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# On the Spectral–Lyapunov Approach to Parametric Optimization of Distributed-Parameter Systems

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The paper is devoted to the discussion of the Riesz-bases application in the analysis of distributed-parameter systems controlled by a finite-dimensional or conventional controller. Both boundary control and boundary observation are allowed. The Riesz bases are constructed from a system of eigenfunctions of the closed-loop system operator. On the basis of the results obtained, they are expected to be an effective tool in proving that the closed-loop system is well-posed; i.e. they give rise to a $C_0$-semigroup. These bases also enable us to construct Lyapunov functionals in the form of series expansions. The analysis is illustrated by a completely worked-out example where the proportional controller setting is optimized with respect to the ISE criterion. The controller is applied to a parabolic plant modelling a resistive–capacitative (noninductive) direct-current transmission line.

## 1. Introduction and formulation of the problem

CONSIDER the system consisting of an RC (resistive–capacitative) transmission line with zero initial conditions and an ideal proportional amplifier with gain coefficient $K$, depicted in Fig. 1. The system is governed by the equations

$$0=-\frac{\partial V(x,\tau)}{\partial x}-RI(x,\tau) \quad \text{and} \quad C\frac{\partial V(x,\tau)}{\partial \tau}=-\frac{\partial I(x,\tau)}{\partial x} \quad (0\leqslant x\leqslant 1;\ \tau\geqslant 0),$$

$$I(1,\tau)=0 \quad \text{and} \quad V(0,\tau)=KV(1,\tau)+e \quad (\tau\geqslant 0), \qquad V(x,0)=0 \quad (0\leqslant x\leqslant 1).$$

*Problem*: Determine the value $K\neq 1$ for which the integral performance index

$$J=\frac{1}{RC}\int_0^\infty \left(V(1,\tau)-\frac{e}{1-K}\right)^2 \mathrm{d}\tau$$

achieves its minimal value.

The substitution $u(x,t)=V(x,RCt)-e/(1-K)$ reduces the system equations to the form

$$\left.\begin{aligned} &\frac{\partial u(x,t)}{\partial t}=\frac{\partial^2 u(x,t)}{\partial x^2} \quad (0\leqslant x\leqslant 1;\ t\geqslant 0),\\ &\left[\frac{\partial u(x,t)}{\partial x}\right]_{x=1}=0 \quad \text{and} \quad u(0,t)=Ku(1,t) \quad (t\geqslant 0),\\ &u(x,0)=e/(K-1) \quad (0\leqslant x\leqslant 1), \end{aligned}\right\} \tag{1.1}$$

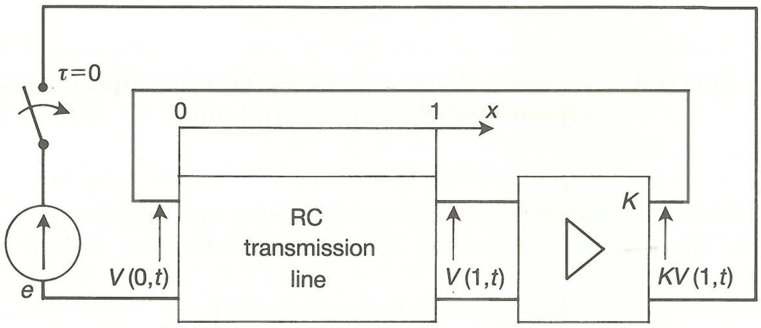


FIG. 1. A simple distributed-parameter feedback system.

and the performance index to the form

$$J = \int_0^\infty u^2(1, t)\,\mathrm{d}t. \tag{1.2}$$

Let $\mathcal{X} = \mathrm{L}^2(0, 1)$, with standard inner product

$$\langle u_1, u_2 \rangle_{\mathcal{X}} = \int_0^1 u_1(x)\overline{u_2(x)}\,\mathrm{d}x.$$

Choosing $\mathcal{X}$ to be the state space, we may write (1.1) as an abstract initial-value problem

$$\dot{u}(t) = Au(t) \quad (t > 0), \qquad u(0) = u_0 \tag{1.3}$$

with the unbounded linear operator

$$Au = u'', \qquad \mathbb{D}(A) = \{u \in \mathrm{H}^2(0, 1) \colon u'(1) = 0,\ u(0) = Ku(1)\} \tag{1.4}$$

and with the initial condition

$$u_0(x) = e/(K-1) \tag{1.5}$$

(note that $u_0 \notin \mathbb{D}(A)$). Simultaneously, the performance index (1.2) can be written formally as

$$J = \int_0^\infty |Qu(t)|^2\,\mathrm{d}t, \tag{1.6}$$

where $Q$ is an unbounded linear functional defined by the formula

$$Qu = u(1). \tag{1.7}$$

Let us notice that there exists a unique $r \in \mathcal{X}$, namely

$$r(x) = x/(K-1), \tag{1.8}$$

such that $Q$ may be represented as

$$Qu = u(1) = \langle Au, r \rangle_{\mathcal{X}} \quad \forall\, u \in \mathbb{D}(A). \tag{1.9}$$

Hence $Q$ is $A$-bounded.

This is a simple but nontrivial example of the determination of the proportional controller setting, optimal with respect to a quadratic integral functional, steering the distributed-parameter plant of parabolic type through the boundary, with feedback from the boundary observation.

To solve our problem, we first have to accomplish two fundamental tasks:

(i) establish the well-posedness of the closed-loop system, i.e. verify that the linear closed-loop-system operator generates a semigroup;

(ii) obtain an explicit expression for the performance index in terms of the system parameters—in the case of a positive solution of (i), this reduces to solving the Lyapunov operator equation.

Contrary to the classical problems met in mathematical physics, the presence of a feedback gives rise to non-selfadjoint operators, which is the source of several complications.

The aim of this paper is to show that elementary spectral methods employing Riesz bases are an effective tool, allowing us to establish semigroup generation as well as express the solution of the Lyapunov operator equation in the series form. The bases are constructed from a system of eigenfunctions of the linear operator describing the closed-loop system. If, as in the given example, we can explicitly determine the spectrum, then a significant further simplification of the formulae expressing the performance index is possible.

Note that an example of parametric optimization of the distributed-parameter system, using the eigenfunctions series expansion, is given also in Romicki (1981). The method used there is based on a somewhat differently understood Lyapunov operator equation, and leads to another computational algorithm. Moreover, the performance index analysed there corresponds to a bounded observation operator $\mathcal{Q}$, and most of the reasoning has only intuitive justification.

## 2. Riesz bases of eigenvectors and the semigroup generation problem

Let $\underline{Y}$ be a Banach space. A family $\{T(t)\}_{t\geq 0}$ of operators in $\mathbb{L}(\underline{Y})$ is said to be a $C_0$-*semigroup* on $\underline{Y}$ if the following conditions are satisfied:

(i) $T(0) = I, \qquad T(t+s) = T(t)T(s) \quad \forall\, t,s \geq 0,$

(ii) $T(t)y \xrightarrow[t\to 0+]{} y \quad \forall\, y \in \underline{Y}.$

If, additionally,

(iii) for each fixed $y \in \underline{Y}$, the map $t \mapsto T(t)y$ is a real analytic function on $(0, \infty)$,

then $\{T(t)\}_{t\geq 0}$ is an *analytic semigroup* on $\underline{Y}$.

The linear operator $D$: $(\mathbb{D}(D) \subset \underline{Y}) \to \underline{Y}$ defined as

$$Dy = \lim_{t\to 0+} [T(t)y - y]/t,$$

with the domain of all $y \in \underline{Y}$ for which this limit exists, is called the *infinitesimal generator* of a linear $C_0$-semigroup $\{T(t)\}_{t\geq 0}$ on $\underline{Y}$.

The following two theorems characterize the generators of analytic and $C_0$-semigroups.

HILLE'S THEOREM (Henry, 1981; Pazy, 1983) *A linear operator* $D$: $(\mathbb{D}(D) \subset \mathcal{Y}) \to \mathcal{Y}$ *acting on a Banach space* $\mathcal{Y}$ *is the infinitesimal generator of a linear analytic semigroup on* $\mathcal{Y}$ *iff* $D$ *is closed and densely defined and there exist* $b \in \mathbb{R}$, $\theta \in (\frac{1}{2}\pi, \pi)$, *and* $M \geqslant 1$ *such that* (i) *the sector* $\mathcal{S}_{b,\theta} = \{\lambda \in \mathbb{C}: \theta \geqslant |\arg(\lambda - b)|, \lambda \neq b\}$ *is contained in the resolvent set* $\rho(D)$ *and* (ii) *the resolvent of* $D$ *satisfies the estimate*

$$\|(\lambda I - D)^{-1}\| \leqslant M/|\lambda - b| \quad \forall \lambda \in \mathcal{S}_{b,\theta}. \tag{2.1}$$

THE HILLE–PHILLIPS–YOSIDA THEOREM *A linear operator* $D$: $(\mathbb{D}(D) \subset Y) \to \mathcal{Y}$ *acting on a Banach space* $\mathcal{Y}$ *is the infinitesimal generator of a linear* $C_0$*-semigroup on* $\mathcal{Y}$ *iff* $D$ *is closed and densely defined, and there exist* $b \in \mathbb{R}$ *and* $M \geqslant 1$ *such that the halfplane* $\Pi_b = \{\lambda \in \mathbb{C}: \operatorname{Re} \lambda > b\}$ *is contained in the resolvent set* $\rho(D)$ *and the resolvent of* $D$ *satisfies the estimate*

$$\|(\lambda I - D)^{-n}\| \leqslant M/(\operatorname{Re} \lambda - b)^n \quad \forall \lambda \in \Pi_b \quad \forall n \in \mathbb{N}.$$

We now turn to the Riesz bases. Let $\mathcal{H}$ be a Hilbert space with the inner product $\langle \bullet, \bullet \rangle_{\mathcal{H}}$. Two vector systems $\{\phi_i\}_{i \in \mathcal{J}}$ and $\{\psi_i\}_{i \in \mathcal{J}}$, where $\mathcal{J}$ is the set of indices, are *biorthogonal* iff

$$\langle \phi_i, \psi_j \rangle_H = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Now let $\{\phi_i\}_{i \in \mathcal{J}}$ be a system for which there exists a biorthogonal system $\{\psi_i\}_{i \in \mathcal{J}}$. We say that $\{\phi_i\}_{i \in \mathcal{J}}$ is a *Riesz basis* of $\mathcal{H}$ if there exists a linear, bounded, and boundedly invertible operator $S$ such that $\phi_i = Se_i$, where $\{e_i\}_{i \in \mathcal{J}}$ is an orthonormal basis of $\mathcal{H}$.

