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THE LYAPUNOV OPERATOR EQUATION WITH UNBOUNDED OPERATORS

1. Introduction

Several problems in control theory of infinite-dimensional systems lead to Lyapunov operator equations with unbounded operators in a Hilbert space. Significant results for equations of this type have been obtained by Datko [1] and Delfour [2] under assumption that the right-hand side of the equation is a bounded operator. However, in some problems [3], [4], [5] the right-hand side is unbounded. The aim of this paper is to generalize the results of Datko and Delfour to Lyapunov equations with an unbounded operator of special type in the right-hand side.

2. Main results

Let $(\mathfrak{X}, \langle \cdot, \cdot \rangle_{\mathfrak{X}})$, $(\mathcal{Y}, \langle \cdot, \cdot \rangle_{\mathcal{Y}})$ be Hilbert spaces. Assume that $\mathcal{A} : (\mathcal{D}(\mathcal{A}) \subset \mathfrak{X}) \longrightarrow \mathfrak{X}$ is the infinitesimal generator of a linear C_0-semigroup $\{S(t)\}_{t \geq 0}$ on $\mathfrak{X}$, $\mathcal{C} : (\mathcal{D}(\mathcal{C}) \subset \mathfrak{X}) \longrightarrow \mathcal{Y}$ is a linear operator such that $\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{C})$ and $\mathcal{C}S(\cdot)u \in L^2_{LOC}(0,\infty; \mathcal{Y})$ for $u \in \mathcal{D}(\mathcal{A})$. The following theorem generalizes the results of [1], [2], where a stronger assumption $\mathcal{C} \in \mathcal{L}(\mathfrak{X}, \mathcal{Y})$ was used.

Theorem 2.1

The Lyapunov operator equation

$$(2.1) \quad \langle \mathcal{A}u, \mathcal{H}u \rangle_{\mathfrak{X}} + \langle u, \mathcal{H}\mathcal{A}u \rangle_{\mathfrak{X}} = -\langle \mathcal{C}u, \mathcal{C}u \rangle_{\mathcal{Y}} \qquad \forall u \in \mathcal{D}(\mathcal{A})$$

has a solution $\mathcal{H} \in \mathcal{L}(\mathfrak{X})$, $\mathcal{H} = \mathcal{H}^*$, $\langle u, \mathcal{H}u \rangle_{\mathfrak{X}} \geq 0$ $\forall u \in \mathfrak{X}$ <u>iff</u> the following conditions hold:

$$(2.2) \quad \exists M > 0 : \int_0^{\infty} \|\mathcal{C}S(t)u\|^2_{\mathcal{Y}} \, dt \leq M \|u\|^2_{\mathfrak{X}} \qquad \forall u \in \mathcal{D}(\mathcal{A}).$$

$$(2.3) \quad \lim_{t \longrightarrow 0+} \frac{1}{t} \int_0^t \|\mathcal{C}S(\tau)u\|^2_{\mathcal{Y}} \, d\tau = \|\mathcal{C}u\|^2_{\mathcal{Y}} \qquad \forall u \in \mathcal{D}(\mathcal{A}).$$

Proof. Necessity. Assume $\exists \mathcal{H} \in \mathcal{L}(\mathfrak{X})$, $\mathcal{H} = \mathcal{H}^*$, $\langle u, \mathcal{H}u \rangle_{\mathfrak{X}} \geq 0$ $\forall u \in \mathfrak{X}$, $\mathcal{H}$ - a solution of (2.1). The functional $V : \mathfrak{X} \ni u \longmapsto V(u) = \langle u, \mathcal{H}u \rangle_{\mathfrak{X}} \in \mathbb{R}$ is Fréchet-differentiable on $\mathfrak{X}$. Moreover,

$$(2.4) \quad (\operatorname{grad} V(u))(h) = \langle u, \mathcal{H}h \rangle_{\mathfrak{X}} + \langle h, \mathcal{H}u \rangle_{\mathfrak{X}}, \quad h \in \mathfrak{X}.$$

Hence [6]

$$(2.5) \quad \dot{V}(u) \overset{df}{=} \lim_{t \longrightarrow 0+} \frac{1}{t} \{ V[S(t)u] - V(u) \} = (\operatorname{grad} V(u))(\mathcal{A}u) \overset{(2.4)}{=}$$

$$= \langle u, \mathcal{H}\mathcal{A}u\rangle_{\mathcal{X}} + \langle \mathcal{A}u, \mathcal{H}u\rangle_{\mathcal{X}} \overset{(2.1)}{=} -\langle \mathcal{C}u, \mathcal{C}u\rangle_{\mathcal{Y}} \leq 0 \quad \forall u \in \mathcal{D}(\mathcal{A}).$$

