Elektrotechnika • Tom 2 • Zeszyt 3 • 1983 517.929.4:517.93

*Piotr Grabowski**

A LYAPUNOV FUNCTIONAL APPROACH TO A PARAMETRIC OPTIMIZATION PROBLEM FOR A CLASS OF INFINITE-DIMENSIONAL CONTROL SYSTEMS

1. A LYAPUNOV APPROACH TO PARAMETRIC OPTIMIZATION OF INFINITE-DIMENSIONAL SYSTEMS

In this section we formulate a parametric optimization problem for an abstract linear systems and we point out the possibility of solving it with the use of a special Lyapunov functional. To this end we shall need some auxiliary notions and results.

Def. 1.

Let $\mathcal{X}$ be a Banach space with norm $\|\cdot\|$ and let $\mathcal{L}(\mathcal{X})$ be the Banach space of bounded linear operators $\mathcal{X} \longrightarrow \mathcal{X}$ with norm $\|\cdot\|_{\mathcal{L}(\mathcal{X})}$. A family $\{S(t)\}_{t\geq 0} \subset \mathcal{L}(\mathcal{X})$ is called a linear C_0-semigroup if

(i) $S(t+\tau) = S(t)\,S(\tau) \quad \forall t,\tau \geq 0$

(ii) $S(0) = I$ where I is an identity in $\mathcal{L}(\mathcal{X})$

(iii) For every $u \in \mathcal{X}$ the mapping $S(\cdot)u : \mathbb{R}^+ \longrightarrow \mathcal{X}$ is continuous (right-continuous at $t=0$), $\mathbb{R}^+ = [0,+\infty)$.

An operater $\mathcal{A} : (\mathcal{D}(\mathcal{A}) \subset \mathcal{X}) \longrightarrow \mathcal{X}$ defined by

$$\mathcal{A}u = \lim_{t \to 0+} \frac{1}{t}\left[S(t)u - u\right], \quad u \in \mathcal{D}(\mathcal{A}) \iff \exists \lim_{t \to 0+} \frac{1}{t}\left[S(t)u - u\right]$$

is called the infinitesimal generator of the linear C_0-semigroup $\{S(t)\}_{t\geq 0}$.

---

*) Instytut Automatyki, Inżynierii Systemów i Telekomunikacji AGH.
Maszynopis poprawiony wpłynął do Redakcji 15 II 1983 r.

Theorem 1 [1,2]

A linear operator $\mathcal{A}: (\mathcal{D}(\mathcal{A}) \subset \mathcal{X}) \longrightarrow \mathcal{X}$, $\mathcal{X}$ -a Banach space with norm $\|\cdot\|$; is the infinitesimal generator of a linear C_0-semigroup $\{S(t)\}_{t \geq 0}$ if and only if

(i) $\mathcal{R}(I - \lambda\mathcal{A}) = \mathcal{X}$ for all sufficiently small $\lambda > 0$, $\mathcal{R}(I - \lambda\mathcal{A})$ is the range of the operator $I - \lambda\mathcal{A}$;

(ii) There exists a norm $\|\cdot\|_0$ in $\mathcal{X}$, equivalent to the norm $\|\cdot\|$ such that for some $\omega \in \mathbb{R}$ $(\omega I - \mathcal{A})$ is $\|\cdot\|_0$-accretive, i.e.

$$\|(u_1 - \lambda\mathcal{A}u_1) - (u_2 - \lambda\mathcal{A}u_2)\|_0 \geq (1 - \lambda\omega)\|u_1 - u_2\|_0 \quad \forall u_1, u_2 \in$$

$$\in \mathcal{D}(\mathcal{A}), \ \forall \lambda > 0, \ \lambda\omega < 1; \tag{1.1}$$

(iii) $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{X}$, i.e. $\overline{\mathcal{D}(\mathcal{A})} = \mathcal{X}$, where $\overline{\mathcal{D}(\mathcal{A})}$ is the closure of $\mathcal{D}(\mathcal{A})$.

Moreover, if the hypotheses (i), (ii), (iii) hold, then we have

$$\|S(t)u\|_0 \leq \|u\|_0 e^{\omega t} \quad \forall t \geq 0, \ \forall u \in \mathcal{X};$$

$S(\cdot)u_0$ is the unique strong solution of the initial-value problem

$$\begin{cases} \dot{u}(t) = \mathcal{A}u(t), & t > 0 \\ u(0) = u_0 \in \mathcal{D}(\mathcal{A}) \end{cases} \tag{1.2}$$

By a strong solution of (1.2), we mean a differentiable function $u(\cdot)$: $\mathbb{R}^* \longrightarrow \mathcal{X}$ such that $u(0) = u_0$, $u(t) \in \mathcal{D}(\mathcal{A})$ $\forall t \in \mathbb{R}^*$ and $\dot{u}(t) = \mathcal{A}u(t)$ for all $t \in \mathbb{R}^*$.

