1 &= \sqrt{\tfrac{1}{2}\left[a^2-b^2+\sqrt{(b^2-a^2)^2+4d^2}\right]} \\ G_1 &= d+K^2\cos Kr-aK\sin Kr \\ G_3 &= (d-ab)K^2+(d^2+b^2K^2)\cos Kr \\ G_5 &= bK^2-aK^2\cos Kr-K^3\sin Kr \\ K &= \sqrt{\tfrac{1}{2}\left[b^2-a^2+\sqrt{(b^2-a^2)^2+4d^2}\right]}\ . \end{aligned}$$

The results of minimization of the performance index (28) for $a=-5$, $r=0.5$ are depicted in Fig.1 and they agree with calculations obtained in [5] with the use of the frequency–domain method.

Example 2: ***RCLG*** **transmission line without distortion – continued.** The conditions ansuring **EXS** reduce to (14) only, and in the expanded form they are

$$\rho-\frac{\kappa}{\rho}>0,\quad |k|<1\ . \tag{29}$$

To get (8) from (13) we take $R=\dfrac{(1+\kappa)^2}{\rho^2}cc^T$, $c^T=\begin{bmatrix} 0 & 1 \end{bmatrix}$. It is more convenient to minimize the normalized performance index

$$j(k)=\frac{vJ(k)\left(\rho-\dfrac{\kappa}{\rho}\right)\left(\rho+\dfrac{\kappa}{\rho}\right)^3}{w_0^2\,(1+\kappa)^2}=\frac{1+k\mu}{(1+k)^3\,(1-k)}$$

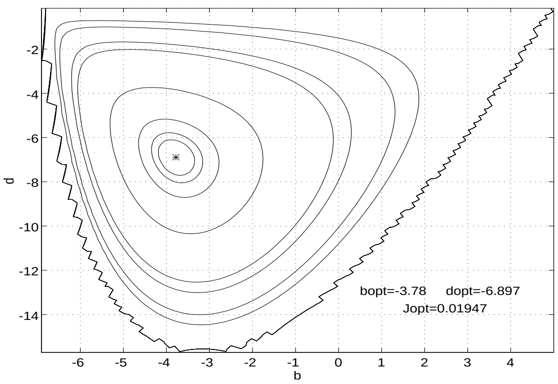

Figure 1: The level curves of the performance index

where $\mu = \dfrac{2\rho^2\kappa}{\rho^4+\kappa^2}$. The first inequality in (29) jointly with $\rho + \frac{\kappa}{\rho} > 0$ yields $\mu \in (-1,1)$ while the second one determines the minimization interval for j. Since

$$j'(k) = \frac{3\mu k^2 + (4-2\mu)k + \mu - 2}{(1+k)^4(1-k)^2}$$

we conclude that j is a unimodal function of k for $k \in (-1,1)$. If $\mu = 0$ (the transmission line is loaded by the wave impedance) then the minimum of j is achieved at $k = \frac{1}{2}$. For $\mu > 0$ the minimum is located in the interval $(0, \frac{1}{2})$, while for $\mu < 0$ in $(\frac{1}{2}, 1)$

5 Approximate optimization

Approximation of the function from $\mathrm{L}^2(0,\infty)$ with exponential sums. In the space $\mathrm{L}^2(0,\infty)$ with the standard scalar product we consider a densely defined unbounded linear operator $\mathcal{L}h = h'$, $D(\mathcal{L}) = \mathrm{W}^{1,2}[0,\infty)$. Its point spectrum is $\Pi^- = \{s \in \mathbb{C} : \mathrm{Re}s < 0\}$. Take n different eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. The corresponding n normalized eigenvectors $f_k(t) = \sqrt{-2\mathrm{Re}\lambda_k}\, e^{\lambda_k t}$, $k = 1, 2 \dots, n$ span an n–dimensional subspace M_n in $\mathrm{L}^2(0,\infty)$. The elements of this subspace will be called *exponential sums*. Consider the problem of approximation of a given function $f \in \mathrm{L}^2(0,\infty)$ with exponential sums. By the orthogonal projection theorem (see [9, Theorem 3.1, p.30 and Theorem 3.2, p.31]), the best approximant of f in M_n is the orthogonal projection of f on M_n. To be more precise,

$$\min_{g\in M_n} \|f-g\|_{\mathrm{L}^2(0,\infty)} = \|f - P_n f\|_{\mathrm{L}^2(0,\infty)} \quad ,$$

where P_n stands for the *orthoprojector* onto M_n.

