Lecture Notes in Control and Information Sciences

Edited by M. Thoma and A. Wyner

IFIP

84

System Modelling and Optimization

Proceedings of 12th IFIP Conference, Budapest, Hungary, September 2-6, 1985

Edited by
A. Prékopa, J. Szelezsán, and B. Strazicky

Springer-Verlag

STABILITY OF SEMILINEAR SYSTEMS IN HILBERT SPACES

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1.Abstract semilinear systems.

Several problems of mathematical physics,control and circuit theories lead to the following semilinear problem

$$\dot{x}(t) = Ax(t) + B\,F[x(t)] \qquad (1.1),$$

where $x(t) \in X$ for every fixed $t \geq 0$, X is a real Hilbert space with scalar product $\langle .,. \rangle_X$; $A : (\mathcal{D}(A) \subset X) \longrightarrow X$ is a linear operator,which is the generator of a linear C_0-semigroup on X ; U is another real Hilbert space with scalar product $\langle .,. \rangle_U$; $B \in \mathcal{L}(U,X)$.

It follows from the earlier results due to Segal [1],Pazy [2] and Ball [3] that for every initial condition $x_0 \in X$ there exists a unique weak (mild) solution of (1.1) prolongable on its right maximal interval of existence $[0,t_{max}(x_0))$,provided that $F:X \longrightarrow U$ is a locally Lipschitz function.

The aim of this paper is to give sufficient conditions for global (as well as global uniform) asymptotic stability of the equilibrium $0 \in X$, independently of F from the prescribed subclass of locally Lipschitz functions from X into U,vanishing at 0.

2.Main results.

The main results will be formulated as theorems and remarks.

Theorem 1.

Let $\mathfrak{M}$ be a class of functions such that :

(i) $\mathfrak{M} \subset \{ P$: P is a locally Lipschitz mapping from X into U, $P(0) = 0 \}$,

(ii) There exist operators $Q \in \mathcal{L}(U,X)$, $M=M^* \in \mathcal{L}(X)$, $L \in \mathcal{L}(U,X)$, $K=K^* \in \mathcal{L}(U)$ such that :

(H1) $\forall F \in \mathfrak{M}$: QF is a gradient type operator,

(H2) $\forall x \in X, \forall F \in \mathfrak{M}$:

$$\langle x,-Mx\rangle_X + \langle x,LF(x)\rangle_X + \langle F(x),L^*x\rangle_U + \langle F(x),-KF(x)\rangle_U \geqslant 0 \qquad (2.1),$$

(H3) There exist $H=H^*\in\mathcal{L}(X)$ and a real positive ε such that

$$\langle Ax,Hx\rangle_X + \langle x,HAx\rangle_X + \langle x,-Mx\rangle_X + \langle x,HBu\rangle_X + \frac{1}{2}\langle u,Q^*Ax\rangle_U +$$
$$+ \frac{1}{2}\langle Q^*Ax,u\rangle_U + \langle u,B^*Hx\rangle_U + \langle x,Lu\rangle_X + \langle u,L^*x\rangle_U + \frac{1}{2}\langle u,Q^*Bu\rangle_U + \qquad (2.2),$$
$$+ \frac{1}{2}\langle u,B^*Qu\rangle_U + \langle u,-Ku\rangle_U \leqslant -\varepsilon[\|x\|_X^2 + \|u\|_U^2] \quad \forall (x,u)\in\mathcal{D}(A)\times U$$

(H4) If $F\in\mathfrak{M}_L=\mathcal{L}(X,U)\cap\mathfrak{M}$ then A + BF generates an exponentially stable semigroup,

(H5) $\forall F\in\mathfrak{M}, \forall x \in X,\ x \neq 0 \quad \exists \varkappa = \varkappa(x,F)\in\mathfrak{M}_L$:

$$\int_0^1 \langle x,Q[F(sx) - \varkappa sx]\rangle_X\, ds = 0 ,$$

Then the equilibrium $0\in X$ is globally asymptotically stable (GAS) for every $F\in\mathfrak{M}$.