Def.2.

Let $\{S(t)\}_{t \geq 0}$ be a linear C_0-semigroup on a Banach space $\mathcal{X}$. We say that a functional $V: \mathcal{X} \longrightarrow \mathbb{R}$ is the continuous Lyapunov functional on a set $G \subset \mathcal{X}$ if and only if

(i) V is continuous on G,

(ii) $\dot{V}(u) \leq 0 \quad \forall u \in G$, where

$$\dot{V}(u) = \lim_{t \to 0+} \frac{1}{t}\{V[S(t)u] - V(u)\}$$

The continuous Lyapunov functionals on $\mathcal{D}(\mathcal{A})$ which are Fréchet-differentiable on $\mathcal{X}$ have some additional properties, as we have.

2

Lemma 1 [1]

Let $\mathcal{A}: (\mathcal{D}(\mathcal{A}) \subset \mathcal{X}) \longrightarrow \mathcal{X}$ be an infinitesimal generator of a linear C_0-semigroup $\{S(t)\}_{t \geq 0}$ on a Banach space $\mathcal{X}$. Let $V: \mathcal{X} \longrightarrow \mathbb{R}$ be Fréchet - differentiable. Then $\dot{V}(u) = (\text{grad } V(u))(\mathcal{A}u)$ $\forall u \in \mathcal{D}(\mathcal{A})$, where grad $V(u)$ is the Fréchet derivative of V at the point u. If furthermore V is the continuous Lyapunov functional on $\mathcal{D}(\mathcal{A})$, then

$$V[S(t)u_0] - V(u_0) = \int_0^t (\text{grad } V(S(\tau)u_0))(\mathcal{A}S(\tau)u_0)\,d\tau \quad \forall u_0 \in \mathcal{D}(\mathcal{A})$$

Now we state the parametric optimization problem for the abstract linear system (1.2) with an output

$$y(t) = \mathcal{C}u(t), \ t \in \mathbb{R}^*, \ \mathcal{C}: (\mathcal{D}(\mathcal{C}) \subset \mathcal{X}) \longrightarrow \mathbb{R}^m \text{ a linear operator} \tag{1.3}$$

It will be assumed that $\mathcal{A}$, u_0, $\mathcal{C}$ depend on a vector of parameters $a \in \Omega \subset \mathbb{R}^r$, and for each $a \in \Omega$ we have:

$\mathcal{A}$ is the infinitesimal generator of a linear C_0-semigroup $\{S(t)\}_{t \geq 0}$ on $\mathcal{X}$, (1.4)

$\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{C})$, (1.5)

The mapping $\mathcal{D}(\mathcal{A}) \ni u_0 \longmapsto \mathcal{C}S(\cdot)u_0 \in L^2([0,\infty), \mathbb{R}^m)$ is continuous. (1.6)

The function (1.6) has a unique extension $\mathcal{X} \ni u_0 \longmapsto y(\cdot, u_0) \in L^2([0,\infty), \mathbb{R}^m)$ with preservation of norm [13]. The function $y(\cdot, u_0)$ is called the output (trajectory) corresponding to the initial state u_0.

The parametric optimization problem consists in the minimization of the performance index

$$J(a) = \int_0^\infty |y(t, u_0)|^2 dt, \ u_0 \in \mathcal{X}, \ |\cdot| \text{-Euclidean norm in } \mathbb{R}^m, \tag{1.7}$$

over the set Ω.

The solution of this problem is considerably simplified if for each fixed $a \in \Omega$

$$S(t)u_0 \xrightarrow[t \to \infty]{} 0 \quad \forall u_0 \in \mathcal{D}(\mathcal{A}) \tag{1.8}$$

and we can explicitly construct a Fréchet-differentiable functional $V: \mathcal{X} \to \mathbb{R}$ such that $V(0) = 0$,

$$\dot{V}(u) = (\mathrm{grad}\, V(u))(\mathcal{A}u) = -|\mathcal{C}u|^2 \qquad \forall u \in \mathcal{D}(\mathcal{A}) \tag{1.9}$$

As $\dot{V}(u) \leq 0 \quad \forall u \in \mathcal{D}(\mathcal{A})$ then V is a continuous Lyapunov functional on $\mathcal{D}(\mathcal{A})$, Fréchet-differentiable on $\mathcal{X}$. Hence in virtue of Lemma 1

$$V[S(t)u_0] - V(u_0) = -\int_0^t |\mathcal{C}S(\tau)u_0|^2 d\tau \quad \forall u_0 \in \mathcal{D}(\mathcal{A}) \tag{1.10}$$

As $V(0) = 0$, V is continuous on $\mathcal{X}$, (1.8) holds then $\lim\limits_{t\to\infty} V[S(t)u_0] = 0$.

