CONTROL-THEORY AND ADVANCED TECHNOLOGY
Vol.7, No.1, pp.117-128, March, 1991

C89032sR

Short Paper

COMMENTS ON "GENERALIZED LYAPUNOV FUNCTIONS FOR LIÉNARD-TYPE NON-LINEAR SYSTEMS"*

P. GRABOWSKI[1]

Abstract. This paper deals with generalized Lyapunov functions for Liénard-type nonlinear systems. Two theorems concerning the estimate of the domain of attraction of a locally asymptotically stable equilibrium are given. The first one generalizes some well-known results, while the other provides a new estimate of the domain of attraction. Several typical examples are treated by the proposed method.

Key Words—Liénard systems, Lyapunov functions, domain of attraction.

1. Introduction

Liénard-type non-linear systems are models of many second order oscillators encountered in electronics, power systems, automatic control, biochemistry etc. They also appear in recent investigations of the steady-states in nonlinear partial differential equations; e.g., travelling wave solutions.

The construction of energy-type and Lur'e-type Lyapunov functions is well-known (LaSalle, 1960; Chang, 1970; Miyagi et al., 1980; 1988). Since the paper of Miyagi et al. (1988), it is expected that the non-Lur'e Lyapunov functions provide more exact estimates for the domains of attraction of an equilibrium.

For the Liénard systems,

$$\dot{x} = y, \quad \dot{y} = -yf(x) - g(x), \tag{1}$$

where f is continuous and g, locally Lipschitzian, $g(0)=0$, (Szegö, 1963; Hewit and Storey, 1969) suggested seeking a Lyapunov functional in the form,

$$\begin{aligned} V(x, y) &= \frac{1}{2}y^2(x) + yp(x) + q(x) \\ &= \frac{1}{2}[y+p(x)]^2 + \int_0^x [q'(s)-p(s)p'(s)]ds, \end{aligned} \tag{2}$$

with coefficients p, q which are differentiable and vanishing at 0. Since

$$\dot{V}(x, y) = [p'(x)-f(x)]y^2 + [q'(x)-f(x)p(x)-g(x)]y - p(x)g(x),$$

by taking two first coefficients equal to arbitrary continuous functions A, B, we

---

* Received by the editors April 10, 1989 and in revised form November 21, 1989.

[1] Institute of Automatics, Academy of Mining and Metallurgy, al. Mickiewicza 30/B1, PL-30-059 Kraków, Poland.

get

$$p(x) = \int_0^x [f(s)+A(s)]ds, \quad q(x) = \int_0^x [p(s)f(s)+g(s)+B(s)]ds$$

and

$$\dot{V}(x, y) = A(x)y^2 + B(x)y - g(x)p(x). \tag{3}$$

The aim of this paper is to show that efficient constructions of Lyapunov functionals can be realized by appropriate choices of functions A and B.

In the next section, two theorems concerning the local asymptotic stability of the origin are proposed. The first of them generalizes the results of Miyagi et al. (1980; 1988), while the second provides a completely new estimate for the domain of attraction. Illustrative examples are presented in Sec. 3. In the Appendix, we discuss briefly the choices of A and B appropriate for the analysis of the ultimate boundedness, existence of a limit cycle and an integrating factor.

2. Main Results

In this section, we present two results providing the conditions under which functional (2) satisfies the assumptions of the LaSalle-Lyapunov Theorem on local asymptotic stability of an equilibrium point (0, 0). The first result makes use of Hurwitz-type inequalities.