<u>Skech of the proof.</u>

Let us consider a continuously Fréchet-differentiable functional

$$V(x) = \langle x,Hx\rangle_X + \int_0^x \langle dy,QF(y)\rangle_X \ \left(= \langle x,Hx\rangle_X + \int_0^1 \langle x,QF(sx)\rangle_X ds\right) \quad (2.3).$$

The second term in (2.3) is the antiderivative of the gradient operator QF.The assumptions (H1),(H2),(H3) allow us to prove the following inequalities :

$$\langle Ax + BF(x),V'(x)\rangle_X \leqslant -\varepsilon[\|x\|_X^2 + \|F(x)\|_U^2] \quad \forall x\in\mathcal{D}(A), \forall F\in\mathfrak{M} \quad (2.4),$$

where $V'(x)$ denotes the gradient of V at x,and

$$V[x(t,x_0)] - V(x_0) \leqslant -\varepsilon\int_0^t \{\|x(\tau)\|_X^2 + \|F[x(\tau)]\|_U^2\}\, d\tau \quad \forall x_0\in X, \quad (2.5).$$
$$\forall t\in[0,t_{max}), \forall F\in\mathfrak{M}$$

If V is nonnegative on X for every $F\in\mathfrak{M}$ then we can prove that $t_{max} = +\infty$ and

$$\varepsilon\int_0^{+\infty}\{\|x(t)\|_X^2 + \|F[x(t)]\|_U^2\}\, dt \leqslant V(x_0) \quad \forall x_0\in X, \forall F\in\mathfrak{M} \qquad (2.6).$$

The conditions assuring that $V(x)\geqslant 0 \quad \forall x\in X, \forall F\in\mathfrak{M}$ can be derived using the linear comparison system technique.The linear comparison system arises from (1.1) by substitution of $\mu\in\mathfrak{M}_L$ instead of F.Using the results known in the stability theory of linear systems defined on Hilbert space,it can be proved that $V(x)\geqslant 0\ \forall x\in X, \forall F\in\mathfrak{M}$ under the assumptions (H4),(H5).Now,(2.6) and the variation-of-constants formula yield the following estimation valid for all weak solutions :

$$\| x(t,x_0)\|_X \leqslant \bar{M}\ \|x_0\|_X\left[1 + \|B\|\sqrt{\tfrac{2}{\varepsilon}}\max\left\{1,\|\mu\|\right\}\sqrt{\|H\| + \tfrac{1}{2}\|Q\|\ell_F(\| x_0\|)}\right],$$

where $\bar{M}\geqslant 1$ is a constant, $\ell_F: \mathbb{R}^+\longrightarrow\mathbb{R}^+$ is a continuous,strictly increasing function,appropriately constructed due to the fact that F is a locally Lipschitz mapping, $\mu\in\mathfrak{M}_L$ is such that A+BF generates an exponentially stable semigroup (2.7).

The global uniform stability follows from (2.7) and the global attractivity follows from some basic properties of convolution operators in L^p spaces (it suffices to use p=2 plus a corollary from the Paley - - Wiener Theorem).

It turns out that we must restrict $\mathfrak{M}$ to the class $\mathfrak{M}^c$,described in the theorem given below,in order to obtain results concerning the global uniform asymptotic stability of the equilibrium $0\in X$. □

Theorem 2.

