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AN APPLICATION OF SHILNIKOV'S THEOREM TO LINEAR SYSTEMS WITH PIECEWISE LINEAR FEEDBACK

Complicated homoclinic trajectories leading to different types of Shilnikov bifurcations may exist in a very simple system.

In many recent papers hopes for understanding the onset of chaotic oscillations by means of Shilnikov's theorem have been expressed.

31.1 **Shilnikov's Theorem** [3], [6], [7], [9], [10]

$$\dot{x}(t) = g(x(t), \mathcal{K}), \qquad (1)$$

where $g: R^3 \times R \to R^3$, $g(0, \mathcal{K}) = 0$ for every $\mathcal{K}$ from some neighbourhood of 0 and g is such that the Hartman-Grobman linearization [11] holds for (1) at 0. Suppose also that the Jacobi matrix at 0, i.e. $[\frac{\partial g}{\partial x}(x, \mathcal{K})]_{x=0}$ has for $\mathcal{K} = 0$ the eigenvalues γ, $\alpha \pm i\beta$, where $\gamma, \alpha, \beta, \in R$, $\beta > 0$, α and γ of opposite signs. Finally, assume that for $\mathcal{K} = 0$ system (1) has a homoclinic orbit (HO) with respect to $0 \in R^3$.

(i) If $|\frac{\gamma}{\alpha}| > 1$, then in a neighbourhood of HO the Poincaré return map for (1) with $\mathcal{K} = 0$ has a countable set of Smale's horseshoes. For all sufficiently small $|\mathcal{K}|$ these horseshoes still exist since they are structurally stable.

(ii) If $|\frac{\gamma}{\alpha}| < 1$, then in an appropriately chosen neighbourhood of HO there is no periodic orbit of (1) with $\mathcal{K} = 0$, but for sufficiently small $|\mathcal{K}|$, $\mathcal{K} \neq 0$ an exactly one periodic orbit is generated from the disappearing HO, stable if $\gamma > 0$ and unstable if $\gamma < 0$.

The properties of the Smale's horseshoe suggest that its existence is close, but not equivalent, to the existence of a strange attractor (compare properties of the Smale's horseshoe discussed below with definition of the strange attractor).

The existence of strange attractor is strictly connected with the existence of transversal HO but it need not be the HO described in (i). In fact, in [8] the generation of a Smale's horseshoe from the HO with respect to a periodic orbit is proved. This periodic orbit may be, for example, the one described in (ii) for $|\mathcal{K}|$ large enough (after its possible destabilization). It seems, however, that the practical importance of Shilnikov's theorem relies on the possibility of better arrangement of physical and/or computer experiments, towards the exhibition of strange attractors, by contraction of "suspected" ranges of the bifurcation parameter $\mathcal{H}$.

31.2 In this paper we shall apply Shilnikov's theorem to the following linear system with piecewise-linear element in the feedback loop,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -B & -A \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + f(x). \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad A>0, \; AB>1, \qquad (2)$$

$f(x) = \{ \begin{matrix} -m_0 x & x \le x_0 \\ m(\lambda)x + n(\lambda), & x \ge x_0 \end{matrix} \}$, m_0, x_0 are fixed constants, $x_0 > 0$, $m(\lambda) = \lambda^3 + A\lambda^2 + B\lambda + 1$, $n/\lambda) = -m_0 x_0 - x_0 m(\lambda)$, λ is a bifurcation parameter. System (2.) describes a class of 3rd order oscillators (see below for biological and electrical examples). Another motivation for the study of the system (2) follows from the fact that if f is an arbitrary locally-Lipschitz function, satisfying the sector conditions: $1 - AB < \frac{f(x)}{x} < 1 \; \forall \; x \neq 0$, $f(0) = 0$, then (2) has the globally asymptotically stable equilibrium point at $0 \in R^3$. Shilnikov's theorem enables us to find an example of a piecewise-linear function f which violates the sector conditions and gives rise to a system with periodic orbits and/or very complicated trajectories. Two classes of systems (2) will be considered:

Class I
$-m_0 < -AB+1$, $\lambda \in \{z \in R: \; z>0, \; 3z^2 + 2Az + (4B-A^2)>0\}$, (3)

Class II
$-m_0 > 1$, $\lambda < -A$, $A^2 - 4B < 0$. (4)

For both classes (2) has two equilibrium points $0, N = (\frac{n}{1-m}, 0, 0) \in R^3$. The system linearized at 0 has eigenvalues γ, $\alpha \pm i\beta$, where

$$\alpha = -0.5 \; (A + \gamma), \; \beta = 0.5 \sqrt{3\gamma^2 + 2A\gamma + (4B - A^2)}. \qquad (5)$$

The same formulas hold for the linearization at N with λ instead of γ.