Theorem 1. Suppose that there exist x_1, x_2, x_3, x_4, $-\infty \le x_1 \le x_3 < 0 < x_4 \le x_2 \le +\infty$, such that

$$xg(x) > 0 \quad \text{for } \forall x \in (x_1, x_2)\backslash\{0\}, \tag{4}$$

$$xp(x) \ge 0, \quad A(x) \le 0 \quad \text{for } \forall x \in (x_1, x_2)$$
$$\text{and at least one inequality is strict on the interval } (x_3, x_4), \tag{5}$$

$$\left.\begin{array}{l} D(x) = B^2(x) + 4g(x)A(x)p(x) \le 0, \quad \text{for } \forall x \in (x_1, x_2) \\ \text{and } g(x) \ne -A(x)p(x) \quad \text{for } x \in (x_3, x_4) \setminus \{0\} \\ \text{for which } D(x) = 0 \text{ and } xB(x) \le 0 \end{array}\right\}. \tag{6}$$

Then,
(i) V is positively-definite in the strip $S=\{(x, y)\in \mathbb{R}^2;\ x_1<x<x_2\}$,
(ii) $\dot{V}(x, y)\le 0$ on S, and
(iii) the set

$$\Omega = \{(x, y)\in S;\ V(x, y)< \min_{i=1,2} V(x_i, 0)\} \tag{7}$$

is contained in the domain of attraction of an equilibrium point (0, 0).

Proof. In an elementary way, we obtain

$$x[q'(x)-p(x)p'(x)] = xg(x) - xA(x)p(x) + xB(x)$$
$$\ge -|xB(x)| + 2\sqrt{xg(x)}\sqrt{-xA(x)p(x)} \text{ on } (x_1, x_2).$$

This inequality is strict for $x \in (x_3, x_4) \backslash \{0\}$ and (i) follows from (2).

If $A(x)=0$, then $B(x)=0$ and $\dot{V}(x, y) = -g(x)p(x) \leq 0$ with equality for $p(x)=0$. If $A(x)<0$, then

$$\dot{V}(x, y) = A(x)\left[y+\frac{B(x)}{2A(x)}\right]^2 - \frac{D(x)}{4A(x)} \leq -\frac{D(x)}{4A(x)} \leq 0 \text{ on } (x_1, x_2).$$

The equality occurs, if $D(x)=0$ and $y=-B(x)/2A(x)$. Thus (ii) holds and then the origin is stable. Moreover,

$$E = \{(x, y) \in S;\ \dot{V}(x, y) = 0\} = E_0 \cup E_-,$$

where

$$E_0 = \{(x, y) \in S;\ A(x) = 0,\ p(x) = 0\},$$

$$E_- = \left\{(x, y) \in S;\ A(x) < 0,\ D(x) = 0,\ y = -\frac{B(x)}{2A(x)}\right\}.$$

By (6), $E_0 \backslash \{(0, 0)\}$ is contained in the strips $x_1 \leq x \leq x_3$, $x_4 \leq x \leq x_2$, and we easily establish that no whole trajectory of (1) can be located in $E_0 \backslash \{(0, 0)\}$. Now suppose that a whole, nonzero trajectory of (1) is located in E_-. Since E_- is contained in a curve in the (x, y) plane, represented by an equation solved explicitly with respect to y, then by the stability of the equilibrium point $(0, 0)$ and directly by (1), only the origin can be a positive limit set of the considered trajectory. Hence, a nonzero sequence $\{(x(t_k), y(t_k))\}$, $k \in \mathbb{N}$ exists, and $V(x(t_k), y(t_k)) = V_0$ for all $k \in \mathbb{N}$. However, the continuity of V implies $V_0 = 0$, contrary to (i). Thus, all assumptions of the LaSalle-Lyapunov Theorem are satisfied, and (7) follows from Willems (1969).