Let $\mathfrak{M}^c$ be a class of functions such that :

(i) $\mathfrak{M}^c\subset\{P$: P is a locally Lipschitz mapping from X into U,P maps every weakly convergent sequence in X into a strongly convergent sequence in U, $P(0)= 0\}$,

(ii) There exist operators $Q\in\mathcal{L}(U,X), M=M^*\in\mathcal{L}(X), L\in\mathcal{L}(U,X), K=K^*\in\mathcal{L}(U)$ such that :

(H1) $\forall F\in\mathfrak{M}^c$: QF is a gradient type operator,

(H2) $\forall x\in X, \forall F\in\mathfrak{M}^c$:

$$\langle x,-Mx\rangle_X + \langle x,LF(x)\rangle_X + \langle F(x),L^*x\rangle_U + \langle F(x),-KF(x)\rangle_U \geqslant 0 \qquad (2.8),$$

(H3) There exist $H=H^*\in\mathcal{L}(X)$ and a positive real ε such that

$$\langle Ax,Hx\rangle_X + \langle x,HAx\rangle_X + \langle x,-Mx\rangle_X + \langle x,HBu\rangle_X + \tfrac{1}{2}\langle u,Q^*Ax\rangle_U +$$
$$+ \tfrac{1}{2}\langle Q^*Ax,u\rangle_U + \langle u,B^*Hx\rangle_U + \langle x,Lu\rangle_X + \langle u,L^*x\rangle_U + \tfrac{1}{2}\langle u,Q^*Bu\rangle_U + \qquad (2.9),$$
$$+ \tfrac{1}{2}\langle u,B^*Qu\rangle_U + \langle u,-Ku\rangle_U \leqslant -\varepsilon\left[\| x\|_X^2 + \|u\|_U^2\right]\ \ \forall(x,u)\in\mathcal{D}(A)\times U$$

(H4) If $F \in \mathfrak{M}^c_L = \mathfrak{M}^c \cap \mathcal{L}(X,U)$ then $A + BF$ generates an exponentially stable semigroup ,

(H5) $\forall\, F \in \mathfrak{M}^c, \forall\, x \in X, x \neq 0\ \exists\, \varkappa = \varkappa(x,F) \in \mathfrak{M}^c_L : \int_1^0 \langle x, Q[F(sx) - \varkappa sx] \rangle_X ds = 0,$

Then the equilibrium $0 \in X$ is globally uniformly asymptotically stable (GUAS) for every $F \in \mathfrak{M}^c$.

Remark 1.

It can be proved that if the assumptions of Theorems 1 and 2 are satisfied with $Q = 0$ then even global exponential stability (GEXS) of the equilibrium $0 \in X$ takes place.

Remark 2.

If A is the generator of an exponentially stable semigroup then the statements in Theorems 1 and 2 remain valid after eliminating the term "$-\varepsilon \|x\|^2_X$" from the right-hand side of (2.2) or (2.9).

Special problems of the form (1.1) have been considered in earlier papers [5-finite dimensional case], [6-time delay systems with $Q=0$], [7-the case $Q=0$], [8-the case $Q \neq 0$, but with A, B, F of very particular forms. The next theorem allows us to obtain more precise conditions for (GAS) or (GUAS), or (GEXS), provided that some additional assumptions are satisfied.

Theorem 3.

Let $\mathfrak{M}$ be a class of functions such that :

(i) $\mathfrak{M} \subset \{ F : F$ is a locally Lipschitz mapping from X into $U, F(0)=0 \}$,

(ii) There exist operators $Q \in \mathcal{L}(U,X), M=M^* \in \mathcal{L}(X), L \in \mathcal{L}(U,X), K=K^* \in \mathcal{L}(U)$ such that :

(H1) $\forall\, F \in \mathfrak{M}$: QF is a gradient type operator,

(H2) $\forall\, x \in X, \forall\, F \in \mathfrak{M}$:

$$\langle x, -Mx \rangle_X + \langle x, LF(x) \rangle_X + \langle F(x), L^* x \rangle_U + \langle F(x), -KF(x) \rangle_U \geqslant 0 \qquad (2.10),$$