For other characterizations of this concept, see Bari (1951), Gohberg & Krein (1969), and Nikolskii (1985). In what follows, some auxiliary results will be useful.

THEOREM 1 (Boas, 1941; Bari, 1951; Gohberg & Krein, 1969; Nikolskii, 1985.) *Let* $\{\phi_i\}_{i \in \mathcal{J}}$ *be a system to which* $\{\psi_i\}_{i \in \mathcal{J}}$ *is biorthogonal. Then* $\{\phi_i\}_{i \in \mathcal{J}}$ *is a Riesz basis iff the Gram matrix of this system, i.e.* $\{\langle \phi_i, \phi_j \rangle\}_{i,j \in \mathcal{J}}$, *defines a linear, bounded, and boundedly invertible operator on* $\ell^2(\mathcal{J})$.

LEMMA 1 (Bari 1951: pp. 73, 77) *If* $\{\phi_i\}_{i \in \mathcal{J}}$ *is a Riesz basis of* $\mathcal{H}$, *then there exist positive constants* $m_{\mathrm{b}}$ *and* $m_{\mathrm{h}}$ *such that*

$$m_{\mathrm{h}}^2 \|f\|^2 \leqslant \sum_{i \in \mathcal{J}} |\langle f, \psi_i \rangle_{\mathcal{H}}|^2 \leqslant m_{\mathrm{b}}^2 \|f\|^2 \quad \forall f \in \mathcal{H},$$

*where* $\{\psi_i\}_{i \in \mathcal{J}}$ *is the (unique) system biorthogonal to* $\{\phi_i\}_{i \in \mathcal{J}}$.

LEMMA 2 *Let* $P$: $(\mathbb{D}(P) \subset \mathcal{H}) \to \mathcal{H}$ *be a closed linear operator having exclusively single eigenvalues* $\mu_i$ $(i \in \mathcal{J})$ *such that the corresponding system of eigenvectors* $\{\phi_i\}_{i \in \mathcal{J}}$ *is minimal and complete in* $\mathcal{H}$. *The unique system biorthogonal to* $\{\phi_i\}_{i \in \mathcal{J}}$, *which then exists, is a system of eigenvectors of* $P^*$, *the adjoint of* $P$.

*Proof.* Since $\{\phi_i\}_{i \in \mathcal{J}}$ is minimal and complete, there is a unique biorthogonal system, which will be denoted by $\{\psi_i\}_{i \in \mathcal{J}}$. Now $(\lambda_i - \lambda_j)\langle \phi_i, \psi_j \rangle_{\mathcal{H}} = 0$, or equivalently

$$\langle \lambda_i \phi_i, \psi_j \rangle_{\mathcal{H}} = \langle P\phi_i, \psi_j \rangle_{\mathcal{H}} = \langle \phi_i, \bar{\lambda}_j \psi_j \rangle_{\mathcal{H}}.$$

The basis $\{\phi_i\}_{i\in\mathcal{J}}$ is complete, and $\phi_i \in \mathbb{D}(P)$ for $i \in \mathcal{J}$. Hence $\overline{\mathbb{D}(P)} = \mathcal{H}$, and the definition of $P^*$ yields $\psi_j \in \mathbb{D}(P^*)$ and $P^*\psi_j = \bar{\lambda}_j\psi_j$ $(j \in \mathcal{J})$. □

The next theorem is a particular case of known results (Röh, 1982: pp. 25–8; Curtain 1984: Thm 2); however, the proof given here is elementary.

THEOREM 2 *Let $\mathcal{H}$ be a Hilbert space with an inner product $\langle \bullet, \bullet \rangle_{\mathcal{H}}$ and let $D$: $(\mathbb{D}(D) \subset \mathcal{H}) \to \mathcal{H}$ be a linear discrete operator (i.e. $(\lambda I - D)^{-1}$ is compact for $\lambda \in \rho(D)$) having exclusively single eigenvalues $\lambda_i$ $(i \in \mathcal{J})$. Assume also that there is a system $\{\phi_i\}_{i\in\mathcal{J}}$ of eigenvectors, corresponding to $\{\lambda_i\}_{i\in\mathcal{J}}$, which forms a Riesz basis of $\mathcal{H}$.*

*A necessary and sufficient condition for $D$ to be the infinitesimal generator of a linear analytic semigroup on $\mathcal{H}$ is the existence of $b \in \mathbb{R}$ and $\theta_0 \in (\frac{1}{2}\pi, \pi)$ such that $\rho(D)$ contains the sector $S_{b,\theta_0} = \{\lambda \in \mathbb{C}: \theta_0 \geq |\arg(\lambda - b)|, \lambda \neq b\}$.*

*A necessary and sufficient condition for $D$ to be the infinitesimal generator of a linear $C_0$-semigroup on $\mathcal{H}$ is the existence of $b \in \mathbb{R}$ such that $\rho(D)$ contains the halfplane $\Pi_b = \{\lambda \in \mathbb{C}: \operatorname{Re}\lambda > b\}$. Moreover, this semigroup is exponentially stable* (ExS) *iff*

$$\sup_{\lambda_i \in \sigma(D)} \operatorname{Re}\lambda_i < 0,$$

*where $\sigma(D)$ denotes the spectrum of $D$.*

*Proof.* The necessary conditions easily follow from Hille's theorem, the Hille–Phillips–Yosida theorem, and exponential stability. We now prove the sufficient condition for the generation of analytic semigroup. $D$ is closed because $(\lambda I - D)^{-1} \in \mathbb{L}(\mathcal{H})$ (Weidmann, 1980: p. 89). Also, span $\{\phi_i\}_{i\in\mathcal{J}} \subset \mathbb{D}(D)$, and $\{\phi_i\}_{i\in\mathcal{J}}$ is complete, which implies the density of $\mathbb{D}(D)$ in $\mathcal{H}$. Now consider the sector $S_{b,\theta}$, defined as $S_{b,\theta_0}$ but with the angle of obtuseness $\theta < \theta_0$, with $\theta \in (\frac{1}{2}\pi, \pi)$. Then $S_{b,\theta} \subset S_{b,\theta_0} \subset \rho(D)$. So, by Hille's theorem, it suffices to show the existence of $M \geq 1$ for which (2.1) holds. Let us inspect Fig. 2. For $\lambda$ belonging to the shaded cone, the inequalities

$$|\lambda - b| \leq |\lambda - \lambda_i| \quad (i \in \mathcal{J})$$

hold. If $\lambda$ belongs to the remaining part of $S_{b,\theta}$, then

$$|\lambda - b|\sin(\theta_0 - \theta) \leq |\lambda - \lambda_i| \quad \forall i \in \mathcal{J}.$$

These estimates together yield

$$\frac{1}{|\lambda - \lambda_i|^2} \leq \frac{\operatorname{cosec}^2(\theta_0 - \theta)}{|\lambda - b|^2} \quad \forall i \in \mathcal{J} \quad \forall \lambda \in S_{b,\theta}.$$

By virtue of Lemma 1,

$$\begin{aligned} \|(\lambda I - D)^{-1}f\|^2 &\leq \frac{1}{m_{\mathrm{h}}^2}\sum_{i\in\mathcal{J}} |\langle(\lambda I - D)^{-1}f, \psi_i\rangle_{\mathcal{H}}|^2 \\ &= \frac{1}{m_{\mathrm{h}}^2}\sum_{i\in\mathcal{J}} |\langle f, (\bar{\lambda} I - D^*)^{-1}\psi_i\rangle_{\mathcal{H}}|^2 = \frac{1}{m_{\mathrm{h}}^2}\sum_{i\in\mathcal{J}} \frac{1}{|\lambda - \lambda_i|^2}|\langle f, \psi_i\rangle_{\mathcal{H}}|^2 \\ &\leq \frac{\operatorname{cosec}^2(\theta_0 - \theta)}{m_{\mathrm{h}}^2|\lambda - b|^2}\sum_{i\in\mathcal{J}} |\langle f, \psi_i\rangle_{\mathcal{H}}|^2 \leq \frac{m_{\mathrm{b}}^2}{m_{\mathrm{h}}^2}\frac{\operatorname{cosec}^2(\theta_0 - \theta)}{|\lambda - b|^2}\|f\|^2 \quad \forall f \in \mathcal{H} \quad \forall \lambda \in S_{b,\theta}, \end{aligned} \tag{2.2}$$

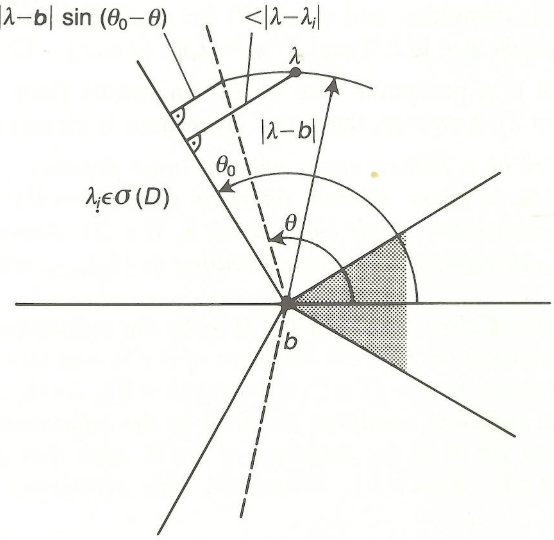


FIG. 2. The auxiliary diagram for the proof of analytic-semigroup generation.

where $\{\psi_i\}_{i\in\mathcal{J}}$ is the system biorthogonal to $\{\phi_i\}_{i\in\mathcal{J}}$. Hence (2.1) holds with

$$M = (m_b/m_h)\operatorname{cosec}(\theta_0 - \theta) \geq 1.$$

Replacing $(\lambda I - D)^{-1}$ by $(\lambda I - D)^{-n}$ or by the semigroup operator, and modifying appropriately the sequence of inequalities in (2.2), one gets the estimate needed to prove the other sufficient conditions. □

## 3. The Lyapunov operator equation

In this section, we prove a few results concerning the existence, uniqueness, and regularity of solution of the Lyapunov operator equation.