Thus V is a Lyapunov functional on $\mathcal{D}(\mathcal{A})$, Fréchet-differentiable on $\mathcal{X}$. By Theorem 3.9 [6] we obtain

$$(2.6) \quad V[S(t)u] - V(u) = -\int_0^t \|\mathcal{C}S(\tau)u\|_{\mathcal{Y}}^2 \, d\tau \quad \forall u \in \mathcal{D}(\mathcal{A}), \forall t \geq 0$$

or

$$\frac{1}{t}\{V[S(t)u] - V(u)\} = -\frac{1}{t}\int_0^t \|\mathcal{C}S(\tau)u\|_{\mathcal{Y}}^2 \, d\tau .$$

Hence in virtue of (2.5), by taking the limit $\lim_{t \to 0+}$ one obtains (2.3).

On the other hand, as $V[S(t)u] = \langle S(t)u, \mathcal{H}S(t)u\rangle_{\mathcal{X}} \geq 0 \quad \forall t \geq 0, \forall u \in \mathcal{X}$, formula (2.6) yields

$$\|\mathcal{H}\|_{\mathcal{L}(\mathcal{X})} \cdot \|u\|_{\mathcal{X}}^2 \geq \langle u, \mathcal{H}u\rangle_{\mathcal{X}} = V(u) \geq \int_0^t \|\mathcal{C}S(\tau)u\|_{\mathcal{Y}}^2 \, d\tau \quad \forall t \geq 0,$$
$$\forall u \in \mathcal{D}(\mathcal{A}).$$

Since the function $\mathbb{R}^{*} \ni t \longmapsto \int_0^t \|\mathcal{C}S(\tau)u\|_{\mathcal{Y}}^2 \, d\tau \in \mathbb{R}$ is weakly increasing and bounded above

$$\exists \lim_{t \to \infty} \int_0^t \|\mathcal{C}S(\tau)u\|_{\mathcal{Y}}^2 \, d\tau \overset{df}{=} \int_0^{\infty} \|\mathcal{C}S(t)u\|_{\mathcal{Y}}^2 \, dt \leq \|\mathcal{H}\|_{\mathcal{L}(\mathcal{X})} \|u\|_{\mathcal{X}}^2 \quad \forall u \in \mathcal{D}(\mathcal{A})$$

and thus (2.2) holds.

Sufficiency. Assume that conditions (2.2), (2.3) hold. Consider the mapping

$$\mathcal{P} : \mathcal{D}(\mathcal{A}) \ni u \longmapsto \mathcal{P}u \in L^2(0,\infty;\mathcal{Y}) = L^2, \ (\mathcal{P}u)(t) \overset{df}{=} \mathcal{C}S(t)u ,$$

which is well-defined due to (2.2). $\mathcal{P}$ is linear and bounded. Indeed,

$$[\mathcal{P}(\alpha_1 u_1 + \alpha_2 u_2)](t) = \mathcal{C}S(t)(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 \mathcal{C}S(t)u_1 + \alpha_2 \mathcal{C}S(t)u_2$$
$$= \alpha_1(\mathcal{P}u_1)(t) + \alpha_2(\mathcal{P}u_2)(t) \qquad \text{(linearity)}.$$

$$\|\mathcal{P}u\|_{L^2}^2 \leq \int_0^{\infty} \|\mathcal{C}S(t)u\|_{\mathcal{Y}}^2 \, dt \overset{(2.2)}{\leq} M\|u\|_{\mathcal{X}}^2 \quad \forall u \in \mathcal{D}(\mathcal{A}), \text{ hence}$$

$$\|\mathcal{P}u\|_{L^2} \leq \sqrt{M}\,\|u\|_{\mathcal{X}} \quad \forall u \in \mathcal{D}(\mathcal{A}) \qquad \text{(boundedness)}.$$