(H3') There exists a triplet of operators (H,G,V), $H=H^* \in \mathcal{L}(X)$, G is a linear, A-bounded operator from X into U, $V \in \mathcal{L}(U)$ such that

$$\begin{cases} \langle Ax, Hx \rangle_X + \langle x, HAx \rangle_X + \langle x, -Mx \rangle_X = -\langle Gx, Gx \rangle_U & \forall\, x \in \mathcal{D}(A) \\ B^* Hx + 0.5 Q^* Ax + L^* x = -V^* Gx & \forall\, x \in \mathcal{D}(A) \\ K - 0.5[Q^* B + B^* Q] = V^* V & (\text{in } \mathcal{L}(U)) \end{cases} \qquad \begin{matrix} (2.11a), \\ (2.11b), \\ (2.11c), \end{matrix}$$

(H6) There exists a positive real α such that for every $x \in X$ and $F \in \mathfrak{M}$ we have the following inequality :

$$V(x) \overset{df}{=} \langle x, Hx \rangle_X + \int_0^1 \langle x, QF(sx) \rangle_X \, ds \geqslant \alpha \| x \|_X^2 \qquad (2.12),$$

(H7) The semigroup $\{S(t)\}_{t \geqslant 0}$, generated by A, is either compact for $t > t_0 \geqslant 0$ or is exponentially stable and each F belonging to $\mathfrak{M}$ is compact,

(H8) For every $F \in \mathfrak{M}$ the null solution is the only one mild solution of (1.1), defined for all $t \in \mathbb{R}$, on which $V[x(t,x_0)] = V(x_0)$ $\forall t \in \mathbb{R}$.

Then the equilibrium $0 \in X$ is (GAS) for every $F \in \mathfrak{M}$.

<u>Sketch of the proof.</u>

In the same way as it was done in the proof of Theorem 1, we can show that the following inequalities are valid for the functional V defined by (2.3) :

$$\langle Ax + BF(x), V'(x) \rangle_X = - \| Gx + VF(x) \|_U^2 - \langle x, -Mx \rangle_X - \langle F(x), L^* F(x) \rangle_U -$$
$$- \langle x, LF(x) \rangle_X - \langle F(x), -KF(x) \rangle_U \leqslant 0 \quad \forall x \in \mathcal{D}(A), \forall F \in \mathfrak{M} \qquad (2.13),$$

where $V'(x)$ denotes the gradient of V at x and

$$V[x(t,x_0)] - V(x_0) \leqslant 0 \quad \forall t \in [0, t_{max}), \forall F \in \mathfrak{M}, \forall x_0 \in X \qquad (2.14).$$

Now, by (H7) and the local Lipschitz property for F,

$$\| x(t,x_0) \| \leqslant \sqrt{\tfrac{1}{\alpha} V(x_0)} \leqslant \sqrt{\tfrac{1}{\alpha}\left[\| H \| + \tfrac{1}{2} \| Q \| \, l(\| x_0 \|) \right]} \; \| x_0 \| \quad \forall t \in [0, t_{max}),$$

$\forall F \in \mathfrak{M}$, $\forall x_0 \in X$, where $l = l(\sigma)$ is a continuous, strictly increasing function from $\mathbb{R}^+$ into $\mathbb{R}^+$, $l(0) = 0$ (see also (2.7)) (2.15).

The unbounded prolongability to the right of all mild solutions and the global uniform stability of the equilibrium $0 \in X$ follow from (2.15) independently of $F \in \mathfrak{M}$. In the case of compactness of $\{S(t)\}$ for all $t > t_0 \geqslant 0$, in (H8), one can apply the generalization of Pazy's criterion for precompactness of bounded trajectories, to prove that all trajectories have precompact ranges, independently of $F \in \mathfrak{M}$. Originally, this criterion was stated for $t_0 = 0$ [2, p.236] or [3, Lemma 5.3] but after a modification of its proof it is possible to handle with $t_0 \geqslant 0$.