$|\frac{\gamma}{\alpha}| > 1$ for Class I and those systems of Class II for which $AB+2A^3+1<-m_0$. (6)

$|\frac{\gamma}{\alpha}| < 1$, $\gamma > 0$ for those systems of Class II for which $AB+2A^3+1>-m_0$. (7)

By Shilnikov's theorem, for the values of λ for which (2) has HO with respect to 0, Smale's horseshoes are generated in the case (6), while in the case (7) stable periodic orbits arise from vanishing HOs. The translation of the origin to N enables us to apply the Shilnikov's theorem once again and now we have

$|\frac{\lambda}{\alpha}| > 1$ for those systems of Class I for which $\lambda > A$ and for Class II, (8)

$|\frac{\lambda}{\alpha}| < 1$, $\lambda > 0$ for those systems of Class I for which $\lambda \in (0, A)$. (9)

Therefore, if the HO with respect to N exists for λ then by Shilnikov's theorem horseshoes are generated if (8) holds while a stable periodic orbit is created from HO if (9) is satisfied. This way the verification of the assumptions of Shilnikov's theorem is reduced to the question of existence of HOs.

31.3 A necessary condition for HO with respect to 0 exist is the existence of a pair (T, λ), $T < 0$ for Class I ($T > 0$ for Class II), λ is such that:

$$x(T;Q_0) = x_0,\ \psi(x(T;Q_0),\ y(T;Q_0),\ z(T;Q_0)) = 0, \qquad (10)$$

where $\psi(x,y,z) = (\gamma^2 + A\gamma + B)x + (\gamma + A)y + z$ and $(x(t;Q_0)$, $y(t;Q_0)$, $z(t;Q_0))$ is the solution to (2) with the initial condition $Q_0 = (x_0, \gamma x_0, \gamma^2 x_0)$, Q_0 is an eigenvector corresponding to γ which ends at the plane $x=x_0$, while $\psi=0$ is the complementary linear manifold. Adding the requirement that the above solution tends to 0 as $t \to -\infty$ (for Class I) or $t \to +\infty$ (for Class II) in the plane $\psi = 0$ one obtains a necessary and sufficient condition for the existence of HO. Similarly, the necessary condition for the existence of HO with respect to N is the existence of a pair (T, λ) satisfying (10) with T of opposite sign (t in the case of additional requirements for necessary and sufficient conditions), Q_0 replaced by $Q_N = (x_0,\ x_0(1+m_0)\lambda^{\frac{1}{1-m}},\ x_0(1+m_0)\lambda^{2\frac{1}{1-m}})$ and ψ redefined as $\psi(x,y,z) = (\lambda^2 + A\lambda + B)x + (\lambda + A)y + z + \frac{n}{\lambda}$. Let us introduce the function $F\colon\ R^4 \to R$, $F(t,c_1,c_2,c_3) = w_1 e^{\alpha t}\cos\beta t + w_2 e^{\alpha t}\sin\beta t + w_3 e^{\gamma t}$, where w_1, w_2, w_3 are the functions of (c_1, c_2, c_3)

defined as follows $w_3 = \dfrac{c_1\gamma^2 + c_2\gamma + c_3}{(\gamma - \alpha)^2 + \beta^2}$, $w_1 = c_1 - w_3$, $w_2 = \frac{1}{\beta}[c_2 + (\alpha+\gamma)c_1 + (\alpha-\gamma)w_3]$, with α, β given by (5). For $\mu = 1+m_0$, the vector

$$(F(t,x(0),Ax(0) + y(0),Bx(0) + Ay(0) + z(0)),\ F(t,y(0),z(0) + Ay(0),\ -\mu\, x(0)),\ F(t,z(0),\ -By(0) - \mu\, x(0)))^T \tag{11}$$

determines the solution of (2) in the halfspace $x \le x_0$, starting from the initial point $(x(0),\ y(0),\ z(0))^T$, $x(0) \le x_0$ for such t for which its first component is less or equal to x_0. Replacing $x(0)$ by $x(0) - \frac{1}{1-m}$, $x(0) \ge x_0$; γ by λ, taking $\mu = 1-m$ and adding