By the Linear Operator Extension Theorem [7, Th.4.2.1, p.117] $\mathcal{P}$ may be uniquely extended to an operator $\bar{\mathcal{P}} \in \mathcal{L}(\mathcal{X}, L^2)$ with preservation of norm. Let us consider, in turn, the bilinear, continuous and symmetric form $a(u,v) \overset{df}{=} \langle \bar{\mathcal{P}}u, \bar{\mathcal{P}}v\rangle_{L^2}$ on $\mathcal{X} \times \mathcal{X}$. In virtue of Theorem 5.4 [8, p.147], there exists a linear, bounded, selfadjoint operator $\mathcal{H}$ such that $\langle u, \mathcal{H}v\rangle_{\mathcal{X}} = \langle \bar{\mathcal{P}}u, \bar{\mathcal{P}}v\rangle_{L^2} \quad \forall u,v \in \mathcal{X}$. Evidently, $\langle u, \mathcal{H}u\rangle_{\mathcal{X}} \geq 0$ on $\mathcal{X}$. Now it remains to prove that $\mathcal{H}$ satisfies (2.1).

On the one hand, for $V(u) = \langle u, \mathcal{H}u\rangle_{\mathcal{X}}$ we have

$$u \in \mathcal{D}(\mathcal{A}) \Longrightarrow \dot{V}(u) = \lim_{t \to 0+} \frac{1}{t}\{V[S(t)u] - V(u)\} = \lim_{t \to 0+} \frac{1}{t} \int_0^{\infty} \{\|\mathcal{C}S(t+\tau)u\|_{\mathcal{Y}}^2 -$$
$$- \|\mathcal{C}S(\tau)u\|_{\mathcal{Y}}^2\} d\tau = -\lim_{t \to 0+} \frac{1}{t}\int_0^t \|\mathcal{C}S(\tau)u\|_{\mathcal{Y}}^2 \, d\tau \overset{(2.3)}{=} -\|\mathcal{C}u\|_{\mathcal{Y}}^2 =$$
$$= -\langle \mathcal{C}u, \mathcal{C}u\rangle_{\mathcal{Y}} .$$

On the other, by Frechet-differentiability of V and by Theorem (3.9) [6]

$$\dot{V}(u) = (\operatorname{grad} V(u))(\mathcal{A}u) = \langle \mathcal{A}u, \mathcal{H}u\rangle_{\mathcal{X}} + \langle u, \mathcal{H}\mathcal{A}u\rangle_{\mathcal{X}} \quad \forall u \in \mathcal{D}(\mathcal{A}) .$$

and thus we have proved that also (2.2) holds. □

Remark 2.1

The pair $(\mathcal{D}(\mathcal{A}), \langle \cdot,\cdot \rangle_{\mathcal{A}})$, where

$$(2.7) \quad \langle u_1, u_2\rangle_{\mathcal{A}} = \langle u_1, u_2\rangle_{\mathcal{X}} + \langle \mathcal{A}u_1, \mathcal{A}u_2\rangle_{\mathcal{X}}, \quad u_i \in \mathcal{D}(\mathcal{A}), i=1,2,$$

has the structure of Hilbert space. If

(2.8) the operator $\mathcal{C}$ is continuous on $\mathcal{D}(\mathcal{A})$ in the topology introduced by $\langle \cdot,\cdot \rangle_{\mathcal{A}}$,

then for each $u \in \mathcal{D}(\mathcal{A})$ the function $\mathbb{R}^{*} \ni t \longmapsto \mathcal{C}S(t)u \in \mathcal{Y}$ is continuous. In this case condition (2.3) holds automatically.

Proof. $\{S(t)|_{\mathcal{D}(\mathcal{A})}\}_{t \geq 0}$ is a semigroup generated on $(\mathcal{D}(\mathcal{A}), \|\cdot\|_{\mathcal{A}})$ by $\mathcal{A}|_{\mathcal{D}(\mathcal{A}^2)}$, hence for each $u \in \mathcal{D}(\mathcal{A})$ the mapping $t \longmapsto S(t)u \in (\mathcal{D}(\mathcal{A}), \|\cdot\|_{\mathcal{A}})$ is continuous. Finally, the function $t \longmapsto \|S(t)u\|_{\mathcal{Y}}^2$ is also continuous and the fulfilment of condition (2.3) follows from the Integral Version of Mean-Value Theorem. □

Remark 2.2

In general, conditions (2.2), (2.3) do not guarantee the uniqueness of the solution to the Lyapunov operator equation (2.1). It can be seen from the following simple example $\mathcal{X} = \mathbb{R}^3, \mathcal{Y} = \mathbb{R}$

$$\mathcal{A} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \in \mathcal{L}(\mathcal{X}) \quad \text{(matrix representation)} ; \quad u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in \mathcal{X} .$$