We now shall discuss briefly the possible choices of A and B. Assumption (6) imposes an upper bound for $|B|$,

$$2\sqrt{g(x)[-A(x)]p(x)} \geq |B(x)| \geq 0, \quad x_1 \leq x \leq x_2. \tag{8}$$

Consider the choices of B, corresponding to the constraints for $|B|$. If $B=0$, then it is not difficult to show that (4) and (5) imply (6). The case where $|B|$ is equal to the upper constraint offers four possibilities; namely $B=B_k$, $k=1,2,3,4$, where

$$\left.\begin{aligned} B_1(x) &= -2\operatorname{sign}x\sqrt{g(x)[-A(x)]p(x)} \\ B_2(x) &= -B_1(x) \\ B_3(x) &= 2\sqrt{g(x)[-A(x)]p(x)} \\ B_4(x) &= -B_3(x) \end{aligned}\right\}, \text{ for } x_1 \leq x \leq x_2. \tag{9}$$

For $B=B_2$, (6) immediately follows from (4) and (5). In other cases, (6) reduces to

$$g(x) \neq -A(x)p(x) \quad \left.\begin{array}{ll} \text{for all } x \in (x_3, x_4)\backslash\{0\} & \text{if } B = B_1 \\ \text{for all } x \in (x_3, 0) & \text{if } B = B_3 \\ \text{for all } x \in (0, x_4) & \text{if } B = B_4 \end{array}\right\}. \tag{10}$$

Let $V=V_k$, $k=1,2,3,4$ denote the functional (2) with $B=B_k$, respectively. Note that if $\min_{i=1,2}V(x_i, 0)$ is achieved only for $i=2$, then V_3 provides the best estimate of the domain of attraction because the value $V(x_2, 0)$ is greatest for $V=V_2$ and for $V=V_3$. However, $V_2(x, y)\geq V_3(x, y)$, and thus the level sets $\{(x, y)\in S;\ V_3(x, y)\leq l\}$ are contained in $\{(x, y)\in S;\ V_2(x, y)\leq l\}$. If $\min_{i=1,2}V(x_i, 0)$ is achieved only for $i=1$ (simultaneously for $i=1,2$), then V_4 (respectively, V_2) yields the best estimate.

From (5), we derive the following integral constraint imposed on A:

$$x\int_0^x f(s)ds \geq x\int_0^x [-A(s)]ds \geq 0, \quad x_1 \leq x \leq x_2. \tag{11}$$

Observe that (11) yields an implicit lower bound for the nonlinear damping coefficient f, justifying why (4) and (5) have been regarded as the Hurwitz inequalities. If A is equal to the convex combination of the lower and upper bounds, then from (11), one finds

$$A(x) = -(1-\alpha)f(x), \quad x_1 \leq x \leq x_2, \quad 0 \leq \alpha \leq 1. \tag{12}$$

The proposed choices of A and B include the well-known Lyapunov function constructions. Indeed, taking A as in (12) and $B=0$, we obtain results coinciding with those of Chang (1970) and Miyagi et al. (1980; 1988) (Theorem 1 and formula (25)). For A as in (12) and $B=B_3$, the results agree with Miyagi et al. (1988) (Theorem 2, formula (29)). To the author's knowledge, the other proposed choices of A and B are new. In the next section, it will be shown that they supplement the above results.

Although the Lyapunov functionals proposed in Theorem 1 are generally not of Lur'e type, as it follows from Miyagi et al. (1988), nevertheless, similarly to Lur's-type functionals, their construction is based on the Hurwitz inequalities. In the global asymptotic stability problem, these conditions are crucial. However, for systems which are only locally asymptotically stable, they seem to be less adequate. The construction of the Lyapunov functional described in Theorem 2 relies on the use of weakened Hurwitz conditions and yields the functional satisfying an equation somewhat similar to Zubov's equation (see (19)).