On the other hand, by (1.6) $\lim\limits_{t\to\infty} \int_0^t |\mathcal{C}S(\tau)u_0|^2 d\tau = \int_0^\infty |\mathcal{C}S(t)u_0|^2 dt < \infty$.

Taking those facts into account we obtain from (1.10)

$$V(u_0) = \int_0^\infty |\mathcal{C}S(t)u_0|^2 dt \qquad \forall u_0 \in \mathcal{D}(\mathcal{A})$$

This formula is valid also for every $u_0 \in \mathcal{X}$. For the proof, we take a sequence $\{u_0^n\}_{n\in\mathbb{N}} \subset \mathcal{D}(\mathcal{A})$, $u_0^n \to u_0 \in \mathcal{X} = \overline{\mathcal{D}(\mathcal{A})}$. Now we have

$$V(u_0^n) = \int_0^\infty |\mathcal{C}S(t)u_0^n|^2 dt = \|\mathcal{C}S(\cdot)u_0^n\|^2_{L^2([0,\infty),\mathbb{R}^m)} \quad \forall n \in \mathbb{N}$$

By continuity of V, and due to (1.6) one obtains in the limit

$$V(u_0) = \|\mathcal{C}S(\cdot)u_0\|^2_{L^2} = \int_0^\infty |\mathcal{C}S(t)u_0|^2 dt = J \qquad \forall u_0 \in \mathcal{X} \tag{1.11}$$

The formula (1.11) enables us to express explicitly the performance index (1.7) by the parameters of the system. Let us note that the above derivations are essentially a generalization of the well-known Lyapunov method to parametric optimization of finite-dimensional systems. Thus, our optimization problem is considerably simplified using Lyapunov approach.

2. PARAMETRIC OPTIMIZATION OF A NEUTRAL SYSTEM

We give now a detalied description of the approach presented in section 1 for the system

$$\begin{cases} \frac{d}{dt}[x(t) - Cx(t-r)] = Ax(t) + Bx(t-r), \; t \geq 0 \\ x(\theta) = \phi(\theta) \\ x(0) - Cx(-r) = v_0, \end{cases} \tag{2.1}$$

$x(t) \in \mathbb{R}^n$ for fixed $t \in [-r, +\infty)$; $A, B, C \in \mathcal{L}(\mathbb{R}^n)$, $0 < r < \infty$, r is a constant

The substitutions

$$v(t) = x(t) - Cx(t-r), \qquad \psi(\theta,t) = x(t+\theta) \tag{2.2}$$

convert formally the system (2.1) into a system of the form

$$\begin{cases} \frac{\partial \psi(\theta,t)}{\partial t} = \frac{\partial x(t+\theta)}{\partial t} = \frac{dx(t+\theta)}{d(t+\theta)} \frac{\partial(t+\theta)}{\partial t} = \frac{dx(t+\theta)}{d(t+\theta)} \frac{\partial(t+\theta)}{\partial \theta} = \frac{\partial \psi(\theta,t)}{\partial \theta} \\ \dot{v}(t) = Av(t) + (AC+B)\psi(-r,t) \end{cases}$$

with initial conditions

$$\psi(\theta,0) = x(\theta) = \phi(\theta), \quad v(0) = v_0$$

and a boundary condition

$$v(t) = x(t) - C\psi(-r,t).$$

This system in turn can be transformed into an abstract initial-value problem of the form (1.2) on a Hilbert space

$$\mathcal{X} = \mathbb{R}^n \times L^2([-r,0],\mathbb{R}^n) \text{ with inner product}$$

$$\langle u_1, u_2 \rangle = v_1^T v_2 + \int_{-r}^0 \psi_1(\theta)^T \psi_2(\theta) d\theta, \quad u_i = (v_i, \psi_i) \in \mathcal{X}, \quad i=1,2 \tag{2.3}$$

In order to represent the initial-value problem (2.1) in the abstract form (1.2) we assume

$u = (v, \psi)$, $u_0 = (v_0, \phi)$, $\mathcal{A}u = (Av + (AC+B)\psi(-r), \dot\psi)$, where $\dot\psi$ is the derivative of ψ;

$$\mathcal{D}(\mathcal{A}) = \{u = (v, \psi) \in \mathfrak{X}:\ \psi \in AC([-r,0], \mathbb{R}^n), \dot\psi \in L^2([-r,0], \mathbb{R}^n),\ v = \psi(0) - C\psi(-r)\} \tag{2.4}$$

$AC([-r,0], \mathbb{R}^n)$ – the space of absolutely continuous functions defined on $[-r,0]$ with values in $\mathbb{R}^n$.