THEOREM 3 *Let $\mathcal{H}$ and $\mathcal{Y}$ be Hilbert spaces with inner products $\langle \bullet, \bullet\rangle_{\mathcal{H}}$ and $\langle \bullet, \bullet\rangle_{\mathcal{Y}}$ respectively. Suppose that a linear operator $D$: $(\mathbb{D}(D)\subset \mathcal{H})\to \mathcal{H}$ is the infinitesimal generator of a linear $C_0$-semigroup $\{T(t)\}_{t\geq 0}$ on $\mathcal{H}$, and that $U$: $(\mathbb{D}(U)\subset H)\to \mathcal{Y}$ is a linear D-bounded operator. Then, for the existence of an operator $W\in \mathbb{L}(\mathcal{H})$, with $W=W^*$ and $W\geq 0$, satisfying the Lyapunov operator equation*

$$\langle Du, Wv\rangle_{\mathcal{H}} + \langle u, WDv\rangle_{\mathcal{H}} = -\langle Uu, Uv\rangle_{\mathcal{Y}} \quad \forall\, u,v\in \mathbb{D}(D), \tag{3.1}$$

*it is necessary and sufficient that a positive number $\gamma$ exists such that*

$$\int_0^\infty \|UT(t)u\|^2_{\mathcal{Y}}\,dt \leq \gamma\,\|u\|^2_{\mathcal{H}} \quad \forall\, u\in \mathbb{D}(D). \tag{3.2}$$

*Proof.* Notice that, for a $D$-bounded operator $U$, the function $[0,\infty)\ni t\mapsto UT(t)u\in Y$ is continuous for every fixed $u\in \mathbb{D}(D)$.

(Necessity). Putting $u = v = T(t)f$, for $f \in \mathbb{D}(D)$ and $t \geq 0$, in (3.1), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle T(t)f, WT(t)f\rangle_{H} = \langle DT(t)f, WT(t)f\rangle_{H} + \langle T(t)f, WDT(t)f\rangle_{H} = -\|UT(t)f\|^2_{\mathcal{Y}},$$

and hence

$$\int_0^t \|UT(\tau)f\|^2_{\mathcal{Y}}\,\mathrm{d}\tau = \langle f, Wf\rangle_{H} - \langle T(t)f, WT(t)f\rangle_{H} \leq \langle f, Wf\rangle_{H} \leq \|W\|\;\|f\|^2_{H}.$$

The integrand is nonnegative and continuous; thus the function

$$t \mapsto \int_0^t \|UT(\tau)f\|^2_{\mathcal{Y}}\,\mathrm{d}\tau$$

is $C^1$, monotone increasing, and bounded from above. This implies the convergence of the improper integral, and (3.2) holds with $\gamma = \|W\|$.
(Sufficiency) By (3.2), the operator $P$: $H \to L^2(0, \infty; \mathcal{Y})$ defined as

$$(Pu)(t) = UT(t)u, \qquad \mathbb{D}(P) = \mathbb{D}(D), \tag{3.3}$$

is linear, densely defined, and bounded. Thus $P$ is closable, and its closure $\bar{P}$ is the unique continuous extension of $P$ to the whole space $H$ (Weidmann, 1980: Thm 4.5, p. 58, Thm 5.2, p. 84, Thm 5.3, p. 90). Now, the bilinear form

$$\langle u, Wv\rangle_{H} = \langle \bar{P}u, \bar{P}v\rangle_{L^2(0, \infty; \mathcal{Y})} \tag{3.4}$$

uniquely determines an operator $W \in \mathbb{L}(H)$, with $W = W^*$ and $W \geq 0$. By virtue of (3.3), (3.4), and the semigroup axiom (i), for $u, v \in \mathbb{D}(D)$ we have

$$\langle T(t)u, WT(t)v\rangle_{H} = \int_t^{\infty} \langle UT(\tau)u, UT(\tau)v\rangle_{\mathcal{Y}}\,\mathrm{d}\tau, \; (t \geq 0).$$

Hence, taking the right derivative at $t = 0$, we get (3.1). □
In Datko (1970), this theorem is proved in the case where $D \in \mathbb{L}(H, \mathcal{Y})$.

If $D$ has an inverse $D^{-1} \in \mathbb{L}(H)$, then (3.1) is equivalent to the following equation in $\mathbb{L}(H)$:

$$(D^*)^{-1}W + WD = -Z^*Z, \qquad Z = UD^{-1} \in \mathbb{L}(H, \mathcal{Y}). \tag{3.5}$$

THEOREM 4 *Under the assumptions of Theorem* 3, *the asymptotic stability of the semigroup* $\{T(t)\}_{t \geq 0}$, *i.e.*

$$T(t)u \xrightarrow[t \to \infty]{} 0 \quad \forall\, u \in H, \tag{3.6}$$

*is sufficient for the uniqueness of solution to* (3.1).

*Proof.* Let $W_1, W_2 \in \mathbb{L}(H)$, with $W_k = W_k^* \geq 0$ $(k = 1, 2)$, be two solutions of (3.1). Then, taking $u = T(t)f$ and $v = T(t)g$, with $f, g \in \mathbb{D}(D)$, and proceeding as in the proof of Theorem 3 (the 'necessity' part), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle T(t)f, (W_1 - W_2)T(t)g\rangle_{H} = 0 \quad (t \geq 0).$$

Hence, by (3.6), and noting that $\overline{\mathbb{D}(D)} = \mathcal{H}$, we get $\langle f, (W_1 - W_2)g \rangle_{\mathcal{H}} = 0 \ \forall f, g \in \mathcal{H}$, and the application of the Hahn–Banach theorem completes the proof. □

It is worth emphasizing that, for semigroups whose infinitesimal generators have compact resolvents, asymptotic stability is equivalent to weak asymptotic stability, i.e. $T(t)u \rightharpoonup_{t\to\infty} 0 \quad \forall u \in \mathcal{H}$ (Slemrod, 1974; Benchimol, 1977: p. 25). Theorems 3 and 4, in somewhat different versions, appeared in Grabowski (1983). The next result relates to the existence of a special type of solution to (3.1).

THEOREM 5 *Suppose that $\mathcal{H}$, $\mathcal{Y}$, and $D$ are as in Theorem* 3, *but now $U$ is a $D$-bounded functional on $\mathcal{H}$ for which there exists $k \in \mathrm{L}^2(0, \infty)$ such that*

$$|UT(t)u| \leqslant k(t)\, \|u\|_{\mathcal{H}} \quad \forall u \in \mathbb{D}(D) \quad \textit{for almost all } t \geqslant 0. \tag{3.7}$$

*Then* (3.1) *has a solution $W \in \mathbb{L}(\mathcal{H})$, with $W = W^*$ and $W \geqslant 0$, and $W$ is a Hilbert–Schmidt* (HS) *operator.*

*Proof.* Since (3.7) implies (3.2), it is sufficient to show that a solution $W$, defined by (3.3)–(3.4), is an HS operator (Weidmann 1980: Thm 6.12, p. 140, Thm 6.10, p. 137). Hence there exists an orthonormal basis $\{e_i\}_{i\in\mathcal{J}}$ in $\mathcal{H}$ for which

$$\sum_{i\in\mathcal{J}} \|\bar{P}e_i\|^2_{\mathrm{L}^2(0,\infty)} = \sum_{i\in\mathcal{J}} \langle e_i, We \rangle_{\mathcal{H}} = \sum_{i\in\mathcal{J}} \|W^{\frac{1}{2}}e_i\|^2_{\mathcal{H}} < \infty.$$

Thus $W^{\frac{1}{2}}$ is an HS operator, which implies that $W$ is an HS operator. □

The assertion of this theorem remains true for finite-rank $D$-bounded operators $U$: $\mathcal{X} \to \mathcal{Y}$ satisfying (3.7). The assumption $\dim \mathcal{Y} < \infty$ is essential. The implication (3.2)$\Rightarrow$(3.7) does not hold generally.

*Remark.* Assume that $D$ is the infinitesimal generator of an ExS linear analytic semigroup, and that $U$ is a linear functional bounded on $\mathcal{H}^\alpha$, where $\mathcal{H}^\alpha$ $(0 \leqslant \alpha \leqslant 1)$ denotes the Banach space equal to $\mathbb{D}(D^\alpha)$, the domain of a fractional power of $D$, equipped with the topology induced by the norm $\|u\|_{\mathcal{H}^\alpha} = \|D^\alpha u\|_{\mathcal{H}}$. Then (3.7) holds for $\alpha < \frac{1}{2}$. Indeed, there exists $\delta > 0$ such that

$$|UT(t)u| \leqslant c_\alpha \, \|T(t)u\|_{\mathcal{H}^\alpha} = c_\alpha \, \|D^\alpha T(t)u\|_{\mathcal{H}} \leqslant \bar{c}_\alpha t^{-\alpha} \mathrm{e}^{-\delta t} \, \|u\|_{\mathcal{H}}$$

(Pazy, 1983: p. 74). For $\alpha < \frac{1}{2}$, the function $k$: $t \mapsto \bar{c}_\alpha t^{-\alpha}\mathrm{e}^{-\delta t}$ belongs to $\mathrm{L}^2(0, \infty)$ (Dwight, 1961: 860.19, 21).

The last theorem of this section enables us to express the solution to the Lyapunov operator equation in the Riesz basis of eigenvectors of the semigroup generator.