If the second case in (H8) holds, the same conclusion can be deduced from Webb's criterion for precompactness of bounded trajectories [4]. Now, by LaSalle Invariance Principle, all trajectories tend to the largest strongly-invariant set contained in the set $\{x_0 \in X: V[x(t,x_0)] = V(x_0)\ \forall t \in \mathbb{R}\}$, but, since (H9) holds, this set is equal to $\{0\}$. Thus, $0 \in X$ is globally attractive, which together with global uniform stability implies (GAS). □

Remark 3.

If the term "$-\varepsilon\langle x,x\rangle_X$" can be added in the right-hand side of (2.11a) where ε is a positive real, then the function $v(t) = V[x(t)]$ satisfies the following differential inequality

$$\dot{v}(t) \leqslant -\varepsilon\{\beta^{-1}[v(t)]\}^2 \text{ for almost all } t \geqslant 0,\ v(0) = V(x_0) \qquad (2.16),$$

where $\beta(\sigma) = [\|H\| + \frac{1}{2}\|Q\|\ell(\sigma)]\sigma^2$ (see (2.15)). Since $\alpha\|x(t)\|^2 \leqslant v(t)\ \forall t \geqslant 0$ then (2.16) frequently allows to obtain an estimate for the rate of decay of solutions or even to convert Theorem 3 into a result concerning (GUAS) or (GEXS) (the last holds f.i. if $Q=0$ or F satisfies a global Lipschitz condition).

Remark 4.

If in (H7) the semigroup $S(t)$ is compact for all $t>0$, then it suffices to assume only $\mathfrak{M} \subset \{$ P: P is continuous and maps bunded sets in X into bounded sets in U, $P(0)=0\}$ in (i) (see [2, p.193]).

3. Example.

Let us consider the electronic circuit shown in Fig.1a. The dynamics of this circuit is governed by the following system of equations

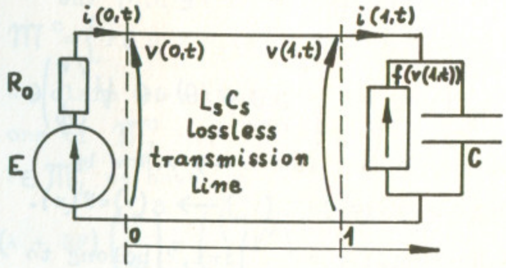

Fig.1a.

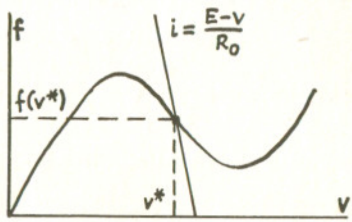

Fig.1b.

$$-L_s\frac{\partial I}{\partial t} = \frac{\partial V}{\partial x},\quad -C_s\frac{\partial V}{\partial t} = \frac{\partial I}{\partial x} \qquad 0 \leqslant x \leqslant 1, t>0 \qquad (3.1),$$

$$0 = V(0,t) + I(0,t)R_0, \quad -C\frac{dV(1,t)}{dt} = -I(1,t) + g[V(1,t)] \qquad (3.2),$$

where $g(y) = f(y+v^*) - f(v^*)$, f is the characteristic of the nonlinear active element (e.g. tunnel diode), v^* is defined in Fig.1b. The system (3.1), (3.2) can be reduced to a simple NFDE in the same way, as it was done by O. Lopes [9],

$$C\frac{d}{dt}\left[\xi(t) + K\xi(t-r)\right] = -g\left[\xi(t) + K\xi(t-r)\right] + \frac{K}{z}\xi(t-r) - \frac{1}{z}\xi(t) \quad t>0$$

where $K = (R_0 - z)\cdot\frac{1}{R_0+z}$, $z = \sqrt{L_s\cdot\frac{1}{C_s}}$, $r = \frac{2}{\sqrt{L_s C_s}}$ (3.3).