The operator $\mathcal{A}$ has the following properties:

$$\mathcal{R}(I - \lambda\mathcal{A}) = \mathfrak{X} \text{ for all sufficiently small } \lambda > 0 \tag{2.5}$$

There exists an $\omega \in \mathbb{R}$ $\left(= \max\left\{\frac{1}{2r}\lambda_{max}(C^TC),\ \frac{1}{2}\lambda_{max}(A+A^T+I) + \frac{1}{2}|AC+B+C|^2\right\}\right)$ such that $(\omega I - \mathcal{A})$ is $\|\cdot\|_e$-accretive (2.6)

where

$$\|u\|_e^2 = v^Tv + \int_{-r}^{0} \psi(\theta)^T\left[I - \frac{\theta}{r}C^TC\right]\psi(\theta)d\theta \tag{2.7}$$

This norm is equivalent to the norm generated by the inner product (2.3);

$$\mathcal{D}(\mathcal{A}) \text{ is a dense subset of } \mathfrak{X}\ (\overline{\mathcal{D}(\mathcal{A})} = \mathfrak{X}) \tag{2.8}$$

Proof of (2.5).

We shall prove that for all sufficiently small $\lambda > 0$ the equation $u - \lambda\mathcal{A}u = \hat u$ has a solution $u \in \mathcal{D}(\mathcal{A})$ for any $\hat u = (\hat v, \hat\varphi) \in \mathfrak{X}$. Taking (2.4) into account we find an equivalent form of this equation

$$\begin{cases} v - \lambda Av - \lambda(AC+B)\psi(-r) = \hat v \\ \psi - \lambda\dot\psi = \hat\varphi \end{cases} \tag{2.9}$$

The second equation has absolutely continuous solution

$$\psi(\theta) = e^{\frac{\theta}{\lambda}}\delta + \int_{\theta}^{0} \frac{1}{\lambda} e^{\frac{\theta-\tau}{\lambda}} \hat\varphi(\tau)d\tau, \tag{2.10}$$

where δ is an arbitrary vector belonging to $\mathbb{R}^n$. In particular,

$$\begin{cases} \psi(0) = \delta \\ \psi(-r) = e^{-\frac{r}{\lambda}}\delta + \int_{-r}^{0}\frac{1}{\lambda}e^{\frac{-r-\tau}{\lambda}}\hat\varphi(\tau)d\tau \end{cases} \tag{2.11}$$

Now we show that δ can be chosen in such a manner that (2.9) will have a solution $(v, \psi) \in \mathcal{D}(\mathcal{A})$ for all sufficiently small $\lambda > 0$. To this end, the system

$$v = \left[I - e^{-\frac{r}{\lambda}}C\right]\delta - C\int_{-r}^{0}\frac{1}{\lambda}e^{\frac{-r-\tau}{\lambda}}\hat\varphi(\tau)d\tau \tag{2.12a}$$

$$(I - \lambda A)v - \lambda(AC+B)\left(e^{\frac{-r}{\lambda}}\delta + \int_{-r}^{0}\frac{1}{\lambda}e^{\frac{-r-\tau}{\lambda}}\hat\varphi(\tau)d\tau\right) = \hat v \tag{2.12b}$$

should have a solution with respect to (δ, v). Eliminating v from (2.12) we obtain an equation determining the vector δ

$$\left[I - e^{\frac{-r}{\lambda}}C - \lambda A - \lambda e^{\frac{-r}{\lambda}}B\right]\delta = \hat v + (\lambda B + C)\int_{-r}^{0}\frac{1}{\lambda}e^{\frac{-r-\tau}{\lambda}}\hat\varphi(\tau)d\tau \tag{2.13}$$

For every sufficiently small $\lambda > 0$ the matrix $[\exp(-\frac{r}{\lambda})C + \lambda A + \exp(-\frac{r}{\lambda})B]$ has the norm less than 1, consequently the system (2.13) has a unique solution δ. Finally, (2.12a) and (2.10) yield the needed solution $u = (v, \psi)$, and obviously $u \in \mathcal{D}(\mathcal{A})$.

Proof of (2.6).