THEOREM 6 *Suppose that $\mathcal{H}$ and $\mathcal{Y}$ are Hilbert spaces with inner products $\langle \bullet, \bullet \rangle_{\mathcal{H}}$ and $\langle \bullet, \bullet \rangle_{\mathcal{Y}}$ respectively. Let $D$: $(\mathbb{D}(D) \subset \mathcal{H}) \to \mathcal{H}$ be a linear operator satisfying the assumptions of Theorem* 2 *and generating an* ExS *linear* $\mathrm{C}_0$*-semigroup* $\{T(t)\}_{t\geqslant 0}$ *on $\mathcal{H}$. Also, let $U$: $\mathcal{X} \to \mathcal{Y}$ be a linear $D$-bounded operator for which* (3.2) *holds. Then the unique solution of the Lyapunov operator equation* (3.1) *has the spectral representation*

$$\langle u, Wv \rangle_{\mathcal{H}} = \sum_{i\in\mathcal{J}} \operatorname*{res}_{\lambda=\lambda_i} \langle U(-\bar{\lambda}I - D)^{-1}u,\; U(\lambda I - D)^{-1}v \rangle_{\mathcal{Y}} \quad \forall u, v \in \mathcal{H}. \tag{3.8}$$

*Proof.* It follows from Theorems 3 and 4 that (3.1) has a unique solution $W \in \mathbb{L}(\mathcal{H})$ with $W = W^*$ and $W \geq 0$. Denote by $\{\phi_i\}_{i \in \mathcal{J}}$ the Riesz basis built from eigenvectors of $D$, and by $\{\psi_i\}_{i \in \mathcal{J}}$ its biorthogonal system. Putting $v = \phi_i$ for $i \in \mathcal{J}$ in (3.1), we obtain

$$\langle (D + \bar{\lambda}_i I)u, W\phi_i \rangle_{\mathcal{H}} = -\langle Uu, U\phi_i \rangle_{\mathcal{Y}} \quad \forall u \in \mathbb{D}(D) \quad \forall i \in \mathcal{J}. \tag{3.9}$$

By exponential stability, $-\bar{\lambda}_i \in \rho(D)$, and (3.9) is equivalent to

$$\langle f, W\phi_i \rangle_{\mathcal{H}} = -\langle U(\bar{\lambda}_i I + D)^{-1} f, U\phi_i \rangle_{\mathcal{Y}} \quad \forall f \in \mathcal{H} \quad \forall i \in \mathcal{J}. \tag{3.10}$$

Taking into account the fact that $\{\phi_i\}_{i \in \mathcal{J}}$ is a Riesz basis and $W \in \mathbb{L}(\mathcal{H})$, we get from (3.10) that

$$\begin{aligned}
\langle f, Wg \rangle_{\mathcal{H}} &= \Big\langle f, \sum_{i \in \mathcal{J}} \langle g, \psi_i \rangle_{\mathcal{H}} W\phi_i \Big\rangle_{\mathcal{H}} = \sum_{i \in \mathcal{J}} \langle f, W\phi_i \rangle_{\mathcal{H}} \overline{\langle g, \psi_i \rangle_{\mathcal{H}}} \\
&= \sum_{i \in \mathcal{J}} \langle U(-\bar{\lambda}_i I - D)^{-1} f, U(\langle g, \psi_i \rangle_{\mathcal{H}} \phi_i) \rangle_{\mathcal{Y}} \\
&= \sum_{i \in \mathcal{J}} \Big\langle U(-\bar{\lambda}_i I - D)^{-1} f, U \operatorname*{res}_{\lambda = \lambda_i} (\lambda I - D)^{-1} g \Big\rangle_{\mathcal{Y}} \quad \forall f, g \in \mathcal{H}. \quad \square
\end{aligned}$$

In the next four sections, we give a complete analysis of the example from Section 1

## 4. Spectral properties of the system

The resolvent of $A$ can be represented in the form

$$\begin{aligned}
(\lambda I - A)^{-1} v(x) &= \frac{-K}{\cosh \lambda^{\frac{1}{2}} - K} \int_0^1 \mathrm{H}(y - x) \frac{\sinh \lambda^{\frac{1}{2}}(x - y)}{\lambda^{\frac{1}{2}}} v(y)\, dy \\
&+ \frac{1}{\cosh \lambda^{\frac{1}{2}} - K} \int_0^1 \begin{Bmatrix} \lambda^{-\frac{1}{2}} \sinh \lambda^{\frac{1}{2}} x \cosh \lambda^{\frac{1}{2}}(1 - y) & (x < y) \\ \lambda^{-\frac{1}{2}} \sinh \lambda^{\frac{1}{2}} y \cosh \lambda^{\frac{1}{2}}(1 - x) & (x > y) \end{Bmatrix} v(y)\, dy \quad (v \in \mathcal{X}),
\end{aligned} \tag{4.1}$$

where $\mathrm{H}$ is the Heaviside function. Let $K \neq 1$. Then, taking $\lambda = 0$, one easily establishes that $A^{-1}$ is compact, since it is an HS operator (Weidmann, 1980: Thm 6.10, p. 137, Thm 6.11, p. 139). Hence $(\lambda I - A)^{-1}$ is compact for all $\lambda \in \rho(A)$ (Pazy, 1983: p. 49). For $K = 1$, a similar conclusion can be derived by taking $\lambda = 1$. Finally, the resolvent $(\lambda I - A)^{-1}$ is compact for all $\lambda \in \rho(A)$ and $K \in \mathbb{R}$, i.e. $A$ is a discrete operator. The spectrum $\sigma(A)$ of $A$ consists exclusively of eigenvalues (the poles of the resolvent), namely

$$\sigma(A) = \{\lambda \in \mathbb{C}: \cosh \lambda^{\frac{1}{2}} = K\} = \{\lambda_n : n \in \mathbb{Z}\},$$

where

$$\lambda_n = \begin{cases} [\ln^2 \Delta - (2n\pi + \pi)^2] + 2\mathrm{i}(2n\pi + \pi) \ln \Delta & \text{if } K < -1, \\ -(\varphi - 2n\pi)^2 & \text{if } |K| \leq 1, \\ [\ln^2 \Delta - 4n^2\pi^2] + 4n\pi\mathrm{i} \ln \Delta & \text{if } K > 1, \end{cases} \tag{4.2}$$

in which

$$\Delta = |K| + (K^2 - 1)^{\frac{1}{2}} \quad (|K| > 1), \qquad \varphi = \arccos K \quad (|K| \leq 1). \tag{4.3a,b}$$

Hence

$$\lambda \in \sigma(A) \quad \Rightarrow \quad \begin{cases} \frac{1}{4}\operatorname{Im}^2\lambda + \operatorname{Re}\lambda \ln^2 \Delta = \ln^4 \Delta & \text{if } |K| > 1, \\ \operatorname{Im}\lambda = 0 \quad \text{and} \quad \operatorname{Re}\lambda \leq 0 & \text{if } |K| \leq 1 \end{cases} \tag{4.4}$$

(so $\sigma(A)$ is contained by a parabola or ray respectively);

$$\sup_{\lambda \in \sigma(A)} \operatorname{Re}\lambda < 0 \quad \text{iff} \quad K \in (-\cosh \pi, 1). \tag{4.5}$$

Moreover, all eigenvalues of $A$ are single if $|K| \neq 1$, and double if $|K| = 1$ (except for the zero eigenvalue, which is single for $K = 1$). The following formula will also be useful:

$$\lambda_n^{\frac{1}{2}} = \begin{Bmatrix} \ln \Delta + (2n\pi + \pi)\mathrm{i} & \text{if } K < -1 \\ \mathrm{i}\,|\varphi - 2n\pi| & \text{if } |K| \leq 1 \\ \ln \Delta + 2n\pi \mathrm{i} & \text{if } K > 1 \end{Bmatrix} \quad (n \in \mathbb{Z}). \tag{4.6}$$

Lemma 3 *For $|K| \neq 1$, a system of eigenvectors of $A$ forms a Riesz basis of $X$.*

*Proof.* We know that $|K| \neq 1$ guarantees that all eigenvalues are simple and $A^{-1}$ is an HS operator. The corresponding eigenfunctions satisfy the boundary-value problem

$$u_n''(x) = \lambda_n u_n(x), \qquad u_n'(1) = 0, \quad u_n(0) = K u_n(1),$$

for $n \in \mathbb{Z}$, and from Biyarov (1989) we deduce that they form a complete system. Let

$$(z, \mu_n) = \begin{Bmatrix} (-\Delta, \ -\ln \Delta - (2n\pi + \pi)\mathrm{i}) & \text{if } K < -1 \\ (\mathrm{e}^{\mathrm{i}\varphi}, \ -\mathrm{i}(\varphi - 2n\pi)) & \text{if } |K| < 1 \\ (\Delta, \ -\ln \Delta - 2n\pi \mathrm{i}) & \text{if } K > 1 \end{Bmatrix} \quad (n \in \mathbb{Z}).$$

By virtue of (4.2)–(4.3), we have

$$\tfrac{1}{2}(z + z^{-1}) = K, \tag{4.7a}$$

$$\mu_n^2 = \lambda_n \quad \text{and} \quad \mathrm{e}^{-\mu_n} = z \quad (n \in \mathbb{Z}). \tag{4.7b}$$

The operator $L$ defined as

$$L\phi(x) = C[z\phi(x) + \phi(1 - x)] \quad (0 \leq x \leq 1), \tag{4.8}$$

where $C$ is an arbitrary fixed nonzero complex number, belongs to $\mathbb{L}(X)$ together with its inverse $L^{-1}$, given by

$$L^{-1}u(x) = \frac{zu(x) - u(1 - x)}{C(z^2 - 1)} \quad (0 \leq x \leq 1);$$

note that $z^2 \neq 1$ because $|K| \neq 1$. It follows from the identities (4.7) that $L$ transforms the system of exponentials

$$\phi_n(x) = \mathrm{e}^{\mu_n x} \quad (0 \leq x \leq 1; n \in \mathbb{Z}) \tag{4.9}$$

into the system of eigenfunctions of $A$:

$$u_n(x) = L\phi_n(x) = C(z\mathrm{e}^{\mu_n x} + \mathrm{e}^{\mu_n(1-x)}) = C(\mathrm{e}^{\mu_n(x-1)} + \mathrm{e}^{\mu_n(1-x)})$$
$$= 2C \cosh [\mu_n(1 - x)] \quad (0 \leq x \leq 1; n \in \mathbb{Z}). \tag{4.10}$$