Theorem 2. Suppose that

A is positive-definite on $\mathbb{R}$ and such that the function

$$x \mapsto \frac{g(x)p(x)}{A(x)} \text{ is continuous at } 0, \tag{13}$$

$$\frac{p(x)}{x} > 0 \quad \text{for a small } |x| \neq 0, \tag{14}$$

(x_1, x_2) for $-\infty \leq x_1 < 0 < x_2 \leq +\infty$ is the maximal interval on which

$$\mathscr{H}(x) = p^2(x) + \frac{g(x)p(x)}{A(x)} > 0. \tag{15}$$

Then, for $B(x) = 2A(x)p(x)$, we get

(i) V is positively-definite on the strip $x_1 < x < x_2$,

(ii) $\dot{V}(x, y) \leq 0$ for $\forall(x, y) \in Q$,

where

$$Q = \{(x, y) \in \mathbb{R}^2;\ V(x, y) < l(x),\ x_1 < x < x_2\} \tag{16}$$

and

$$l(x) = \frac{\mathscr{H}(x)}{2} + \int_0^x [g(s) + A(s)p(s)]ds, \tag{17}$$

(iii) the set

$$\Omega = \{(x, y) \in \mathbb{R}^2;\ x_1 < x < x_2,\ V(x, y) < \min_{x_1 \leq x \leq x_2} l(x) = l_0\} \tag{18}$$

is contained in the domain of attraction of an equilibrium point (0, 0).

Proof. By (13), (14), (15), $x[g(x) + A(x)p(x)] > 0$ for $\forall x \in (x_1, x_2)$, $x \neq 0$. Hence, (i) follows easily from (2). Since $l(x) - V(x, y) = -[y + p(x)]^2/2 + \mathscr{H}(x)/2$, then by (15) we conclude that ∂Q is composed of graphs of two functions, $y^{\pm}(x) = -p(x) \pm \sqrt{\mathscr{H}(x)}$, for $x_1 \leq x \leq x_2$. Now, $y^+(0) = -y^-(0) = \sqrt{\mathscr{H}(0)} > 0$, so Q is an open neighborhood of (0, 0). $\dot{V}$ can be represented in the form,

$$\dot{V}(x, y) = -2A(x)[l(x) - V(x, y)], \tag{19}$$

and other statements are obvious.

Condition (13) shows that the choice of A depends on the growth rate of f and g in the neighborhood of 0. In the simplest case where $f(0) > 0$, and g is differentiable at 0, one may take $A(x) = cx^2/2$ $(c > 0)$. Then, (13) is satisfied, and instead of (14) and (15), we may equivalently assume that (x_1, x_2) is the maximal interval on which the weakened Hurwitz conditions hold:

$$\frac{p(x)}{x} = \frac{1}{x}\int_0^x f(s)ds + \frac{c}{6}x^2 > 0, \tag{20}$$

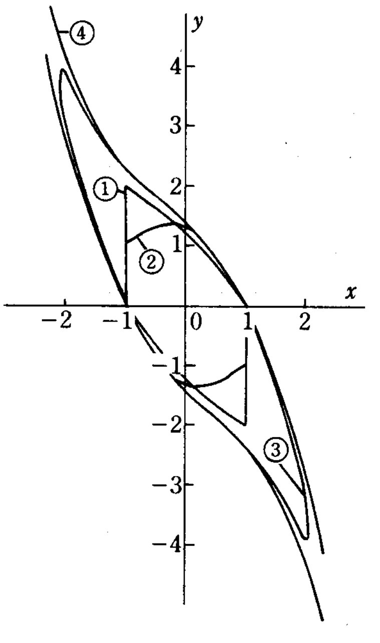
$$\frac{g(x)}{x} + \frac{cxp(x)}{2} > 0. \tag{21}$$

3. Examples

The results presented in the previous section will be illustrated by six comparative examples listed in Table 1. The obtained estimates of the domain of attraction are depicted in Figs. 1 to 6. According to considerations after Theorem 1, the functional V_3 gives the best estimate in the synchronous machine examples, i.e., in the examples analyzed by Miyagi et al. (1980; 1988). However, this is not the general case, which is documented by our Examples 1 and 3, for which V_2 and V_4 turned out to be optimal. For Example 2, the functionals V_k, $k=1,2,3,4$, are of little value. It should be emphasized that Theorems 1 and 2 yield mutually completing estimates, and that there are also some differences in the numerical procedures used to get the estimates. In the case of Theorem 1, both tracing the boundary of Ω and calculating the minimum in (7) require numerical integration while for Theorem 2, the numerical minimization of a single-variable function (17), which is not unimodal, is needed. The tracing of the boundary of Ω is very simple.