By applying the Gram–Schmidt orthonormalization procedure to the system $\{f_k\}_{k=1}^n$ spanning M_n, we determine an orthonormal basis $\{e_k\}_{k=1}^n$ in M_n and express the orthoprojector P_n as follows

$$P_n f = \sum_{k=1}^{n} \langle f, e_k\rangle_{\mathrm{L}^2(0,\infty)} e_k \quad .$$

The sequence $\{e_k\}_{k=1}^n$ can be represented by their Laplace transforms, i.e., *Malmquist functions* [7]

$$\hat{e}_1(s) = \frac{\sqrt{-2\mathrm{Re}\lambda_1}}{s-\lambda_1}, \qquad \hat{e}_k(s) = \frac{\sqrt{-2\mathrm{Re}\lambda_k}}{s-\lambda_k} \prod_{i=1}^{k-1} \frac{s+\overline{\lambda_i}}{s-\lambda_i} \quad ,$$

$k \ge 2$. In virtue of the Parseval theorem we have

$$\|P_n f\|^2_{\mathrm{L}^2(0,\infty)} = \sum_{k=1}^{n} \left| \langle f, e_k\rangle_{\mathrm{L}^2(0,\infty)} \right|^2 \quad .$$

Formulae for $\langle f, e_k\rangle_{\mathrm{L}^2(0,\infty)}$, the Fourier expansion coefficients are given in [7].

Another way of determining the approximants follows from the observation that any element of the subspace M_n can be regarded as an output Y_n of the *observed* linear system

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & Ax(t) \\ x(0) & = & b \in \mathbb{C}^n \\ Y_n(t) & = & c^* x(t) \end{array} \right\} \quad ,$$

where $A = \mathrm{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}$, $\sigma(A) \subset \Pi^-$, $c^* = [\begin{array}{cccc} 1 & 1 & \cdots & 1 \end{array}]$. The pair (A, c^*) is *observable*. From the orthogonal projection theorem we get

$$\|f - P_n f\|^2_{\mathrm{L}^2(0,\infty)} = \min_{b\in\mathbb{C}^n} \|f - Y_n\|^2_{\mathrm{L}^2(0,\infty)} \quad . \tag{30}$$

Observe that

$$\|Y_n\|^2_{\mathrm{L}^2(0,\infty)} = \int_0^\infty b^* e^{tA^*} c c^* e^{tA} b\, dt = b^* H b \quad ,$$

where $H = \left[\dfrac{-1}{\overline{\lambda_i} + \lambda_j} \right]_{i,j=1,2,\dots,n} = H^* > 0$ is a unique solution of the *Lyapunov matrix equation*

$$A^* H + HA = -cc^* \tag{31}$$

or the *Gram matrix* of the system $\{e^{\overline{\lambda_k}(\cdot)}\}_{k=1}^n \subset \mathrm{L}^2(0,\infty)$. In virtue of the Paley – Wiener theory, $f \in \mathrm{L}^2(0,\infty)$ iff its Laplace transform $\hat{f}$, belongs to $\mathrm{H}^2(\Pi^+)$, the Hardy space of functions φ analytic on the right complex half–plane $\Pi^+ = \{s \in \mathbb{C} :\ \mathrm{Re}s > 0\}$, such that

$$\sup_{x>0} \int_{-\infty}^{\infty} |\varphi(x+iy)|^2\, dy < \infty \quad .$$

Since $\sigma(-A^*)$ is located in Π^+, the domain of analyticity of $\hat{f}$, we have (see [8, Theorem 5.3.2])

$$\int_0^\infty f(t) e^{tA^*} dt = \hat{f}(-A^*) = \mathrm{diag}\ \{\hat{f}(-\overline{\lambda_1}), \dots, \hat{f}(-\overline{\lambda_n})\} \quad .$$

Hence

$$\|f - Y_n\|^2_{\mathrm{L}^2(0,\infty)} = \|f\|^2_{\mathrm{L}^2(0,\infty)} - 2\mathrm{Re}\left[b^* \hat{f}(-A^*) c \right] + b^* H b$$

and the minimal value in (30) is achieved on $b \in \mathbb{C}^n$ being a solution of the equation

$$Hb = \hat{f}(-A^*) c \quad . \tag{32}$$

If b_0 is the solution of (32) then $\|P_n f\|^2_{\mathrm{L}^2(0,\infty)} = b_0^* H b_0 = b_0^* \hat{f}(-A^*) c$.