Now the proof reduces to showing that the system of exponentials forms a Riesz basis of $\mathcal{X}$. To construct the system biorthogonal to (4.9) we notice that the functions of (4.9) are eigenfunctions of the operator $B$, where

$$B\phi = \phi', \qquad \mathbb{D}(B) = \{\phi \in \mathrm{H}^1(0,1): \phi(0) = z\phi(1)\}. \tag{4.11}$$

By Lemma 2, for (4.9) to define a Riesz basis of $\mathcal{X}$, it is necessary that $\{\phi_n\}_{n\in\mathbb{Z}}$ has a unique biorthogonal system which is a system of eigenvectors of $B^*$, where

$$B^*\psi = -\psi', \ \mathbb{D}(B^*) = \{\psi \in \mathrm{H}^1(0,1): \bar{z}\psi(0) = \psi(1)\}. \tag{4.12}$$

This allows us to determine easily the biorthogonal system

$$\psi_n(x) = \mathrm{e}^{-\bar{\mu}_n x} \quad (0 \leqslant x \leqslant 1; n \in \mathbb{Z}). \tag{4.13}$$

Since for $|K| < 1$ we have $\mu_n = -\overline{\mu_n}$, then $\phi_n = \psi_n$ $(n \in \mathbb{Z})$, and the system of exponentials is an orthonormal basis (!), which is obviously a Riesz basis of $\mathcal{X}$. For $|K| > 1$, we find

$$\langle \phi_n, \phi_m \rangle_{\mathcal{X}} = \frac{\mathrm{e}^{\mu_n + \bar{\mu}_m} - 1}{\mu_n + \overline{\mu_m}} = \frac{|z|^2 - 1}{2|z|^2 \ln \Delta} \cdot \frac{1}{1 + a\mathrm{i}(n-m)},$$

$$\langle \psi_n, \psi_m \rangle_{\mathcal{X}} = \frac{1 - \mathrm{e}^{-(\bar{\mu}_n + \mu_m)}}{\overline{\mu_n} + \mu_m} = \frac{|z|^2 - 1}{2 \ln \Delta} \cdot \frac{1}{1 + a\mathrm{i}(m-n)},$$

where $a = \pi/\ln \Delta > 0$. Therefore, the Gram matrices of (4.9) and (4.13) are expressed by the doubly infinite Toeplitz matrix

$$T_c = \left[\frac{1}{1 + a\mathrm{i}(n-m)}\right]_{m,n\in\mathbb{Z}} = T_c^*,$$

corresponding to the symbol $c$, given by

$$c(x) = \frac{2\pi/a}{1 - \mathrm{e}^{-2\pi/a}} \mathrm{e}^{-(2\pi/a)x} \quad (0 \leqslant x \leqslant 1).$$

But $c \in \mathrm{C}[0,1] \subset \mathrm{L}^\infty(0,1)$, and $c(x) > 0$ for $0 \leqslant x \leqslant 1$; thus $T_c$ is a matrix representation of a linear, bounded, and boundedly invertible operator on $\ell^2(\mathbb{Z})$ (Halmos & Sunder 1978: Thm 5.6, p. 35; Kashin & Saakyan 1984: pp. 179–80). By virtue of Theorem 1, the system of exponentials (4.9) is a Riesz basis of $\mathcal{X}$.

*Remarks*

1. It is known from Bari (1951: Lemma 2, p. 81) that the Gram matrices for the Riesz basis $\{\phi_i\}_{i\in\mathcal{J}}$ and its biorthogonal system $\{\psi_i\}_{i\in\mathcal{J}}$ are mutually inverse. In the case of (4.9) and (4.13), this property follows easily from the formula

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n-p)(n-q)} = -\pi^2 \frac{\cot \pi p - \cot \pi q}{\pi p - \pi q} \xrightarrow[q\to p]{} \frac{\pi^2}{\sin^2 \pi p}$$

for $p, q \in \mathbb{C}\backslash\mathbb{Z}$ (Krzyż, 1972: Problems 4.6.4 and 4.6.5).

2. Lemma 2, Theorem 2, and the fact that the system biorthogonal to a Riesz basis is also a Riesz basis imply that, for $|K| \neq 1$, a system of eigenfunctions of the

adjoint $A^*$ is a Riesz basis of $\mathcal{X}$, where

$$A^*v = v'', \qquad \mathbb{D}(A^*) = \{v \in \mathrm{H}^2(0\,,1)\colon v(0) = 0,\ v'(1) = Kv'(0)\}.$$

It is shown in Ionkin (1977) that a system of eigenfunctions and generalized eigenfunctions of $A^*$ forms a Riesz basis of $\mathcal{X}$ for $K = 1$.

3. The problem of showing the system of exponentials appearing in the proof to be a Riesz basis was a topic of several papers, and finally solved in Pavlov (1979); see also Khruščov (1979), Nikolskii (1985), and Gubreev (1984). The idea of considering the eigenproblem for (4.11) to prove the basis-forming property of the exponentials coincides with the results of Gubreev & Kovalenko (1981) and Gubreev (1984: Thm 1).

4. Equations (1.4), (4.7), (4.8), and (4.11) yield an identity $L^{-1}AL = B^2$, i.e. the similarity of $A$ and $B^2$. By (4.12), $\mathrm{i}B$ is selfadjoint for $|K| < 1$.

## 5. Semigroup generation

From (4.4), Theorem 2, and Lemma 3, the following result can be deduced.

LEMMA 4 *If* $|K| \neq 1$, *then the operator* $A$ *defined by* (1.4) *is the infinitesimal generator of a linear analytic semigroup* $\{S(t)\}_{t\geq 0}$ *on* $\mathcal{X}$. *It has the representation*

$$S(t)u_0 = \sum_{n=-\infty}^{\infty} \mathrm{e}^{\lambda_n t}\langle u_0, v_n\rangle u_n = \sum_{n=-\infty}^{\infty} \mathrm{e}^{\lambda_n t} \operatorname*{res}_{\lambda=\lambda_n} (\lambda I - A)^{-1}u_0 \quad \forall\, t \geq 0 \quad \forall\, u_0 \in X,$$

*where* $\{u_n\}_{n\in\mathbb{Z}}$ *is the system of eigenfunctions given by* (4.10) *and* $\{v_n\}_{n\in\mathbb{Z}}$ *is its biorthogonal system. Moreover,* (4.5) *is necessary and sufficient for exponential stability of the semigroup* $\{S(t)\}_{t\geq 0}$.

Lemma 4 permits us to strengthen the results concerning the regularity of the functional $Q$ defined in (1.7).

LEMMA 5 *For* $|K| \neq 1$ *and* $\alpha > \frac{1}{4}$, *the functional* $Q$ *defined by* (1.7) *is linear and bounded on the Banach space* $\mathcal{X}^\alpha$, *where*

$$\mathcal{X}^\alpha = \mathbb{D}(A^\alpha) \quad (0 \leq \alpha \leq 1),$$

*the domain of a fractional power of* $A$, *equipped with a topology induced by the norm* $\|u\|_{\mathcal{X}^\alpha} = \|A^\alpha u\|_{\mathcal{X}}$.

*Proof.* It follows from Lemma 4 that, for $|K| \neq 1$, the fractional powers $A^\alpha$ $(0 \leq \alpha \leq 1)$ of $A$ and the corresponding spaces $X^\alpha$ are well defined (Henry: 1981, pp. 24–30). Since for $\alpha > \frac{1}{4}$ we have $\mathcal{X}^\alpha \subset \mathrm{C}(0\,,1)$ (Henry 1981: Thm 1.6.1, p. 39), then there exists $M_\alpha > 0$ such that

$$|Qu| = |u(1)| \leq \|u\|_{\mathrm{C}(0\,,1)} \leq M_\alpha \, \|u\|_{\mathcal{X}^\alpha} \quad \forall\, u \in \mathcal{X}^\alpha. \quad \square$$

*Remark.* There is another way to obtain the above result: By (1.8)–(1.9), we have

$$Qu = \langle A^{*\,1-\alpha}r, A^\alpha u\rangle_{\mathcal{X}} \quad \forall\, u \in \mathcal{X}^\alpha,$$

where $\alpha \in (0, 1)$ is such that $r \in \mathbb{D}(A^{*\,1-\alpha})$. Now

$$A^* v = \sum_{n=-\infty}^{\infty} \overline{\lambda_n} \langle v, u_n \rangle_X v_n \quad \forall\, v \in \mathbb{D}(A^*),$$

and Lemma 1 implies $r \in \mathbb{D}(A^{*\beta})$ iff $\sum_{n=-\infty}^{\infty} |\langle r, u_n \rangle_X|^2 \, |\lambda_n|^{2\beta} < \infty$. Hence, taking (1.8), (4.10), and (4.2) into account, we get $\beta < \frac{3}{4}$.