First we show that the norms $\|\cdot\|_e$ and $\|\cdot\|$, defined by (2.7) and (2.3) respectively, are equivalent. Indeed, $0 \leqslant -\frac{\theta}{r}\psi(\theta)^TC^TC\psi(\theta) \leqslant \lambda_{max}(C^TC)\psi(\theta)^T\psi(\theta) \Longrightarrow$

$$0 \leqslant \int_{-r}^{0}\left(-\frac{\theta}{r}\right)\psi(\theta)^TC^TC\psi(\theta)d\theta \leqslant \lambda_{max}(C^TC)\int_{-r}^{0}\psi(\theta)^T\psi(\theta)d\theta, \text{ thus}$$

$$\|u\|^2 = v^Tv + \int_{-r}^{0} \psi(\theta)^T\psi(\theta)d\theta \le \|u\|_e^2 \le \left[1+\lambda_{max}(C^TC)\right]\int_{-r}^{0} \psi(\theta)^T\psi(\theta)d\theta$$

$$+ v^Tv \le \left[1+\lambda_{max}(C^TC)\right]\|u\|^2$$

The norm $\|\cdot\|_e$ satisfies the parallelogram law and thus defines a new inner product in $\mathcal{X}$

$$<u_1,u_2>_e = v_1^T v_2 + \int_{-r}^{0} \psi_1(\theta)^T\left[I - \frac{\theta}{r} C^TC\right]\psi_2(\theta)d\theta, \quad u_i = (v_i,\psi_i)\in\mathcal{X},$$

$i=1,2$. Thus the pair $(\mathcal{X}, <\cdot,\cdot>_e)$ is also a Hilbert space. It can be proved [12] that the condition (1.1) of $\|\cdot\|_e$-accretivity of $(\omega I - \mathcal{A})$ takes in this case an equivalent form $<u,\mathcal{A}u>_e \le \omega<u,u>_e = \omega\|u\|_e^2$ $\forall u\in\mathcal{D}(\mathcal{A})$ for some $\omega\in\mathbb{R}$. For $u\in\mathcal{D}(\mathcal{A})$ we have

$$<u,\mathcal{A}u>_e = v^TAv + v^T(AC+B)\psi(-r) + \int_{-r}^{0} \psi(\theta)^T\left[I - \frac{\theta}{r} C^TC\right]\dot{\psi}(\theta)d\theta =$$

$$= \frac{1}{2} v^T(A+A^T)v + \frac{1}{2} v^T(AC+B)\psi(-r) + \frac{1}{2}\psi(-r)^T(AC+B)^Tv +$$

$$+ \frac{1}{2}\int_{-r}^{0} \frac{d}{d\theta}\left[\psi(\theta)^T\cdot\left(I - \frac{\theta}{r}C^TC\right)\psi(\theta)\right]d\theta + \frac{1}{2r}\int_{-r}^{0}\psi(\theta)^TC^TC\psi(\theta)d\theta =$$

$$= \left[v^T, \psi(-r)^T\right]\begin{bmatrix} \frac{1}{2}(A+A^T+I) & \frac{1}{2}(AC+B) + \frac{1}{2}C \\ \frac{1}{2}C^T + \frac{1}{2}(B^T + C^TA^T) & -\frac{1}{2}I \end{bmatrix}\begin{bmatrix} v \\ \psi(-r) \end{bmatrix} +$$

$$+ \frac{1}{2r}\int_{-r}^{0}\psi(\theta)^TC^TC\psi(\theta)d\theta \le \left\{\frac{1}{2}\lambda_{max}(A+A^T+I) + \frac{1}{2}|AC+B+C|^2\right\}v^Tv +$$

$$+ \frac{1}{2r}\int_{-r}^{0}\lambda_{max}(C^TC)\psi(\theta)^T\psi(\theta)d\theta \le \max\left\{\frac{1}{2r}\lambda_{max}(C^TC), \frac{1}{2}\lambda_{max}(A+\right.$$

$$\left. + A^T+I) + \frac{1}{2}|AC+B+C|^2\right\}\|u\|_e^2.$$ Hence (2.6) holds.

Proof of (2.8)

As $(\mathcal{X}, <\cdot,\cdot>_e)$ is a Hilbert space then by (2.5), (2.6) and Lemma [2,p.115] we conclude that $\mathcal{D}(\mathcal{A})$ is a dense subset of $\mathcal{X}$ in the topology introduced by $\|\cdot\|_e$ and thus $\mathcal{D}(\mathcal{A})$ is a dense subset of $\mathcal{X}$ in the topology introduced by $\|\cdot\|$.

The properties (2.5), (2.6), (2.8) lead to the following proposition.

Proposition 1

The operator $\mathcal{A}$ defined by (2.4) satisfies all hypotheses of theorem 1 and thus it is the infinitesimal generator of a linear C_0-semi-group $\{S(t)\}_{t\ge 0}$ on $\mathcal{X} = \mathbb{R}^n \times L^2([-r,0], \mathbb{R}^n)$.