The system (3.3) can be interpreted as the abstract semilinear system (1.1). To do this, we take

$X = \mathbb{R}\times L^2(-r,0)$ with scalar product $\langle x_1, x_2\rangle = v_1v_2 + \int_{-r}^{0}\psi_1(\theta)\psi_2(\theta)\,d\theta$,

$x_i = \begin{pmatrix} v_i \\ \psi_i \end{pmatrix} \in X, i=1,2$; $U = \mathbb{R}$, $A\begin{pmatrix} v \\ \psi \end{pmatrix} = \begin{pmatrix} -\frac{1}{zC}v + \frac{2K}{zC}\psi(-r) \\ \psi' \end{pmatrix}$, $\mathcal{D}(A) = \left\{\begin{pmatrix} v \\ \psi \end{pmatrix} \in\right.$ $\left.\in X : \psi \in AC[-r,0], \psi' \in L^2(-r,0), \psi(0) + K\psi(-r) = v\right\}$, $Bu = \begin{pmatrix} -\frac{1}{C} \\ 0 \end{pmatrix}u$, $F\begin{pmatrix} v \\ \psi \end{pmatrix} = g(v)$

It can be proved that A is the infinitesimal generator of a linear C_0-semigroup on X (see [10] for details) while $B \in \mathcal{L}(U,X)$. Now it is quite natural to consider a class of mappings

$$D = \left\{F : F\begin{pmatrix} v \\ \psi \end{pmatrix} = g(v),\ g: \mathbb{R} \longrightarrow \mathbb{R} \text{ is a locally Lipschitz function}, g(0)=0\right\}.$$

Notice that if $F \in D$ then $F(0)=0$ and F maps every weakly convergent sequence $\{x_n\}_{n\in\mathbb{N}}$ in X into a strongly convergent sequence of values of F, $\{F(x_n)\}$ in U. Indeed, it is obvious that $F(0)=0$ for $F \in D$. By the definition of weak limit we have

$$X \ni \begin{pmatrix} v_n \\ \psi_n \end{pmatrix} = x_n \rightharpoonup x = \begin{pmatrix} v \\ \psi \end{pmatrix} \in X \text{ iff } \left\langle \begin{pmatrix} v_n \\ \psi_n \end{pmatrix}, \begin{pmatrix} w \\ \phi \end{pmatrix} \right\rangle = v_n w + \int_{-r}^{0}\phi(\theta)\psi_n(\theta)\,d\theta \quad \forall \begin{pmatrix} w \\ \phi \end{pmatrix} \in$$

$\in X$. Thus, in particular, for $w=1$ and $\phi=0$ one obtains $v_n \longrightarrow v$. Now by the continuity of g and since $F \in D$ we have $F(x_n) = g(v_n) \longrightarrow g(v) = F(x)$.

Let R be an arbitrary real positive number. If $x_i = \begin{pmatrix} v_i \\ \psi_i \end{pmatrix}$ $i=1,2$ belong to the ball $\{x \in X : \|x\| \leqslant R\}$ then v_i, $i=1,2$ belong to the ball $\{v \in \mathbb{R} : |v| \leqslant R\}$ and hence by the fact that g is a locally Lipschitz function there exists $l(R)>0$ such that

$\| F(x_1)-F(x_2)\|_U = |g(v_1)-g(v_2)| \leqslant l(R)|v_1-v_2| \leqslant l(R)\|x_1-x_2\|_X$ and

therefore the local Lipschitz property for F takes place. It follows from the above facts that if $\mathfrak{M}^c$ is any subset of D then the assumption (i) in Theorem 2 is satisfied.