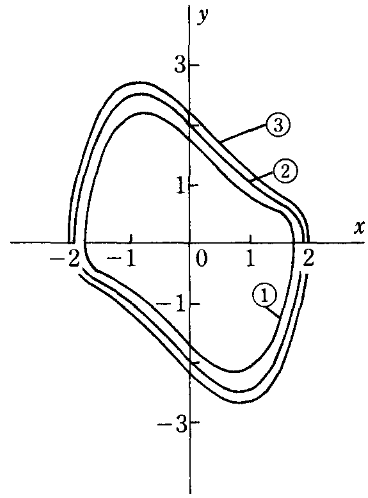
Table 1.

Name of example	$f(x)$	$g(x)$
Duffing's oscillator	1	$x-x^3$
Van der Pol's oscillator	$1-x^2$	x
LaSalle's asymmetric oscillator	0.5	$2x+x^2$
Single-machine power system I (Miyagi et al., 1980)	0.3	$\sin(x+x_0)-\sin x_0$ $x_0=0.412$
Single-machine power system II (Miyagi et al., 1980)	0.5	$\sin(x+x_0)-\sin x_0$ $x_0=0.412$
Single-machine power system III (Miyagi et al., 1988)	$0.32+0.28\cos 2x$	$\sin\left(x+\frac{\pi}{6}\right)-\sin\frac{\pi}{6}$



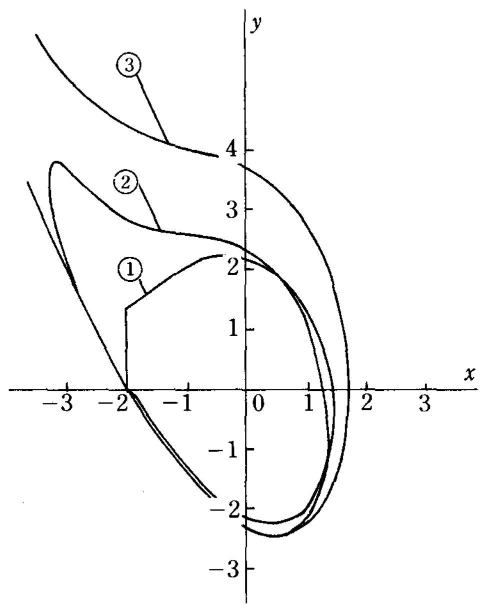
① Theorem 1 with $A=B=0$.
② Theorem 1 with $A(x)=-0.5f(x)$, $B(x)=B_2(x)$.
③ Theorem 2 with $A(x)=0.5cx^2$, $c=1$.
④ The true stability boundary.

Fig. 1. Duffing's oscillator.



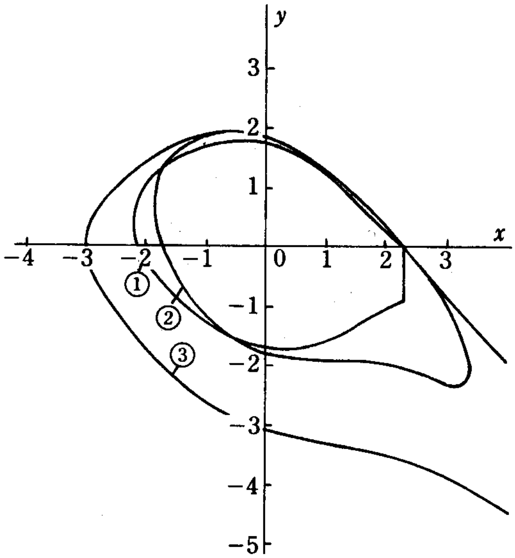
① Theorem 1 with $A=B=0$.
② Theorem 2 with $A(x)=0.5cx^2$, $c=1/3$.
③ The true stability boundary.