Note that the semigroup $\{S(t)\}_{t\ge 0}$ has a form $S(t)u = (v(u)(t), x_t(u))$ where $x_t(u)(\theta) = x(u)(t+\theta)$ for $\theta\in[-r,0]$ and $t\ge 0$; $v(u)(\cdot)$, $x(u)(\cdot)$ are the solution to the initial-value problem

$$\begin{cases} \dot{v}(t) = Av(t) + (AC+B)x(t-r) \\ v(0) = v_0 \\ x(\theta) = \phi(\theta), \phi\in L^2([-r,0], \mathbb{R}^n) \\ x(t) = C\,x(t-r) + v(t), \quad u = (v_0, \phi) \end{cases} \qquad (2.14)$$

equivalent to the problem (2.1). This solution can be obtained by the method of steps.

Lemma 2 [3]

The semigroup $\{S(t)\}_{t\ge 0}$, generated by the operator $\mathcal{A}$ of the form (2.4) is exponentially stable (EXS), i.e. $\exists M \ge 1, \exists \mu > 0: \|S(t)\| \le Me^{-\mu t}$ $\forall t \ge 0$, if and only if

$|\lambda(C)| < 1$ (the eigenvalues of the matrix C lie inside the unit circle), (2.15)

the roots of the characteristic quasipolynomial $\det[\lambda I - \lambda e^{-\lambda r}C - A - e^{-\lambda r}B]$ of the system (2.1) has negative real parts. (2.16)

This result is a slight extension of a result by Henry [14] (see also Hale [9]). A detailed proof will be given in [3].

Proposition 2

Assume that (2.15), (2.16) hold. Then the solution $x(u)(\cdot)$ of (2.14) belongs to the space $L^2([-r,\infty), \mathbb{R}^n)$, and the mapping $\mathcal{X}\ni u \longmapsto x(u)(\cdot)\in L^2([-r,\infty), \mathbb{R}^n)$ is continuous.

Proof

Let $|\cdot|$ be a norm in $\mathbb{R}^n$. Then

$$\int_{-r}^{\infty}|x(u)(t)|^2dt = \int_{-r}^{0}|\phi(\theta)|^2d\theta + \int_{0}^{r}|x(u)(t)|^2dt + \int_{r}^{2r} + \ldots \le$$

$$\leqslant \|u\|^2 + \int_{-r}^{0} |x_r(u)(\theta)|^2 d\theta + \int_{-r}^{0} |x_{2r}(u)(\theta)|^2 d\theta + \dots .$$

In virtue of Lemma 2 this yields

$$\int_{-r}^{\infty} |x(u)(t)|^2 dt \leqslant \|u\|^2 \left[1 + M^2 e^{-2\mu r} + M^2 e^{-4\mu r} + \dots\right] =$$

$$= \|u\|^2 \left[1 + M^2 \frac{e^{-2\mu r}}{1-e^{-2\mu r}}\right] < \infty \tag{2.17}$$

thus $x(u)(\cdot) \in L^2([-r,\infty), \mathbb{R}^n)$. On the other hand the mapping $\mathfrak{X} \ni u \longmapsto x(u)(\cdot) \in L^2([-r,\infty), \mathbb{R}^n)$ is linear, so it is continuous if it is continuous at $0 \in \mathfrak{X}$, but this is obvious due to (2.17).

Suppose now that the output equation is

$$y(t) = H_0 x(t) + H_1 x(t-r), \quad H_0, H_1 \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \tag{2.18}$$

(2.18) can be rewritten in the abstract form (1.3) with

$$\mathcal{C}u = H_0[v + C\psi(-r)] + H_1\psi(-r) = H_0 v + (H_1 + H_0 C)\psi(-r), \quad u = (v, \psi). \tag{2.19}$$

The domain $\mathcal{D}(\mathcal{C})$ of $\mathcal{C}$ includes the set $\mathbb{R}^n \times C([-r,0], \mathbb{R}^n)$ where $C([-r,0], \mathbb{R}^n)$ is the space of continuous functions defined o $[-r,0]$ with values in $\mathbb{R}^n$, then $\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\mathcal{C})$.

Let $a \in \mathbb{R}^r$ be the vector of all parameters which occur in the (2.1), (2.18). In general, the matrices A, B, C, H_0, H_1, and the initial condition u_0 may depend on a. Assume that for each $a \in \Omega$ the conditions (2.15), (2.16) are fulfilled.