Convergence of the approximation. The infinite sequence $\{f_k\}_{k\in\mathbb{N}}$ is *complete* if its linear span is dense in $\mathrm{L}^2(0,\infty)$. Clearly, $\{f_k\}_{k\in\mathbb{N}}$ is complete iff $\{e^{\lambda_k(\cdot)}\}_{k\in\mathbb{N}}$ is complete and thus the next result is useful for the verification of completeness of the system $\{f_k\}_{k\in\mathbb{N}}$.

Lemma 5.1. The system $\{e^{\lambda_k(\cdot)}\}_{k\in\mathbb{N}} \subset \mathrm{L}^2(0,\infty)$ is complete if

$$\sum_{k=1}^{\infty} \frac{-2\mathrm{Re}\lambda_k}{1+|\lambda_k|^2} = \infty, \qquad |\lambda_k| \nearrow \infty \ \text{ as } \ k \to \infty \ . \qquad (33)$$

If $\{f_k\}_{k\in\mathbb{N}}$ is a complete system in $\mathrm{L}^2(0,\infty)$ then the system $\{e_k\}_{k\in\mathbb{N}}$ resulting from the Gram–Schmidt orthonormalization applied to $\{f_k\}_{k\in\mathbb{N}}$ is an orthonormal basis in $\mathrm{L}^2(0,\infty)$ (the corresponding sequence $\{\hat{e}_k\}_{k\in\mathbb{N}}$ consists of the Malmquist functions) and thus

$$\lim_{n\to\infty} \|f - P_n f\|_{\mathrm{L}^2(0,\infty)} = 0 \qquad \forall f \in \mathrm{L}^2(0,\infty) \ .$$

In particular, (33) holds for $\lambda_k = -k$, $k \in \mathbb{N}$. Consequently, the system $\{e^{-k(\cdot)}\}_{k\in\mathbb{N}}$ is complete and the proposed algorithm of L^2–approximation with exponential sums is convergent. Moreover, in this case we have

$$\hat{f}(-A^*) = \hat{f}(-A) = \mathrm{diag}\ \{\hat{f}(1), \hat{f}(2), \ldots, \hat{f}(n)\} \ ,$$

and the solution of (31) is the *finite Hankel – Hilbert matrix* $H = \left[\frac{1}{i+j}\right]_{i,j=1,2,\ldots,n}$. In practice, since the *finite* symmetric positive definite Hankel – Hilbert matrix H tends to the symmetric, but only positive semidefinite Hankel – Hilbert *infinite* matrix as $n \to \infty$, the *bad conditioning* of the linear system (32) will occur. Several numerical tests showed that a minor improvement in solving (32) can be achieved by dealing with its equivalent form $Zb = A^*\hat{f}(-A^*)c$, $Z = A^*H$.

Application to parametric optimization. We wish to solve the parametric optimization problem of finding minimum of the function

$$J(B) = \|f\|^2_{\mathrm{L}^2(0,\infty)}, \quad \hat{f}(s) = -\frac{A}{s - A - Be^{-sr}}$$

in the interval $S = \left(-\frac{1}{\omega}\sin\omega r, -A\right)$ being the stability region in the space of the proportional controller gain B. Here ω is the smallest positive solution of the equation $\omega = A\tan\omega r$.

The explicit exact expression for the performance index as a function of B is given in [5] and it can be also deduced from the results presented in Section 3 upon substituting $n = 1$, $C = 0$, $v_0 = -A$, $\phi \equiv 0$, $P = 1$, $Q = R = 0$

$$\frac{1}{A^2}J(B) = \left\{ \begin{array}{ll} \dfrac{B\sin Kr - K}{2K(A + B\cos Kr)}, & B^2 \geq A^2 \\ \dfrac{B\sinh Kr - K}{2K(A + B\cosh Kr)}, & B^2 < A^2 \end{array} \right\} \quad (34)$$

where $K = \sqrt{|B^2 - A^2|}$.

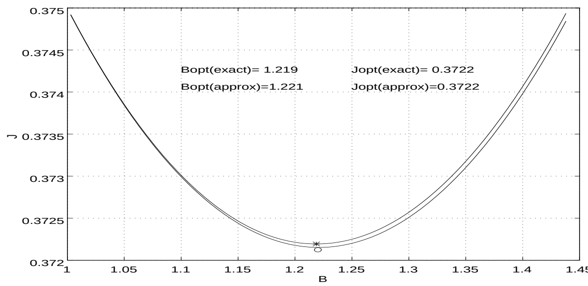

Figure 2: Plots of the performance index

The exact values of the performance index calculated from (34) are compared in Fig.2 with the results derived by applying the approximate method. The relative error of finding the optimal value of B is less than 0.2%. The main barrier in getting better accuracy is the bad conditioning of the Hankel – Hilbert matrix.

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