$\mathcal{C}u = u_1 - u_2 + u_3$. The symmetric (selfadjoint), nonnegative solution of (2.1) has the form

$$\mathcal{H} = \begin{bmatrix} 1-\alpha & -0.5 & \alpha \\ -0.5 & 1-\alpha & -0.5 \\ \alpha & -0.5 & 1-\alpha \end{bmatrix}, \quad \alpha \leq 0.5 .$$

On the other hand, $\mathcal{C}S(t)u = (u_1 - u_2 + u_3)\, e^{-t}$ and the integral

$\int_0^\infty \| \mathcal{C} S(t)u \|_{\mathfrak{Y}}^2 dt$ gives only the solution corresponding to $\alpha = 0.5$. The formulation of a necessary and sufficient condition for the existence of a unique bounded, selfadjoint and nonnegative solution of the Lyapunov operator equation (2.1) is rather complicated, however a simple, sufficient condition is available.

Lemma 2.1

Suppose that conditions (2.2) and (2.3) in Theorem 2.1 hold and, furthermore,

$$(2.9) \qquad S(t)u \xrightarrow[t \to \infty]{} 0 \qquad \forall u \in \mathcal{D}(\mathcal{A})$$

then there exists a unique $\mathcal{H} \in \mathcal{L}(\mathfrak{X})$, $\mathcal{H} = \mathcal{H}^*$, $\langle u, \mathcal{H}u \rangle_{\mathfrak{X}} \geqslant 0$ $\forall u \in \mathfrak{X}$, $\mathcal{H}$ - the solution of (2.1).

Proof. By theorem 2.1 we know that $\exists \mathcal{H} \in \mathcal{L}(\mathfrak{X})$, $\mathcal{H} = \mathcal{H}^*$, $\langle u, \mathcal{H}u \rangle \geqslant 0$ on $\mathfrak{X}$, $\mathcal{H}$ is the solution to (2.1). Suppose that there is also $\mathcal{H}_1 \in \mathcal{L}(\mathfrak{X})$, $\mathcal{H}_1 = \mathcal{H}_1^*$, $\langle u, \mathcal{H}_1 u \rangle_{\mathfrak{X}}$ on $\mathfrak{X}$, $\mathcal{H}_1$ - a solution to (2.1). Define the functions

$t \longmapsto \psi(t) \overset{df}{=} \langle S(t)u, \mathcal{H}S(t)u \rangle_{\mathfrak{X}}$, $t \longmapsto \psi_1(t) \overset{df}{=} \langle S(t)u, \mathcal{H}_1 S(t)u \rangle_{\mathfrak{X}}$, $u \in \mathcal{D}(\mathcal{A})$. They are absolutely continuous on $[0, \infty)$ and we have

$$\dot{\psi}(t) = \langle \dot{S}(t)u, \mathcal{H}S(t)u \rangle_{\mathfrak{X}} + \langle S(t)u, \mathcal{H}\dot{S}(t)u \rangle_{\mathfrak{X}} = \langle \mathcal{A}S(t)u, \mathcal{H}S(t)u \rangle_{\mathfrak{X}} + \langle S(t)u, \mathcal{H}\mathcal{A}S(t)u \rangle_{\mathfrak{X}} = -\langle \mathcal{C}S(t)u, \mathcal{C}S(t)u \rangle_{\mathfrak{Y}} \leqslant 0 \quad \forall t \geqslant 0 .$$

By the same arguments

$$(2.10) \quad \dot{\psi}_1(t) = -\langle \mathcal{C}S(t)u, \mathcal{C}S(t)u \rangle_{\mathfrak{Y}} = \dot{\psi}(t) \leqslant 0 \quad \forall t \geqslant 0.$$

$$0 \leqslant \psi(t) = |\langle S(t)u, \mathcal{H}S(t)u \rangle_{\mathfrak{X}}| \leqslant \|\mathcal{H}\|_{\mathcal{L}(\mathfrak{X})} \|S(t)u\|_{\mathfrak{X}}^2 ,$$

$$0 \leqslant \psi_1(t) = |\langle S(t)u, \mathcal{H}_1 S(t)u \rangle_{\mathfrak{X}}| \leqslant \|\mathcal{H}_1\|_{\mathcal{L}(\mathfrak{X})} \|S(t)u\|_{\mathfrak{X}}^2 .$$