Fig. 2. Van der Pol's oscillator.



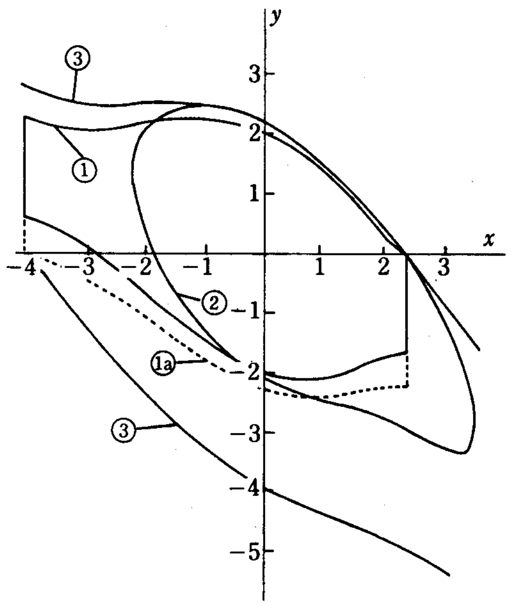
① Theorem 1 with $A(x)=-(1-\alpha)f(x)$, $B(x)=B_4(x)$, $\alpha=0.6573$.
② Theorem 2 with $A(x)=0.5cx^2$, $c=0.295$.
③ The true stability boundary.

Fig. 3. LaSalle's asymmetric oscillator.



① Theorem 1 with $A(x) = -(1-\alpha)f(x)$, $B(x) = B_3(x)$, $\alpha = 0.635$ (Miyagi and Taniguchi, 1980, Ex. 1).
② Theorem 2 with $A(x) = 0.5cx^2$, $c = 0.1555$.
③ The true stability boundary.

Fig. 4. The single-machine power system I.



① Theorem 1 with $A(x) = -(1-\alpha)f(x)$, $B(x) = B_3(x)$, $\alpha = 0.712$ (Miyagi and Taniguchi, 1980, Ex. 2).
1a An improvement of ①: $V_3(x, y) = V_3(x_1, 0) = V_3(-\pi - 2x_0, 0)$ (justified by LaSalle's Invariance Principle).
② Theorem 2 with $A(x) = 0.5cx^2$, $c = 0.156$.
③ The true stability boundary.

Fig. 5. The single-machine power system II.

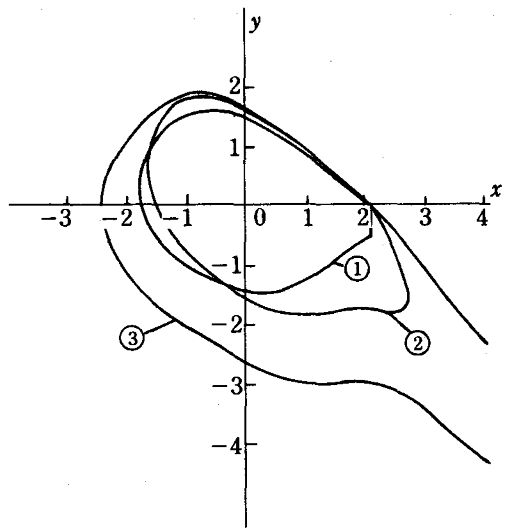


Fig. 6. The single-machine power system III.

4. Conclusions

We have presented several choices of A and B, ensuring the Lyapunov function construction in the class of functionals (2), which provide good estimates of the domains of attraction. Further improvement of these estimates is possible by taking them as an initial approximation for the algorithm proposed by Loparo and Blankenship (1978) and Hayashi and Ohsawa (1979). Evidently, a better initial approximation reduces computational expenses needed to employ the algorithm.