Let us take $Qu=\begin{pmatrix} q \\ 0 \end{pmatrix}, q\in\mathbb{R}$. Then the operator QF is of gradient type for every $F\in D$ having the antiderivative $q\int_0^v g(s)ds$. Hence (H1) holds if $\mathfrak{M}^c$ is any subset of D. Assume that $|K|<1$, ε is a sufficiently small positive real and take the operators M,K,L of the forms :

$$M\begin{pmatrix} v \\ \psi \end{pmatrix}=\begin{pmatrix} -\frac{1-|K|}{zC(1+|K|)}v+\varepsilon v \\ 0 \end{pmatrix}, Lu=\begin{pmatrix} \left(\frac{1}{2C}+\frac{\varepsilon zr}{2(1-K^2)}\right)u \\ 0 \end{pmatrix}, Ku=\varepsilon u .$$

Then (H2) holds iff $\frac{g(v)}{v}$ lies between or at the roots of the following quadratic equation

$$Cy^2-y\left(\frac{1}{\varepsilon}+\frac{Czr}{1-K^2}\right)-\left(\frac{1-|K|}{\varepsilon z(1+|K|)}-C\right)=0 \qquad (3.4).$$

It is not difficult to check that if $|K|<1, q=0$ and ε is a sufficiently small positive real, then the assumption (H3) is satisfied, since for

$$H\begin{pmatrix} v \\ \psi \end{pmatrix}(\theta)=\begin{pmatrix} \left(\frac{1}{2}+\frac{Czr\varepsilon}{2(1-K^2)}\right)v \\ \left(\frac{|K|}{zC}+\varepsilon\left(\frac{r}{1-K^2}+\theta\right)\right)\psi(\theta) \end{pmatrix}$$

inequality (2.9) is fulfilled for every $(x,u)\in\mathcal{D}(A)\times U$. Now, in order to have the assumptions (H1), (H2), (H3) satisfied, we assume that $|K|<1$ and define the class $\mathfrak{M}^c$ is Theorem 2 as follows

$$\mathfrak{M}^c=\left\{F\in D : y_1(\varepsilon)\leqslant \frac{g(v)}{v}\leqslant y_2(\varepsilon)\ \ \forall v\neq 0, y_1 y_2 \text{ are the roots of } (3.4)\right\}$$

We claim that (H4) and (H5) also hold in this case. Indeed, by definitions of $\mathfrak{M}^c_L$ and $\mathfrak{M}^c$, $F\in\mathfrak{M}^c_L$ iff $F(x)=\mu v$, where $\mu\in[y_1,y_2]$. Thus, for $F\in\mathfrak{M}^c_L$ we have

$$(A+BF)\begin{pmatrix} v \\ \psi \end{pmatrix}=\begin{pmatrix} \left(-\frac{1}{zC}+\frac{\mu}{C}\right)v+\frac{2K}{zC}\psi(-r) \\ \psi' \end{pmatrix}, \mathcal{D}(A+BF)=\mathcal{D}(A).$$

A+BF generates an exponentially stable semigroup, independently of the delay r, iff the spectrum of A+BF lies in the open left complex half-

-plane and $|K|<1$. If $|K|<1$, then by Pontriagin's theorem this holds for $\mu \in \left(-\frac{1-|K|}{z(1+|K|)}, +\infty\right)$. As $y_1 > -\frac{1-|K|}{z(1+|K|)}$ (H4) is fulfilled. (H5) holds evidently, since Q=0. Now by Theorem 2 and Remark 1 we establish that if $|K|<1$ then the equilibrium $0 \in X$ is (GEXS) for every locally Lipschitz function $g: \mathbb{R} \longrightarrow \mathbb{R}$ such that $y_1 \leqslant \frac{g(v)}{v} \leqslant y_2 \ \forall v \neq 0$, $g(0)=0$, where y_1, y_2 are the roots of (3.4).