Hence by (2.9) we conclude that $\psi(t) \xrightarrow[t \to \infty]{} 0$, $\psi_1(t) \xrightarrow[t \to \infty]{} 0$, what together with (2.10) yields $\psi(t) \equiv \psi_1(t)$. In particular, for $t=0$ we obtain $\langle u, \mathcal{H}u \rangle_{\mathfrak{X}} = \langle u, \mathcal{H}_1 u \rangle_{\mathfrak{X}}$ $\forall u \in \mathcal{D}(\mathcal{A})$. Since $\overline{\mathcal{D}(\mathcal{A})} = \mathfrak{X}$ and $\mathcal{H}$, $\mathcal{H}_1$ are selfadjoint operatore one obtains $\mathcal{H} = \mathcal{H}_1$. ☐

3. Example

$\mathfrak{X} = \mathbb{R}^n \times L^2(-h, 0; \mathbb{R}^n)$, $\langle u_1, u_2 \rangle_{\mathfrak{X}} = x_1^T x_2 + \int_{-h}^0 \psi_1(\theta)^T \psi_2(\theta) d\theta$, $u_i = (x_i, \psi_i) \in \mathfrak{X}$, $i = 1,2$; $0 < h < \infty$.

$\mathfrak{Y} = \mathbb{R}^m$ with standard inner product.

$\mathcal{A}u = (Ax + B\psi(-h), \dot{\psi})$, $A, B \in \mathcal{L}(\mathbb{R}^n)$.

$\mathcal{D}(\mathcal{A}) = \{ u = (x, \psi) \in \mathfrak{X} : \psi \in AC(-h, 0; \mathbb{R}^n), \dot{\psi} \in L^2(-h, 0; \mathbb{R}^n), \psi(0) = x \}$;

$AC(-h, 0; \mathbb{R}^n)$ - the space of absolutely continuous on $[-h, 0]$ functions, taking values in $\mathbb{R}^n$.

The operator $\mathcal{A}$ defined above is the infinitesimal generator of a linear semigroup $\{S(t)\}_{t \geqslant 0}$ [9]. $\mathbb{R}^n$-component (point component) of the function $t \longmapsto S(t)u_0$ coincides on the interval $[0, \infty)$ with the solution and L^2-component (function component) with segment of solution of the retarded system

$$(3.1) \quad \left\{ \begin{array}{l} \dot{x}(t) = Ax(t) + Bx(t-h) , \; t > 0 \\ x(0) = x_0, x(\theta) = \phi(\theta) , u_0 = (x_0, \phi) \in \mathfrak{X} \end{array} \right\} .$$

The solution of (3.1) can be represented in the form

$$(3.2) \quad x(t) = X(t)x_0 + \int_{-h}^0 X(t-h-\theta) B \phi(\theta) d\theta , \; t \geqslant 0,$$

where

$$(3.3) \quad \left\{ \begin{array}{l} \dot{X}(t) = X(t) A + X(t-h)B , \; t \geqslant 0 \\ X(t) = 0 \quad t < 0, \; X(0) = I \end{array} \right\} .$$

Consider now the linear operator $\mathcal{C} : (\mathcal{D}(\mathcal{C}) \subset \mathfrak{X}) \longrightarrow \mathfrak{Y}$ defined as follows

$$(3.4) \quad \mathcal{C}u = H_0 x + H_1 \psi(-h), \; u = (x, \psi) \in \mathcal{D}(\mathcal{C}), H_0, H_1 \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m).$$

Evidently $\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{C})$, $\mathcal{C} \notin \mathcal{L}(\mathfrak{X}, \mathfrak{Y})$, $u \in \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{C}S(\cdot)u \in L^2_{Loc}(0, \infty; \mathbb{R}^m)$

We now prove, that $\mathcal{C}$ satisfies condition (2.8). If $|\cdot|$ is an Euclidean norm in $\mathbb{R}^m$ and $u \in \mathcal{D}(\mathcal{A})$ than