5. An Additional Discussion

If we allow A, B to belong to $L^{\infty}_{\text{loc}}(\mathbb{R})$, instead of being continuous, then V is no longer of class C^1, but is still a locally Lipschitzian functional. Thus, replacing $\dot{V}$ in (3) by the right upper derivative D^+V,

$$D^+V(x, y) = \limsup_{h\to 0_+} \frac{1}{h}\{V(x+hy, y+h[-yf(x)-g(x)])-V(x, y)\},$$

we can take, e.g.,

$$A(x) = \begin{Bmatrix} 0, & |x| \geq a \\ -\dfrac{\varepsilon}{a}, & |x| < a \end{Bmatrix} \text{ with } a, \varepsilon > 0; \quad B = 0. \tag{22}$$

By a slight modification of the arguments given by LaSalle (1960), one can prove the following theorem:

Theorem 3. Suppose that there exist a, $\varepsilon>0$, such that

$$xg(x) > 0 \quad \text{for} \quad |x| \geq a, \tag{23}$$

$$\operatorname{sign} x \int_0^x f(s)ds > \varepsilon \quad \text{for} \quad |x| \geq a \tag{24}$$

and

$$\int_0^x g(s)ds \to \infty \quad \text{as} \quad |x| \to \infty. \tag{25}$$

Then,

$$D^+V(x, y) < 0 \quad \text{on} \quad \mathbb{R}^2 \setminus \Omega, \tag{26}$$

where Ω denotes the rectangle $|x|\leq a$, $|y|\leq\sqrt{M\varepsilon/a}$, and where

$$M = \max_{|x|\leq a}\left|g(x)\left[F(x)-\frac{\varepsilon}{a}x\right]\right|, \quad F(x) = \int_0^x f(s)ds$$

and

$$V(x, y)\to\infty, \quad \text{if } |x|+|y|\to\infty. \tag{27}$$

Hence, the solutions of (1) are ultimately bounded (or (1) is a dissipative system in another notation) by the Yoshizawa Theorem.

Adding to (23), (24), (25), sufficient conditions for instability, with the property of repelling from a neighborhood of the origin,

$$xg(x) > 0 \quad \text{for all } x \neq 0, \tag{28}$$

$$x\int_0^x f(s)ds < 0 \quad \text{for a sufficiently small } |x| \neq 0, \tag{29}$$

one can construct the positively invariant, attracting annular region. By the standard Brouwer fixed point arguments, the existence of a limit cycle located therein may be shown. This result is known as the Dragilev Theorem (compare Reissig et al., 1963, Theorem 4.2.4). It is applicable to many practical problems, such as the Wien bridge generator or the Lefever-Nicholis biochemical oscillator (see Mees and Chua, 1979; Ponzo and Wax, 1978, for the dynamic equations).

We end with a remark on the existence of an integrating factor for (1). From Greenberg (1970), it is known that if there exist $c\in\mathbb{R}$, A, B such that

$$\dot{V}(x, y) = -cf(x)V(x, y), \tag{30}$$

then $M(x, y)=[V(x, y)]^{-1/c}$ is an integrating factor for the Pfaff form of (1), $[yf(x)+g(x)]dx+ydy=0$.

If there exists $c\in\mathbb{R}$, such that the following identity:

$$\left.\begin{aligned}\frac{Fg}{f}\cdot\frac{2-c}{2c} &= G + \frac{(c-1)(c-2)}{4}F^2 \\ F(x) = \int_0^x f(s)ds, &\quad G(x) = \int_0^x g(s)ds\end{aligned}\right\} \quad (31)$$

holds, then taking $B(x)=2A(x)p(x)$ (i.e., as in Theorem 2) and $A=-cf/2$, we obtain (30).