## 6. Determination of the Integral Performance Index and the Parametric Optimization

For the evaluation of the performance index (1.6), we put $\mathcal{H} = \mathcal{X}$, $D = A$, $T(t) = S(t)$, $\mathcal{Y} = \mathbb{R}$, and $U = Q$ throughout Section 3, and assume $K \in (-\cosh \pi, 1)$, with $K \neq -1$. From Lemmas 4 and 5, and the remark after Theorem 5, it follows that $Q \in \mathbb{L}(\mathcal{X}^\alpha, \mathbb{R})$ for $\alpha \in (\frac{1}{4}, \frac{1}{2})$. Then (3.7) is satisfied and, by Theorems 3–5, the Lyapunov operator equation

$$\langle Au, Hv \rangle_X + \langle u, HAv \rangle_X = -Qu\overline{Qv} \quad \forall\, u, v \in \mathbb{D}(A)$$

has a unique solution $H \in \mathbb{L}(\mathcal{X})$ satisfying $H = H^*$ and $H \geq 0$, which is an HS operator. Therefore $H$ is the integral operator (Weidmann, 1980: Thm 6.11, p. 139) defined by

$$Hu(x) = \int_0^1 h(x, y) u(y)\, \mathrm{d}y \quad (u \in \mathcal{X}) \tag{6.1}$$

with a symmetric and positive-definite kernel $h \in \mathrm{L}^2\big((0, 1) \times (0, 1)\big)$ (see Gohberg & Krein (1969: § III, 10.1) for explanation). Moreover, since $\mathbb{D}(P) = \mathcal{X}$, (3.3)–(3.4) give

$$J = \int_0^\infty |QS(t)u_0|^2\, \mathrm{d}t = \langle u_0, Hu_0 \rangle_X \quad \forall\, u_0 \in \mathcal{X}. \tag{6.2}$$

Theorem 6 and formulae (1.7) and (4.1) yield an additional characterization of $H$:

$$\langle u, Hv \rangle_X = \sum_{n=-\infty}^{\infty} \operatorname*{res}_{\lambda=\lambda_n} \frac{L_u(-\lambda) L_v(\lambda)}{M(-\lambda) M(\lambda)}, \tag{6.3}$$

where

$$L_f(\lambda) = \frac{1}{\lambda^{\frac{1}{2}}} \int_0^1 \sinh \lambda^{\frac{1}{2}} x\, f(x)\, \mathrm{d}x \quad (f \in \mathcal{X}), \qquad M(\lambda) = \cosh \lambda^{\frac{1}{2}} - K,$$

with $\lambda_n$ $(n \in \mathbb{Z})$ as given in (4.2). For each fixed $f \in X$, the mapping $L_f$ is an entire function:

$$\overline{L_f(-\bar{\lambda})} = L_f(-\lambda).$$

$M$ is also entire, i.e. $\overline{M(-\bar{\lambda})} = M(-\lambda)$.

We determine the value of $J$ at the constant function $u_0(x) = 1$ $(0 \leq x \leq 1)$, since finding the value of $J$ at initial condition (1.5) requires only a slight modification. To do this, we put $u = v = u_0$ in (6.2)–(6.3), which yields

$$J = \langle u_0, Hu_0 \rangle_X = \sum_{n=-\infty}^{\infty} \operatorname*{res}_{\lambda=\lambda_n} \frac{(\cosh \mathrm{i}\lambda^{\frac{1}{2}} - 1)(\cosh \lambda^{\frac{1}{2}} - 1)}{-\lambda^2(\cosh \mathrm{i}\lambda^{\frac{1}{2}} - K)(\cosh \lambda^{\frac{1}{2}} - K)}$$

$$= 2(1-K) \sum_{n=-\infty}^{\infty} \frac{1}{\lambda_n^{\frac{3}{2}} \sinh \lambda_n^{\frac{1}{2}}} \left(1 - \frac{1-K}{\cosh \lambda_n^{\frac{1}{2}} - K}\right), \tag{6.4}$$

where $\lambda_n^{\frac{1}{2}}$ are given by (4.6). In particular,

$$J = \frac{2(1-K)}{(1-K^2)^{\frac{1}{2}}} \sum_{n=-\infty}^{\infty} \frac{1}{(\varphi - 2n\pi)^3} \left(1 - \frac{1-K}{\cosh(\varphi - 2n\pi) - K}\right),$$

where $|K| < 1$ and $\varphi = \arccos K$. Employing the partial-fraction expansion of the function $z \mapsto (\cosh z - \cos z)^{-1}$ (Krzyż, 1972: Problem 5.2.7), we obtain

$$J = \frac{2(1-K)}{(1-K^2)^{\frac{1}{2}}} \sum_{n=-\infty}^{\infty} \frac{1}{(\varphi - 2n\pi)^3} - \frac{2(1-K)}{(1-K^2)^{\frac{1}{2}}} \sum_{n=-\infty}^{\infty} \frac{1}{(\varphi - 2n\pi)^5}$$
$$- \frac{8\pi(1-K)^2}{(1-K^2)^{\frac{1}{2}}} \sum_{k=1}^{\infty} \frac{(-1)^k k}{\sinh k\pi} \sum_{n=-\infty}^{\infty} \frac{1}{(\varphi - 2n\pi)[4k^4\pi^4 + (\varphi - 2n\pi)^4]}. \tag{6.5}$$

By the method described in Krzyż (1972: Problem 4.1.11), and by some well-known formulae (Krzyż, 1972: Problems 4.5.15, 4.6.4), we have

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\varphi - 2n\pi)^3} = \frac{\cos \frac{1}{2}\varphi}{8 \sin^3 \frac{1}{2}\varphi}, \qquad \sum_{n=-\infty}^{\infty} \frac{1}{(\varphi - 2n\pi)^5} = \frac{2\cos \frac{1}{2}\varphi + \cos^3 \frac{1}{2}\varphi}{96 \sin^5 \frac{1}{2}\varphi}. \tag{6.6}$$

We now apply the following identity found by the residue method (Krzyż (1972, Problem 4.6.1)):

$$\sum_{n=-\infty}^{\infty} \frac{1}{(\varphi - 2n\pi)[4k^4\pi^4 + (\varphi - 2n\pi)^4]}$$
$$= \frac{\cos \frac{1}{2}\varphi}{8k^4\pi^4 \sin \frac{1}{2}\varphi} - \frac{(-1)^k \sin \varphi}{8k^4\pi^4[\cosh k\pi - (-1)^k \cos \varphi]} \quad (k \in \mathbb{N}). \tag{6.7}$$

Taking (6.6)–(6.7) into account in (6.5), we come to

$$J = \frac{(1-K)\cos \frac{1}{2}\varphi}{4(1-K^2)^{\frac{1}{2}} \sin^3 \frac{1}{2}\varphi} - \frac{(1-K)^2(2\cos \frac{1}{2}\varphi + \cos^3 \frac{1}{2}\varphi)}{48(1-K^2)^{\frac{1}{2}} \sin^5 \frac{1}{2}\varphi}$$
$$- \frac{(1-K)^2 \cos \frac{1}{2}\varphi}{\pi^3(1-K^2)^{\frac{1}{2}} \sin \frac{1}{2}\varphi} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3 \sinh k\pi}$$
$$+ \frac{(1-K)^2 \sin \varphi}{\pi^3(1-K^2)^{\frac{1}{2}}} \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3 \sinh k\pi[(-1)^k \cosh k\pi - K]}. \tag{6.8}$$

Consider the partial-fraction expansion of the function

$$z \mapsto (z^2 - \cosh z + \cos z)/(z^4 \cosh z - z^4 \cos z)$$

which can be easily derived from the expansion of $z \mapsto (\cosh z - \cos z)^{-1}$. Passing

TABLE 1
*Performance-related indices for various values of K*

| $K$ | $J$ for $u_0(x)\equiv 1$ | $J/(1-K)^2e^2$ for $u_0$ as in (1.5) | $K$ | $J$ for $u_0(x)\equiv 1$ | $J/(1-K)^2e^2$ for $u_0$ as in (1.5) |
|---|---|---|---|---|---|
| 0·95 | 5·0418 | 2016·7 | −5·0 | 0·1153 | 0·0032 |
| 0·9 | 2·5419 | 254·19 | −5·5 | 0·1176 | 0·0028 |
| 0·75 | 1·0424 | 16·678 | −6·0 | 0·1213 | 0·0025 |
| 0·5 | 0·5431 | 2·1725 | −6·5 | 0·1267 | 0·0023 |
| 0·25 | 0·3772 | 0·6706 | −7·0 | 0·1341 | 0·0021 |
| 0 | 0·2947 | 0·2947 | −7·5 | 0·1440 | 0·0020 |
| −0·25 | 0·2455 | 0·1571 | −8·0 | 0·1574 | 0·0019 |
| −0·5 | 0·2131 | 0·0947 | | | |
| −0·75 | 0·1902 | 0·0621 | −8·2078 | | 0·00193863 |
| −1·0 | 0·1733 | 0·0433 | | | |
| −1·5 | 0·1503 | 0·0241 | −8·5 | 0·1759 | 0·0019 |
| −2·0 | 0·1360 | 0·0151 | −9·0 | 0·2022 | 0·0020 |
| −2·5 | 0·1266 | 0·0103 | −9·5 | 0·2418 | 0·0022 |
| −3·0 | 0·1205 | 0·0075 | −10·0 | 0·3072 | 0·0025 |
| −3·5 | 0·1167 | 0·0058 | −10·5 | 0·4336 | 0·0038 |
| −4·0 | 0·1148 | 0·0046 | −11·00 | 0·7752 | 0·0054 |
| | | | −11·25 | 1·3216 | 0·0088 |
| −4·4049 | 0·114287 | | −11·50 | 4·8418 | 0·0310 |
| | | | −11·59 | 226·72 | 1·4303 |
| −4·5 | 0·1143 | 0·0038 | −11·5919 | 8311·4 | 52·420 |

to the limit $z \to 0$ yields

$$\frac{1}{\pi^3}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3\sinh k\pi} = -\frac{1}{360}. \tag{6.9}$$

Eliminating $\varphi$ from (6.8) and taking (6.9) into account, we obtain the final formula for $J$, valid for all $K \in (-\cosh\pi, 1)$:

$$J = \frac{K^2 - 17K + 106}{360(1-K)} + \frac{(1-K)^2}{\pi^3}\sum_{k=1}^{\infty}\frac{1}{k^3\sinh k\pi[\cosh k\pi - (-1)^k K]}. \tag{6.10}$$

In comparison with (6.4), the formula (6.10) is more convenient for numerical computation of $J$ as a function of $K$, since it contains rapidly convergent series. The results of calculations are given in the first column of Table 1. Those values divided by $(1-K)^2$ yield the values of $J$ at the initial condition (1.5) and create the second column of Table 1. The optima are solutions to the Problem posed in Section 1.