$|\mathcal{C}u| = |H_0 x + H_1 \psi(-h)| \leqslant \alpha \sqrt{x^T x + \psi(-h)^T \psi(-h)}$, where

$$\alpha = \sqrt{\lambda_{max}\left(\begin{bmatrix} H_0^T H_0 & H_0^T H_1 \\ H_1^T H_0 & H_1^T H_1 \end{bmatrix} \right)} \geqslant 0.$$ Farther

$$|\mathcal{C}u| \leqslant \alpha \sqrt{x^T x + x^T x + \int_{-h}^0 [\psi(\theta) + \dot{\psi}(\theta)]^T [\psi(\theta) + \dot{\psi}(\theta)] d\theta + [Ax + B\psi(-h)]^T [Ax + B\psi(-h)] - \psi(0)^T \psi(0) + \psi(-h)^T \psi(-h)} = \alpha \sqrt{x^T x + \langle u, u \rangle_{\mathfrak{X}} + \langle \mathcal{A}u, \mathcal{A}u \rangle_{\mathfrak{X}}} \leqslant$$
$$\leqslant \alpha \sqrt{2} \sqrt{\langle u, u \rangle_{\mathfrak{X}} + \langle \mathcal{A}u, \mathcal{A}u \rangle_{\mathfrak{X}}} = \alpha \sqrt{2} \|u\|_{\mathcal{A}}$$

and therefore $\mathcal{C}$ is continuous on $\mathcal{D}(\mathcal{A})$ in the topology introduced by the inner product (2.7).

Assume that

$$(3.5) \quad \{ s \in \mathbb{C} : \det [sI - A - e^{-sh}B] = 0 \} \subset \{ s \in \mathbb{C} : \mathrm{Re}\, s < 0 \} .$$

It is a necessary and sufficient condition for the (EXS) of the semigroup $\{S(t)\}_{t \geqslant 0}$. In our problem, (3.5) implies that (2.2) and (2.9) hold. Thus by Theorem 2.1, Remark 2.1 and Lemma 2.1 condition (3.5) is sufficent for the existence of a unique bounded, selfadjoint, nonnegative solution of the Lyapunov equation

$$(3.6)\quad \langle \mathcal{A}u, \mathcal{H}u\rangle_{\mathfrak{X}} + \langle u, \mathcal{H}\mathcal{A}u\rangle_{\mathfrak{X}} = -x^TPx - x^TQ\psi(-h) - \psi(-h)^TQ^Tx - \psi(-h)^TR\psi(-h)\ ;\ \forall u=(x\ \psi)\in\mathcal{D}(\mathcal{A})\ ; P=H_0^TH_0=P^T, Q=H_0^TH_1, R=H_1^TH_1=R^T.$$

On the basis of the representation of this solution,obtained in the pro-of of Theorem 2.1,we give its detailed description

$$(3.7)\quad \langle u_0, \mathcal{H}u_0\rangle_{\mathfrak{X}} = \int_0^\infty \|(\mathcal{F}u_0)(t)\|^2_{\mathcal{Y}}\,dt = \int_0^\infty [x(t)^T, x(t-h)^T]\begin{bmatrix} P & Q \\ Q^T & R\end{bmatrix}\begin{bmatrix} x(t) \\ x(t-h)\end{bmatrix}dt =$$

$$= \int_0^\infty x(t)^T(P+R)x(t)\,dt + \int_0^\infty x(t)^TQx(t-h)\,dt + \int_0^\infty x(t-h)^TQ^Tx(t)\,dt + \int_{-h}^0 \phi(\theta)^TR\phi(\theta)\,d\theta .$$

Substituting (3.2) into (3.7),using the Fubini-Tonelli Theorem and changing the variables of integration several times we find that

$$(3.8)\quad \langle u_0, \mathcal{H}u_0\rangle_{\mathfrak{X}} = x_0^T\Big[\int_0^\infty X(t)^T(P+R)X(t)\,dt + \int_0^\infty X(t+h)^TQX(t)\,dt +$$

$$+\int_0^\infty X(t)^TQ^TX(t+h)\,dt\Big]x_0 + \int_{-h}^0 x_0^T\Big[\Big[\int_0^\infty X(t)^TQ^TX(t-\theta)\,Bdt + X(h+\theta)^TQ +$$

$$+\int_0^\infty X(t+h)^TQX(t-h-\theta)\,Bdt + \int_0^\infty X(t)^T(P+R)X(t-h-\theta)B\,dt\Big]\phi(\theta)\,d\theta +$$

$$+\int_{-h}^0 \phi(\theta)^T\begin{bmatrix}\text{transposition of the matrix} \\ \text{from the preceding term}\end{bmatrix} x_0\,d\theta + \int_{-h}^0\int_{-h}^0 \phi(\theta)^T\Big[Q^TX(\theta-\sigma)B$$

$$+\int_0^\infty B^TX(t-h-\theta)^TQ^TX(t-\sigma)\,Bdt + B^TX(\sigma-\theta)^TQ + \int_0^\infty B^TX(t-\theta)^TQX(t-h-\sigma)\,Bdt +$$

$$+\int_0^\infty B^TX(t-h-\theta)^T(P+R)X(t-h-\sigma)\,Bdt\Big]\phi(\sigma)\,d\theta\,d\sigma + \int_{-h}^0 \phi(\theta)^TR\phi(\theta)\,d\theta ,$$

$u_0=(x_0,\phi)\in\mathfrak{X}$.