As $\mathcal{C}S(t)u = H_0 v(u)(t) + (H_0 C + H_1)x(u)(t-r)$ then in virtue of Lemma 2 and Proposition 2, (1.6) and (1.8) hold. (1.4) is fulfilled by proposition 1, and also (1.5) holds. Hence, in accordance with the results given in section 1 the performance index in the parametric optimization problem

$$J(a) = \int_0^{\infty} |y(t)|^2 dt = \int_0^{\infty} [x(t)^T, x(t-r)^T] \begin{bmatrix} H_0^T H_0 & H_0^T H_1 \\ H_1^T H_0 & H_1^T H_1 \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-r) \end{bmatrix} dt \tag{2.20}$$

may be explicitly expressed by a if we can construct the Fréchet-differentiable functional $V: \mathfrak{X} \longrightarrow \mathbb{R}$ which is the Lyapunov functional on $\mathcal{D}(\mathcal{A})$, and such that $V(0) = 0$,

$$\dot V(u) = -|H_0 v + (H_0 C + H_1)\psi(-r)|^2 = [v^T, \psi(-r)^T] \begin{bmatrix} -H_0^T H_0 & -H_0^T H_1 \\ -H_1^T H_0 & -H_1^T H_1 \end{bmatrix} \begin{bmatrix} v \\ \psi(-r) \end{bmatrix}$$

$$\forall u = (v, \psi) \in \mathcal{D}(\mathcal{A}). \tag{2.21}$$

The construction of this functional will be given in the next section.

3. THE CONSTRUCTION OF THE SPECIAL LYAPUNOV FUNCTIONAL FOR THE SYSTEM (2.1)

In this section we present the construction of the special Lyapunov functional V for the system (2.1), for which $V(0) = 0$ and

$$\dot V(u) = -[v^T, \psi(-r)^T] \begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix} \begin{bmatrix} v \\ \psi(-r) \end{bmatrix} \forall u = (v, \psi) \in \mathcal{D}(\mathcal{A}) \tag{3.1}$$

for some

$$\begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix} \geqslant 0_{2n}$$

This form of functional is somewhat more general than we need for the evaluation of the integral in (2.20) and it allows us to determine the integrals of the form

$$\int_0^{\infty} [x(t)^T, x(t-r)^T] \begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-r) \end{bmatrix} dt$$

We seek for the Lyapunov functional, for which (3.1) holds, in the class of functionals of the following general form:

$$V(u) = v^T \alpha v + \int_{-r}^{0} v^T \beta(\theta)\psi(\theta) d\theta + \int_{-r}^{0}\int_{\theta}^{0} \psi(\theta)^T \delta(\theta,\sigma)\psi(\sigma)\, d\sigma d\theta + \int_{-r}^{0} \psi(\theta)^T \gamma(\theta)\psi(\theta) d\theta \tag{3.2}$$

where $\alpha = \alpha^T \in \mathcal{L}(\mathbb{R}^n)$; $\beta, \gamma \in C^1([-r,0], \mathcal{L}(\mathbb{R}^n))$, $\gamma = \gamma^T$; $\delta \in C^1(\Delta, \mathcal{L}(\mathbb{R}^n))$, Δ - a triangle with the vertices (0,0), (-r,-r), (-r,0). Here $C^1(Z, \mathcal{L}(\mathbb{R}^n))$ denotes the space of $n \times n$, real matrices with differentiable elements, defined on the set Z.

Taking into account Lemma 1, by partial integration and the Green-Stokes theorem we obtain

$$\dot V(u) = [v^T, \psi(-r)^T]\left[\begin{array}{c|c} A^T\alpha + \alpha A + \frac{1}{2}\beta(0) + \frac{1}{2}\beta(0)^T + \gamma(0) & \alpha(AC+B) + \frac{1}{2}\beta(0)C + \gamma(0)C - \frac{1}{2}\beta(-r) \\ \hline (AC+B)^T\alpha + \frac{1}{2}C^T\beta(0)^T - \frac{1}{2}\beta(-r)^T + C^T\gamma(0) & C^T\gamma(0)C - \gamma(-r) \end{array}\right]\left[\begin{array}{c} v \\ \psi(-r) \end{array}\right] + \tag{3.3}$$

$$+ \int_{-r}^{0} v^T\left[A^T\beta(\theta) - \frac{d\beta(\theta)}{d(\theta)} + \delta(\theta,0)^T\right]\psi(\theta)d\theta -$$

$$- \int_{-r}^{0} \psi(\theta)^T \frac{d\gamma(\theta)}{d(\theta)}\psi(\theta)d\theta + \int_{-r}^{0} \psi(-r)^T\left[(AC+B)^T\beta(\theta) -\right.$$

$$\left.- \delta(-r,\theta) + C^T\delta(\theta,0)^T\right]\psi(\theta)d\theta - \int_{-r}^{0}\int_{\theta}^{0} \psi(\theta)^T\left[\frac{\partial\delta(\theta,\sigma)}{\partial(\theta)} +\right.$$

$$\left.+ \frac{\partial\delta(\theta,\sigma)}{\partial\sigma}\right]\psi(\sigma)d\sigma d\theta, \quad u = (v,\psi) \in \mathcal{D}(\mathcal{A})$$