## 7. Determination of the integral operator kernel of the solution to the Lyapunov operator equation

To find the kernel $h$ of the integral operator (6.1), we put $u(x) = \delta(x-a)$, $v(x) = \delta(x-b)$, with $0 \leq a,b \leq 1$, where $\delta$ is the Dirac pseudofunction, in (6.3), which yields

$$h(a,b)=h(b,a)=\sum_{n=-\infty}^{\infty}\operatorname*{res}_{\lambda=\lambda_n}\frac{\sinh\lambda^{\frac{1}{2}}a}{\lambda(\cosh\lambda^{\frac{1}{2}}-K)}\frac{\sin\lambda^{\frac{1}{2}}b}{(\cos\lambda^{\frac{1}{2}}-K)}$$
$$=\sum_{n=-\infty}^{\infty}\frac{2\sinh\lambda_n^{\frac{1}{2}}a\sin\lambda_n^{\frac{1}{2}}b}{(\lambda_n^{\frac{1}{2}}\sinh\lambda_n^{\frac{1}{2}})(\cos\lambda_n^{\frac{1}{2}}-K)}\quad(0\leq a,b\leq 1),$$

where $\lambda_n^{\frac{1}{2}}$ ($n\in\mathbb{Z}$) are given by (4.6). In particular,

$$h(a,b)=\frac{2}{(1-K^2)^{\frac{1}{2}}}\sum_{n=-\infty}^{\infty}\frac{\sin[a(\varphi-2n\pi)]\sinh[b(\varphi-2n\pi)]}{(\varphi-2n\pi)[\cosh(\varphi-2n\pi)-\cos(\varphi-2n\pi)]},$$

where $|K|<1$ and $\varphi=\arccos K$. Employing the partial-fraction expansion of the meromorphic function $z\mapsto(\sin az\sinh bz)/z(\cosh z-\cos z)$ (Krzyż, 1972: Problem 5.2.1), we obtain

$$\begin{aligned}h(a,b)=\frac{2}{(1-K^2)^{\frac{1}{2}}}\Big(&\sum_{n=-\infty}^{\infty}\frac{ab}{\varphi-2n\pi}+\sum_{k=1}^{\infty}\sum_{n=-\infty}^{\infty}\frac{A_k}{\varphi-2n\pi-z_k}\\&+\sum_{k=1}^{\infty}\sum_{n=-\infty}^{\infty}\frac{\overline{A_k}}{\varphi-2n\pi+\overline{z_k}}+\sum_{k=1}^{\infty}\sum_{n=-\infty}^{\infty}\frac{A_k}{\varphi-2n\pi+z_k}\\&+\sum_{k=1}^{\infty}\sum_{n=-\infty}^{\infty}\frac{\overline{A_k}}{\varphi-2n\pi-\overline{z_k}}\Big),\end{aligned}\tag{7.1}$$

where $A_k=(\sin az_k\sinh bz_k)/z_k(\sinh z_k+\sin z_k)$ and $z_k=k\pi(1+\mathrm{i})$ ($k\in\mathbb{N}$). Taking into account the identity (Krzyż, 1972: Problem 5.2.1)

$$\sum_{n=-\infty}^{\infty}\frac{1}{\varphi-2n\pi}=\tfrac{1}{2}\cot\tfrac{1}{2}\varphi$$

and eliminating $\varphi$ in (7.1), and after some transformations, we obtain the final formula for $h$, valid for all $K\in(-\cosh\pi,1)$:

$$\begin{aligned}h(a,b)&=\frac{ab}{1-K}+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\cos(a-b)k\pi\cosh(a+b)k\pi-\cosh(a-b)k\pi\cos(a+b)k\pi}{k\sinh k\pi[\cosh k\pi-(-1)^kK]}\\&=\frac{ab}{1-K}+\frac{2}{\pi}\sum_{k=1}^{\infty}\Big(\frac{[\cos(a-b)k\pi][\mathrm{e}^{(a+b-2)k\pi}+\mathrm{e}^{-(a+b+2)k\pi}]}{k(1-\mathrm{e}^{-2k\pi})[1+\mathrm{e}^{-2k\pi}-2K(-1)^k\mathrm{e}^{-k\pi}]}\\&\qquad-\frac{[\cos(a+b)k\pi][\mathrm{e}^{(a-b-2)k\pi}-\mathrm{e}^{-(a-b+2)k\pi}]}{k(1-\mathrm{e}^{-2k\pi})[1+\mathrm{e}^{-2k\pi}-2K(-1)^k\mathrm{e}^{-k\pi}]}\Big)\quad(0\leq a,b\leq 1).\end{aligned}\tag{7.2}$$

The series in (7.2) converges in the square $|a|+|b|<2$. From the divergence of the harmonic series and the estimate

$$\frac{\sinh k\pi}{\cosh k\pi-(-1)^kK}\geq\frac{\sinh\pi}{\cosh\pi+|K|}\quad(k\in\mathbb{N}),$$

it follows that the series in (7.2) is divergent at $a=b=1$. Hence, the kernel $h$ is unbounded: $h(1,1)=+\infty$. Numerical calculations show that the folding of $h$

TABLE 2

*Values of 100h (a, b) (=100h (b, a)) for two values of K and various a,b ∈ [0 , 1]*

$K = 0{\cdot}75$

| a  b: | 0·0 | 0·1 | 0·2 | 0·3 | 0·4 | 0·5 | 0·6 | 0·7 | 0·8 | 0·9 | 1·0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1·0 | 0 | 38·4959 | 77·2644 | 116·6073 | 156·8968 | 198·6521 | 242·6980 | 290·5547 | 345·6279 | 418·7748 | ∞ |
| 0·9 | 0 | 35·2954 | 70·8501 | 106·9495 | 143·9399 | 182·2863 | 222·6768 | 266·2204 | 314·7696 | 370·3214 | |
| 0·8 | 0 | 31·8397 | 63·9008 | 96·4193 | 129·6619 | 163·9423 | 199·6259 | 237·0595 | 276·1602 | | |
| 0·7 | 0 | 28·1649 | 56·4991 | 85·1731 | 114·3559 | 144·1985 | 174·7817 | 205·9692 | | | |
| 0·6 | 0 | 24·3211 | 48·7575 | 73·4160 | 98·3806 | 123·6836 | 149·2481 | | | | |
| 0·5 | 0 | 20·3607 | 40·7904 | 61·3451 | 82·0501 | 102·8732 | | | | | |
| 0·4 | 0 | 16·3293 | 32·6936 | 49·1146 | 65·5837 | | | | | | |
| 0·3 | 0 | 12·2608 | 24·5353 | 36·8264 | | | | | | | |
| 0·2 | 0 | 8·1768 | 16·3567 | | | | | | | | |
| 0·1 | 0 | 4·0886 | | | | | | | | | |
| 0·0 | 0 | | | | | | | | | | |

$K = -8$

| a  b: | 0·0 | 0·1 | 0·2 | 0·3 | 0·4 | 0·5 | 0·6 | 0·7 | 0·8 | 0·9 | 1·0 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1·0 | 0 | −4·2114 | −7·3311 | −8·0938 | −4·9088 | 4·2513 | 21·9221 | 51·2674 | 96·7630 | 168·9833 | ∞ |
| 0·9 | 0 | −1·4264 | −1·8576 | −0·2102 | 4·7547 | 14·4596 | 30·4890 | 54·5319 | 88·2909 | 132·3048 | |
| 0·8 | 0 | 0·3787 | 1·5643 | 4·3878 | 9·7065 | 18·3607 | 31·0493 | 48·0460 | 68·4790 | | |
| 0·7 | 0 | 1·3814 | 3·3539 | 6·4899 | 11·3032 | 18·1656 | 27·1412 | 37·6778 | | | |
| 0·6 | 0 | 1·7904 | 3·9716 | 6·8929 | 10·8035 | 15·7614 | 21·4864 | | | | |
| 0·5 | 0 | 1·8020 | 3·8336 | 6·2760 | 9·2024 | 12·4993 | | | | | |
| 0·4 | 0 | 1·5761 | 3·2675 | 5·1447 | 7·1828 | | | | | | |
| 0·3 | 0 | 1·2275 | 2·5001 | 3·8271 | | | | | | | |
| 0·2 | 0 | 0·8280 | 1·6661 | | | | | | | | |
| 0·1 | 0 | 0·4145 | | | | | | | | | |
| 0·0 | 0 | | | | | | | | | | |

increases for $K$ decreasing from 0 to $-\cosh\pi$. Table 2 shows the values of $100h$ at the points $(0.1m, 0.1n)$ for $m,n=0,\dots,10$ and for two admissible values of $K$.

*Remark* Using (3.5), it is possible to characterize the kernel $h$ by the integral equation

$$\int_0^1 [k(w,y)h(y,x)+h(y,w)k(w,x)]\,dw = -r(x)r(y) \quad (0\leqslant x,y\leqslant 1);$$

here, $k$ denotes the kernel of $A^{-1}$ (see (4.1)), and $r$ is defined by (1.8). An approximate solution of this equation can be found with the aid of the inflation technique (Halmos & Sunder, 1978) which allows us to pass to the matrix Lyapunov equation.