It can be readily seen that the operator $\mathcal{H}$ has the following general form

$$(3.9)\quad \mathcal{H}u = \Big(\alpha x + \int_{-h}^0 \beta(\theta)\psi(\theta)\,d\theta,\ \beta(\theta)^Tx + \int_{-h}^0 \delta(\theta,\sigma)\psi(\sigma)\,d\sigma + \gamma\psi(\theta)\Big), u=(x,\psi)\in\mathfrak{X} ;$$

$\alpha=\alpha^T, \gamma=\gamma^T\in\mathcal{L}(\mathbb{R}^n)$; $\theta\longmapsto\beta(\theta)$, $\theta\longmapsto\delta(\theta,\cdot)$ are absolutely continuous functions on $[-h,0]$ with values in $\mathcal{L}(\mathbb{R}^n)$, $\delta(\theta,\sigma) = \delta(\sigma,\theta)^T$.

Taking the forms of the operators $\mathcal{A}$ and $\mathcal{H}$ into account,using integration-by-parts and the Green-Stokes theorem we find

$$(3.10)\quad \langle \mathcal{A}u, \mathcal{H}u\rangle_{\mathfrak{X}} + \langle u, \mathcal{H}\mathcal{A}u\rangle_{\mathfrak{X}} = [x^T, \psi(-h)^T]\left[\begin{array}{c|c} A^T\alpha + \alpha A + \beta(0) + \beta(0)^T + \gamma & \alpha B - \beta(-h) \\ \hline B^T\alpha - \beta(-h)^T & -\gamma\end{array}\right]\begin{bmatrix} x \\ \psi(-h)\end{bmatrix} + \int_{-h}^0 x^T\Big[A^T\beta + \delta(0,\theta) - \frac{d\beta(\theta)}{d\theta}\Big]\psi(\theta)\,d\theta +$$

$$+\int_{-h}^0 \psi(\theta)^T\Big[-\frac{d\beta(\theta)^T}{d\theta} + \delta(\theta,0) + \beta(\theta)^TA\Big]x\,d\theta + \int_{-h}^0 \psi(-h)^T\Big[B^T\beta(\theta) -$$

$$-\delta(-h,\theta)\Big]\psi(\theta)\,d\theta + \int_{-h}^0 \psi(\theta)^T\Big[\beta(\theta)^TB - \delta(\theta,-h)\Big]\psi(-h)\,d\theta -$$

$$-\int_{-h}^0\int_{-h}^0 \psi(\theta)^T\Big[\frac{\partial\delta(\theta,\sigma)}{\partial\theta} + \frac{\partial\delta(\theta,\sigma)}{\partial\sigma}\Big]\psi(\sigma)\,d\theta\,d\sigma \quad \forall u=(x,\psi)\in\mathcal{D}(\mathcal{A}).$$

By the arguments similar to given in [9] it is not difficult to see , comparing (3.6) and (3.10),that the elements $\alpha,\beta,\gamma,\delta$ satisfy the system of equations

$$(3.11)\quad \begin{cases} A^T\alpha + \alpha A + \beta(0) + \beta(0)^T + \gamma = -P & (a) \\ \alpha B - \beta(-h) = -Q & (b) \\ -\gamma = -R & (c) \\ A^T\beta(\theta) + \delta(0,\theta) - \dfrac{d\beta(\theta)}{d\theta} = 0 & (d)\ \text{a.e.on } [-h,0] \\ B^T\beta(\theta) - \delta(-h,\theta) = 0 & (e)\ \text{– " –} \\ \dfrac{\partial\delta}{\partial\theta} + \dfrac{\partial\delta}{\partial\sigma} = 0 & (f)\ \text{a.e.on } [-h,0]\times[-h,0] \end{cases}$$