As an example, we consider (1) with $f(x)=2\mu-1$, $g(x)=\mu(\mu-1)x+x^n$. Such a system is related to the familiar Emden equation (see Kamke, 1959). For

$$c = \frac{2(n+1)}{n+3}, \quad (c-1)\left\{\frac{\mu(\mu-1)}{c} + \frac{(2\mu-1)^2(c-2)}{4}\right\} = 0,$$

(31) is satisfied. Hence, extracting the case $c=n=1$, previously analyzed in Kamke (1959), we find that M is an integrating factor for (1) with

$$f(x) = \pm\frac{n+3}{n-1}, \quad g(x) = \frac{2(n+1)}{(n-1)^2}x + x^n.$$

This is a new case, where the Emden equation is integrable. The above results demonstrate the flexibility of Szegö's idea for Lyapunov function construction.

References

Chang, J.A. and C.-C. Hang (1970). An algorithm for construction Lyapunov functions based on variable gradient method. *IEEE Trans. Automatic Control*, **AC-15**, 4, 510–512.

Greenberg, M. (1976). Limit cycle city. *Quart. Appl. Math.*, **34**, 1, 103–105.

Hayashi, M. and Y. Ohsawa (1979). Transient stability of power systems using series expansion of Lyapunov functions. *Int. J. Control*, **29**, 6, 1059–1066.

Hewit, J.R. and C. Storey (1969). Numerical applications of Szegö's method for constructing Lyapunov functions. *IEEE Trans. Automatic Control*, **AC-14**, 1, 106–108.

Kamke, E. (1959). *Differentialgleichungen. Lösungsmethoden und Lösungen, I.* Akademische Verlagsgesellschaft, Leipzig.

LaSalle, J.P. (1960). Some extensions of Lyapunov's second method. *IRE Trans. Circuit Theory*, **7**, 4, 520–527.

Loparo, K.A. and G.L. Blankenship (1978). Estimating the domain of attraction of nonlinear feedback systems. *IEEE Trans. Automatic Control*, **AC-23**, 4, 602–608.

Mees, A.I. and L.O. Chua (1979). The Hopf bifurcation theorem and its application to nonlinear oscillations in circuit and systems. *IEEE Trans. Circuit and Systems*, **CAS-26**, 4, 235–254.

Miyagi, H. and T. Taniguchi (1980). Applications of the Lagrange-Charpit method. *Int. J. Control*, **32**, 2, 371–379.

Miyagi, H., T. Ohshiro and K. Yamashita (1988). Generalized Lyapunov functions for Liénard-type non-linear systems. *Int. J. Control*, **48**, 2, 805–812.

Ponzo, P. and N. Wax (1978). Note on a model of biochemical reaction. *J. Math. Anal. Appl.*, **66**, 2, 354–357.

Reissig, R., G. Sansone and R. Conti (1963). *Qualitative Theorie Nichtlinearer Differentialgleichungen.* Edizione Cremonese, Roma.

Szegö, G.P. (1963). A contribution to Lyapunov's 2nd method, nonlinear autonomous systems. J.P. LaSalle and S. Lefschetz (ed.), *Int. Simp. Nonlinear Diff. Eqs. Nonlinear Mechanics*. Academic Press, N.Y., 421–430.

Willems, J.L. (1969). The computation of finite stability regions by means of open Lyapunov surfaces. *Int. J. Control*, **10**, 5, 537–544.

Piotr Grabowski was born in Kraków on 13th February 1952. He received an M.S. degree in control engineering and a Ph.D. degree in technical sciences, both from the Academy of Mining and Metallurgy in 1974 and 1979, respectively. From 1971 to 1975 he studied mathematics at the Jagiellonian University. He is co-author (jointly with H. Górecki, S. Fuksa, A. Korytowski) of the monograph Analysis and Synthesis of Time Delay Systems. J. Wiley & Sons and the Polish Scientific Publishers. Chichester and Warsaw, 1989. His main interests include dynamical systems, stability theory and applications of functional analysis in control theory. Dr. Piotr Grabowski is a member of the American Mathematical Society.