## 8. Discussion of results

### *Comparison of the Direct Spectral Approach with Perturbation Methods*

The characteristic feature of the proposed spectral approach to the semigroup generation problem is the direct verification of whether a system of eigenfunctions of the linear closed-loop system operator forms a Riesz basis in the state space. Perturbation methods rely on proving that the generation of a semigroup, or the existence of a Riesz basis, is a property of the open-loop system which is stable with respect to a class of perturbations necessarily including the feedback one. In the perturbation method worked out in Lasiecka & Triggiani (1983), this is realized by the use of the Kato perturbation theorem, and the results are applicable to problems with the Neumann trace operator being a dominating part of the boundary control observation. In the example discussed in the present paper, the control operator is a Dirichlet trace-type operator, which does not permit that method to be employed. To be more precise, the first step of Lasiecka & Triggiani's method is to write the closed-loop system in the form of the so-called Fattorini model (Slemrod, 1976); here,

$$\dot{u}=A^0[u-Kb\langle A^0u,h\rangle_{X}], \qquad h(x)=-x \quad\text{and}\quad b(x)=1 \quad (0\leqslant x\leqslant 1), \tag{8.1}$$

where $A^0$ is the operator given by

$$A^0u=u'', \qquad \mathbb{D}(A^0)=\{u\in \mathrm{H}^2(0,1)\colon u'(1)=0,\, u(0)=0\},$$

which corresponds to the open-loop system. Since $A^0=A^{0*}<0$ and $\overline{\mathbb{D}(A^0)}=X=\mathrm{L}^2(0,1)$, it generates a linear analytic semigroup on $X$. The next step is to establish regularity of $b$ and $h$ by determination of the fractional powers of $A$, to the domain of which $b$ and $h$ belong. In our example, we have $b\in\mathbb{D}[(-A)^{\theta}]$ for $\theta\in[0,\frac14)$, and $h\in\mathbb{D}[(-A)^{\varkappa}]$ for $\varkappa\in[0,\frac34)$. This permits to rewrite (8.1) in the form

$$\dot{u}=(A^0)^{\frac34+\varepsilon}[(A^0)^{\frac14-\varepsilon}u-K(A^0)^{\frac14-\varepsilon}b\langle(A^0)^{\frac34-\varepsilon}h,(A^0)^{\frac14+\varepsilon}u\rangle_{X}], \tag{8.2}$$

where $\varepsilon$ is a sufficiently small positive number. The operator $A^0$ is now extended to $\tilde{A}^0$: $(\mathbb{D}(\tilde{A}^0)\subset\tilde{X})\to\tilde{X}$, where $\tilde{X}$ denotes the space dual to $\mathbb{D}[(-A^0)^{\frac34+\varepsilon}]$. The operator $\tilde{A}^0$ also generates a linear and analytic semigroup, but on $\tilde{X}$. Due to this, we can write (8.2) in the form with an additive perturbation, i.e. in a form convenient for the application of Kato's theorem:

$$\dot{u} = \tilde{A}^0 u - K(A^0)^{\frac{3}{4}+\varepsilon} b \langle (A^0)^{\frac{3}{4}-\varepsilon} h, (A^0)^{\frac{1}{4}+\varepsilon} u \rangle .$$

As a consequence of the fact that $b$ is not sufficiently regular, the perturbation is of higher order than $\tilde{A}^0$-bounded, which eliminates the possibility of this theorem being applied.

A comparison of the direct spectral approach with recent results on the stability of the Riesz basis of eigenfunctions, with respect to some classes of perturbations, leads to similar inferences. Setting $m = n = 1$, $G = A^0$, and $T_0 u = Kb\langle u, h \rangle_X$ in Akhmedov (1987: Thm 4, p. 522 jointly with Remark 5, p. 523), we notice that, since $b \in \mathbb{D}[(-A^0)^\theta]$ for $\theta \in [0, \frac{1}{4})$, then $G^\alpha T_0$ is a well-defined compact operator for $\alpha \in [0, \frac{1}{4})$. However, then $\liminf_{j \to \infty} j\, |\mu_j|^{-\alpha} = +\infty$, where $\{\mu_j\}_{j \in \mathbb{N}} = \sigma(A^0)$ and $|\mu_1| \leqslant |\mu_2| \leqslant \ldots$, and the second condition in Akhmedov's Theorem 4 does not hold. Thus, again as a consequence of the fact that $b$ is not sufficiently regular, no conclusion on the Riesz basis of eigenfunctions of the closed-loop system operator can be drawn.

For the example discussed, perturbation methods cannot be applied, whereas the direct spectral approach is useful.

### *Prospect of Further Investigations*

The direct spectral approach to the semigroup generation problem is effective if it is possible to establish whether a system of eigenfunctions of the linear closed-loop operator forms a Riesz basis in the state space. To get some idea about the applicability of different Riesz-basis criteria, we consider a generalization of (1.1) with a finite-dimensional system considered instead of the proportional controller:

$$\left.\begin{aligned} &\dot{\boldsymbol{v}}(t) = \boldsymbol{F}\boldsymbol{v}(t) + u(1, t)\boldsymbol{g} \quad (t \geqslant 0), \\ &\frac{\partial u(x, t)}{\partial t} = \frac{\partial^2 u(x, t)}{\partial x^2} \quad (t \geqslant 0; 0 \leqslant x \leqslant 1), \\ &\left[\frac{\partial u(x, t)}{\partial x}\right]_{x=1} = 0 \quad \text{and} \quad \boldsymbol{h}^\mathsf{T}\boldsymbol{v} + du(1, t) = u(0, t) \quad (t \geqslant 0), \end{aligned}\right\} \tag{8.3}$$

where $\boldsymbol{F} \in \mathbb{L}(\mathbb{R}^n)$, $\boldsymbol{g}, \boldsymbol{h} \in \mathbb{R}^n$, and $d \in \mathbb{R}$, with appropriate initial conditions.

The space $X = \mathbb{C}^n \oplus \mathrm{L}^2(0, 1)$ and the operator $A$ given by

$$A(\boldsymbol{v}, u) = (\boldsymbol{F}\boldsymbol{v} + u(1)\boldsymbol{g}, u''),$$

$$\mathbb{D}(A) = \{(\boldsymbol{v}, u) \in X: u \in \mathrm{H}^2(0, 1), u'(1) = 0, \boldsymbol{h}^*\boldsymbol{v} + du(1) = u(0)\},$$

are naturally connected with the system (8.3). An eigenvalue problem for $A$ takes the form

$$-(\lambda \boldsymbol{I} - \boldsymbol{F})\boldsymbol{v} + u(1)\boldsymbol{g} = \boldsymbol{0}, \qquad \boldsymbol{h}^*\boldsymbol{v} + du(1) = u(0), \qquad \boldsymbol{v} \in \mathbb{C}^n,$$

$$u'(1) = 0, \qquad u'' = \lambda u, \qquad u \in \mathrm{H}^2(0, 1).$$

Premultiplying the first equation by $\boldsymbol{h}^*$ adj $(\lambda \boldsymbol{I} - \boldsymbol{F})$ and the second equation by det $(\lambda \boldsymbol{I} - \boldsymbol{F})$, and adding the resulting equations, we come to 'the reduced eigenproblem':

$$\left.\begin{array}{c} u'' = \lambda u, \\ u'(1) = 0, \qquad u(1)[\boldsymbol{h}^* \operatorname{adj}(\lambda \boldsymbol{I} - \boldsymbol{F})\boldsymbol{g} + d \det(\lambda \boldsymbol{I} - \boldsymbol{F})] = u(0) \det(\lambda \boldsymbol{I} - \boldsymbol{F}). \end{array}\right\} \tag{8.4}$$

The polynomials of the spectral parameter $\lambda$, appearing in the boundary conditions, are easily identified as the numerator and denominator of the transfer function of the finite-dimensional part. Similar equations can be derived for the conventional PID controller.

Earlier results providing conditions for a system of eigenfunctions of the $n$th order ordinary differential operator to be a basis, with a variety of boundary conditions, are known as the Birkhoff–Tamarkin theory (Tamarkin, 1927). Mikhaylov (1962) and Kesel'man (1964), refining the results due to J. Schwartz, have obtained a Riesz-basis criterion for such operators in the case of boundary conditions not containing the spectral parameter. Their criterion reduces to checking the so-called strong regularity of boundary conditions. Lemma 3 very easily follows from the theorem of Mikhaylov & Kesel'man, and the inequality $|K| \neq 1$ guarantees the strong regularity of the boundary conditions. If the boundary conditions are not strongly regular but only regular, then the system of eigenvectors does not generally constitute a Riesz basis (see Kesel'man, 1964), however, a Riesz basis of subspaces can be constructed from an appropriately chosen sequence of subspaces (Shkalikov, 1979); Bushmakin, 1985). The boundary conditions in (1.4) are regular for any $K$. It will be shown elsewhere that the statements of Theorem 2 remain valid for some classes of Riesz bases of subspaces, though without the spectral mapping theorem.

Shkalikov (1983) gives an extensive discussion of the Riesz-basis problem of a system of eigenvectors or subspaces in the case of a spectral parameter $\lambda$ polynomially entering the boundary conditions. Some criteria are established for basisness in various spaces, particularly related to (8.4) The results directly concern the operator (8.4) or its realizations in those spaces. Application of Shkalikov's theory requires, however, a separate detailed presentation. Here we note only that, if the pairs $(\boldsymbol{F}, \boldsymbol{g})$ and $(\boldsymbol{F}, \boldsymbol{h})$ in (8.3) are controllable and observable respectively, then a spectral parameter $\lambda$ fails to enter the boundary conditions in (8.4) only in the case of a proportional controller (the case of a static feedback).

In the proof of Lemma 3, to justify that a system of eigenfunctions forms a Riesz basis, we use a linear transformation, reducing the problem to the analysis of a system of exponentials. Although, in this way, several amazing relationships are shown with the methods and results of modern harmonic analysis, its efficiency seems to be limited to problems with an explicitly computable spectrum. The appearance of a spectral parameter in boundary conditions does not essentially affect this approach.

If the series (3.8) admits an explicit summation, then it is possible to obtain a compact formula expressing the performance index in terms of the system parameters. From the example discussed, one may conclude that an exact knowledge of the spectrum simplifies these attempts. In the opposite case, the series (3.8) should be approximated by its truncations, with eigenvalues calculated numerically.

If $L^2(0,1)$ is the state space, and the assumptions of Theorem 5 hold, then the information about the general form of a solution to the Lyapunov equation is sufficient to characterize the kernel of this solution by an integral equation. For computational purposes, this equation can be converted, with the aid of the inflation technique, into a matrix Lyapunov equation. Further investigations should be focused on a maximal elimination of the need of exact knowledge of a spectrum.

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