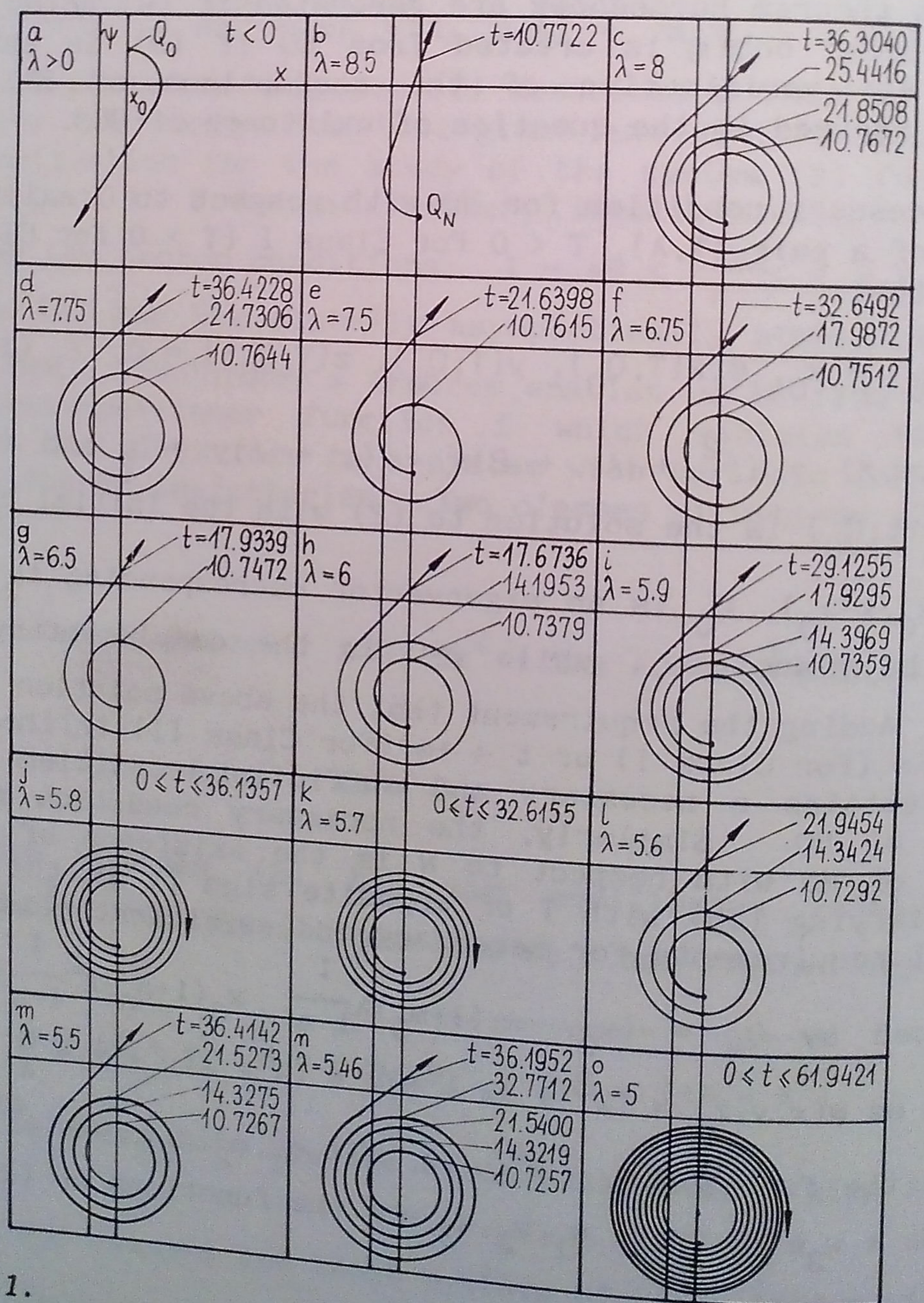

Figure 1.

the term $\frac{n}{1-m}$ to the first component of the vector (11) one obtains a formula for the solution of (2) in the halfspace $x \geq x_0$, valid for such t for which the first component is greater or equal to x_0. The Newton procedure can be used to find $\bar{T}$ such that $x(\bar{T}, x(0), y(0), z(0)) = x_0$. This will be needed for gluing together the segments of trajectories. To localize the solutions of (10) we fix a sequence of values of λ and observe the trajectories, startng from Q_0 with appropriate direction of time, in the plane (x, ψ). If for two different values λ_1, λ_2 of λ there exists a time interval $[t_1, t_2]$ such that for λ_1 the trajectory crosses the straight line $x = x_0$ with the opposite sign of ψ which it does the same for λ_2 then by continuity arguments, a solution to (10) exists in the rectangle $[t_1, t_2] \times [\lambda_1, \lambda_2]$. By the simple HO with respect to 0 we mean HO for which (10) holds and additionally $x(t; Q_0) \geq x_0 \ \forall\, t \in [0, T]$. If the pair (T, λ) corresponding to the simple HO is localized then the exact solution to (10) can be determined by careful application (since this system has countably many solutions) of the iterative scheme

$$T_{j+1} = T_j - \frac{x(T_j; Q_0) - x_0}{y(T_j; Q_0)}, \quad \lambda_{j+1} =$$

$$- \frac{n(\lambda_j)}{\psi(x(T_j; Q_0), y(T_j; Q_0), z(T_j; Q_0)) - \frac{n(\lambda_j)}{\lambda_j}} \qquad (12)$$

$j = 0, 1, 2, \ldots$, $(x(.; Q_0), y(.; Q_0), z(.; Q_0))^T$ is a solution to (2) in the halfspace $x \geq x_0$, starting from Q_0.