In the case of the tunnel diode characteristic f (Fig.1b) the fulfilment of the above conditions depends on the behaviour of f in a neighbourhood of v^* (in this case the lower constraint for $\frac{g(v)}{v}$ becomes active) and on the rate of growth of g for large $|v|$ (in this case the upper bound for $\frac{g(v)}{v}$ becomes active). To obtain the best possible conditions for (GEXS) one therefore should take ε as a very small number for we have $y_1(\varepsilon) \searrow -\frac{1-|K|}{z(1+|K|)}$, $y_2(\varepsilon) \nearrow +\infty$ as $\varepsilon \longrightarrow 0+$. If for large $|v|$ g behaves as, say, v in the power 3 then the upper bound for $\frac{g(v)}{v}$ can be reached for any $\varepsilon > 0$.

Now, we see the need for removing the upper bound for $\frac{g(v)}{v}$ from our conditions for (GEXS). To do this we use Theorem 3 and Remark 3, taking X,U,A,B,Q,M,L,H as previously and

$$\mathfrak{M} = \left\{ F : F\begin{pmatrix} v \\ \psi \end{pmatrix} = g(v);\ g: \mathbb{R} \longrightarrow \mathbb{R} \text{ is a locally Lipschitz function,} \right.$$

$$\left. g(0)=0,\ \frac{g(v)}{v} \geqslant -\frac{\frac{1-|K|}{z(1+|K|)}}{\frac{Czr\varepsilon}{1-K^2}+1} \left(\underset{\varepsilon \longrightarrow 0+}{\longrightarrow} -\frac{1-|K|}{z(1+|K|)} \right) \forall v \neq 0 \right\},$$

$K=0 \in \mathcal{L}(U)$ in Theorem 3. It can be readily seen that the assumption (i) and (H1), (H2) in (ii) hold. If one takes the following linear, A-bounded functional on X

$$G\begin{pmatrix} v \\ \psi \end{pmatrix} = \sqrt{\frac{1-|K|}{Cz}} \left[\sqrt{\frac{|K|}{1+|K|}}\ v - (\operatorname{sign} K)\sqrt{|K|(1+|K|)}\ \psi(-r) \right]$$ and

$V = 0 \in \mathcal{L}(U)$ then

$$\langle Ax, Hx \rangle + \langle x, HAx \rangle + \langle x, -Mx \rangle = -\langle Gx, Gx \rangle - \varepsilon \langle x, x \rangle \quad \forall x \in \mathcal{D}(A),$$

$$HB + L = 0 \ (\text{in } \mathcal{L}(U,X)), \quad K = V^* V \ (\text{in } \mathcal{L}(U)).$$

Thus the strengthened version of (H3'), mentioned in Remark 3, holds.

As for $\theta \in [-r, 0]$: $\theta + \frac{r}{1-K^2} \geqslant \frac{rK^2}{1-K^2} > 0$ then $\langle x, Hx \rangle \geqslant \alpha \|x\|_X^2$ $\forall x \in X$, where $\alpha = \min\left\{ \frac{1}{2} + \frac{\varepsilon Czr}{2(1-K^2)},\ \frac{|K|}{zC} + \frac{\varepsilon rK^2}{1-K^2} \right\} > 0$, so (H6) is fulfilled.

Let us recall that the semigroup generated by A is exponentially stable and every F belonging to $\mathfrak{M}$ is compact, therefore (H7) holds. In our example, we can take (2.14) with $-\varepsilon\int_0^t \|x(\tau,x_0)\|^2\,d\tau$ in the right-hand side, which yields (H8).

By Theorem 3 and Remark 3, the equilibrium $0\in X$ is (GEXS) for every g determined by $\mathfrak{M}$, provided that $|K|<1$. This result is equivalent to that given by O.Lopes [9, Theorem 3.5], who assumed that $C[-r,0]$ is the state space for (3.3).

More sophisticated considerations lead to the additional result conserning (GAS) of the equilibrium $0\in X$. The (GAS) takes place for all locally Lipschitz functions $g:\mathbb{R}\longrightarrow R$ such that $\frac{g(v)}{v} > -\frac{1-|K|}{2(1+|K|)}$ $\forall v \neq 0, g(0)=0$.

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