Equation (3.11f) has the solution

$$\delta(\theta,\sigma) = \begin{cases} \varphi(\theta-\sigma) & ,\ \sigma\geq\theta \\ \varphi(\sigma-\theta)^T, & \sigma\leq\theta \end{cases}\quad \varphi : [-h,0] \longrightarrow \mathcal{L}(\mathbb{R}^n)\text{-an absolutely continuous function.}$$

This enables us to reduce system (3.11) to the boundary-value problem

$$(3.12)\quad \begin{cases} \dfrac{d\beta(\theta)}{d\theta} = \beta(-h-\theta)^TB + A^T\beta(\theta) & \text{a.e. on } [-h,0] \\ \beta(-h) = \alpha B + Q,\ \beta(0) + \beta(0)^T = -P - R - \alpha A - A^T\alpha \end{cases}$$

and to relationships

$$(3.13)\quad \gamma = R\ ,\quad \delta(\theta,\sigma) = \begin{cases} B^T\beta(-h-\theta+\sigma)\ , & \sigma\geq\theta \\ \beta(-h+\theta-\sigma)^TB, & \sigma\leq\theta \end{cases} .$$

The quadruplet $(\alpha,\beta,\gamma,\delta)$ obtained from the above systems of equat-ions can be substituted into (3.9) to give the desired solution of the Lyapunov operator equation (3.6) .The comparision of the formulas (3.8) and (3.9) yields explicit expressions for $\alpha,\beta,\gamma,\delta$ in terms of $X(t)$, P,Q,R.It is not difficult to check,using (3.2) ,that $\alpha,\beta,\gamma,\delta$ obtained in this way satisfy (3.12) and (3.13).

Remark 3.1

The above results mean that under condition (3.5) the hypotheses of Proposition 3.1 [4] hold for system (3.1).

Remark 3.2

Evidently, instead of (3.12) we may consider its equivalent form

$$
(3.14)\left\{\begin{aligned}
\vartheta(\theta) &= \beta(-h-\theta)^T \\
\frac{d\beta(\theta)}{d\theta} &= A^T\beta(\theta) + \vartheta(\theta)B \\
\frac{d\vartheta(\theta)}{d\theta} &= -B^T\beta(\theta) - \vartheta(\theta)A \\
\beta(0) + \beta(0)^T &= -P - R - A^T\alpha - \alpha A \\
\vartheta(0) &= B^T\alpha + Q^T
\end{aligned}\right\}
$$

Details connected with solving system (3.14) can be found in [4] and [10]. In [4] formulas similar to (3.12)÷(3.14) for neutral systems are given.

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ОСОБЫЕ ОПТИМАЛЬНЫЕ ПРОЦЕССЫ ДЛЯ ИНТЕГРОДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

I. Постановка задачи. Рассмотрим динамическую систему, состояние которой в дискретные моменты времени $t \in T_1 = \{0, 1, \ldots, t_1\}$, t_1 - фиксированное число, описывается n -вектором $x(t)$. Пусть состояние системы изменяется под воздействием управляющих сигналов $u(t), t \in T = T_1 \setminus t_1$. Связь между указанными группами переменных задается уравнениями

$$x(t+1) = Ax(t) + bu(t), \ t \in T. \qquad (I)$$

В уравнении (I) A - $n \times n$ -матрица, b - n -вектор. Задано начальное состояние системы в начальный момент времени

$$x(0) = x_0. \qquad (2)$$

Последовательность управляющих сигналов $u = (u(t), t \in T)$ назовем управлением, если выполняется ограничение

$$f_*(t) \le u(t) \le f^*(t), \ t \in T. \qquad (3)$$

Управление u и соответствующая траектория $x = (x(t), t \in T_1)$ называются допустимыми, если при заданной $m \times n$ -матрице H, $\operatorname{rank} H = m$ и m -векторе g в конечный момент времени выполняется терминальное ограничение

$$Hx(t_1) = g. \qquad (4)$$

На допустимых управлении и траектории определим критерии качества

$$J(u) = \min_{i \in K} (c_i' x(t_1) + d_i). \qquad (5)$$

Здесь $c_i, i \in K$, n -векторы, $d_i, i \in K$, -скаляры, K - конечное множество индексов.

Предполагается, что система (I)-(4) относительно управляема, т.е. выполнено условие $\operatorname{rank} P = m$, где $P = \{HA^{t_1-t-1}b, t \in T\}$.

Определения: I. Допустимое управление $u^0(t), t \in T$, доставляющее максимум критерию качества (5): $J(u^0) = \max J(u)$ называ-