The similar procedure can be applied to the HO with respect to N. In this case Q_0 should be replaced by Q_N and for simple HO we have $x(t; Q_N) \leq x_0 \ \forall t \in [0, T]$.

31.4 Example 1: O.E. Rössler's system [1,2]. The substitutions $x = x_3 - \frac{67}{188}$, $y = x_2 - x_3$, $z = x_1 - 2x_2 + x_3$ transform the standard Rössler system with bifurcation parameter r to the form (2) with A=B=3, $m_0 = 8.4$, $x_0 = \frac{95}{1316}$, $m(\lambda) = (\lambda+1)^3 = 8.4r$. In a wide range of allowed (by (3)) values of λ the trajectories have the general shape shown in Figure 1a. Hence we conclude that there is no NO with respect to 0. Figure 1b-1o show the trajectories starting from Q_N,

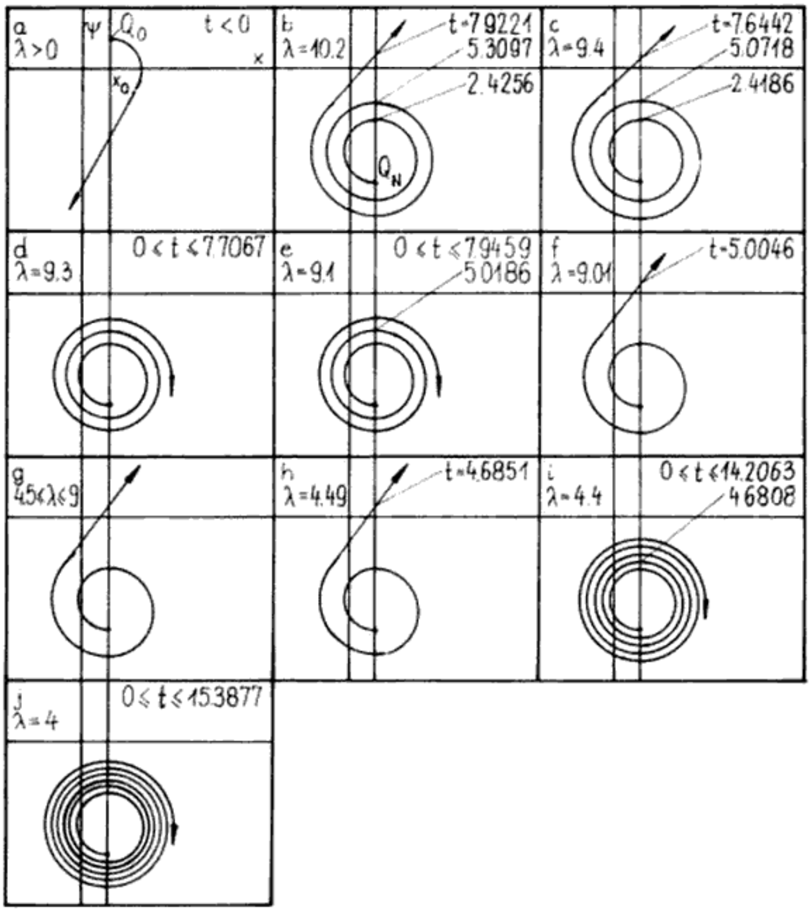

Figure 2.

observed in (x,ψ)-plane (the scale and precise shape are neglected as unessential) for different fixed λ. This allows us to deduce that there are HO in the intervals, (8,8.5) (the simple HO), (7.5,7.75), (6.5,6.75), (5.9,6), (5.5,5.6). The exact data for simple HO are found to be: $T = 10.77216478$, $\lambda = 8.498305443$ ($r = 102.0138429$). All estimated values of λ correspond to the generation of Smale's horseshoe (by (8)). Note that in [1,2] chaotic oscillations were found by computer simulations for $\lambda = 4.4243074$ ($r = 19$).

Example 2. A transformed model of asymmetric RC generator of chaos, considered in [4] has the form (2) with A=5, B=6, $m_0 = \frac{16913}{512}$, $x_0 = \frac{1}{5}$. The results of calculations are presented in Figure 2. Again there is no HO with respect to 0, while in the intervals (10.2,10.3) (simple HO with $T=2.426254614$, $\lambda = 10.287103039$), (9.3,9.4), (9.01,9.1), (4.4,4.49), HOs with respect to N have been found. By (8), (9) the last value corresponds to the stable periodic orbit (its existence has been confirmed by computer simulation) and other values to Smale's horseshoes.

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