If the system of equations

$$\begin{cases} \text{(a)} \; A^T\beta(\theta) - \frac{d\beta(\theta)}{d\theta} + \delta(\theta,0)^T = 0 \\ \text{(b)} \; (AC+B)^T\beta(\theta) - \delta(-r,\theta) + C^T\delta(\theta,0)^T = 0 \\ \text{(c)} \; \frac{\partial\delta(\theta,\sigma)}{\partial\theta} + \frac{\partial\delta(\theta,\sigma)}{\partial\sigma} = 0 \end{cases} \tag{3.4}$$

has a solution with respect to β, δ and $\gamma(\theta) \equiv \gamma$ = const then $\dot V(u)$ takes a form:

$$\dot V(u) = [v^T, \psi(-r)^T]\left[\begin{array}{c|c} A^T\alpha + \alpha A + \frac{1}{2}\beta(0) + \frac{1}{2}\beta(0)^T + \gamma & \alpha(AC+B) + \frac{1}{2}\beta(0)C + \gamma C - \frac{1}{2}\beta(-r) \\ \hline (AC+B)^T\alpha + \frac{1}{2}C^T\beta(0)^T + C^T\gamma - \frac{1}{2}\beta(-r)^T & C^T\gamma C - \gamma \end{array}\right]\left[\begin{array}{c} v \\ \psi(-r) \end{array}\right].$$

Solving the equation (3.4c) and taking into account this solution in (3.4a), (3.4b) one obtains the system

$$\begin{array}{ll} \text{(a)} & \frac{d\beta(\theta)}{d\theta} - A^T\beta(\theta) = \varphi(\theta)^T \\ \text{(b)} & (AC+B)^T\beta(\theta) - \varphi(-r-\theta) + C^T\varphi(\theta)^T = 0 \\ \text{(c)} & \delta(\theta,\sigma) = \varphi(\theta-\sigma), \varphi \in C^1([-r,0], \mathcal{L}(\mathbb{R}^n)) \end{array} \tag{3.5}$$

After elimination of φ from (3.5a), (3.5b) one readily obtains

$$\begin{cases} \frac{d}{d\theta}\left[\beta(\theta) + \beta(-r-\theta)^T C\right] = A^T\beta(\theta) + \beta(-r-\theta)^T B \\ \delta(\theta,\sigma) = \left[\frac{d\beta(\varkappa)^T}{d\varkappa} - \beta(\varkappa)^T A\right] \quad \varkappa = \theta - \sigma \end{cases}$$

On the other hand we require

$$\dot V(u) = -[v^T, \psi(-r)^T]\begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix}\begin{bmatrix} v \\ \psi(-r) \end{bmatrix}$$

hence the following system of algebraic equations must be satisfied

$$\begin{cases} A^T\alpha + \alpha A + \frac{1}{2}\beta(0)^T + \frac{1}{2}\beta(0)^T + \gamma = -P \\ \alpha(AC+B) + \frac{1}{2}\beta(0)C + \gamma C - \frac{1}{2}\beta(-r) = -Q \\ C^T\gamma C - \gamma = -R \end{cases}$$

Thus we come to thefollowing proposition

Proposition 3

Assume that $P, Q, R, P=P^T$, $R=R^T$ are given $n \times n$-dim. real constant matrices. Assume also there exist $n \times n$-dim. real, symmetric matrices α, γ and $\beta \in C^2([-r,0], \mathcal{L}(\mathbb{R}^n))$, such that

$$C^T \gamma C - \gamma = -R \tag{3.6}$$

and

$$\begin{cases} \frac{d}{d\theta}\left[\beta(\theta) + \beta(-r-\theta)^T C\right] = A^T \beta(\theta) + \beta(-r-\theta)^T B, \quad \theta \in [-r,0] \\ \beta(0) + \beta(0)^T = -2P - 2\gamma - 2(\alpha A + A^T \alpha) \\ \beta(-r) = 2Q + 2\gamma C + \beta(0) C + 2\alpha(AC+B) \end{cases} \tag{3.7}$$

Then substituting these α, β, γ and

$$\delta(\theta,\sigma) = \left[\frac{d\beta(\varkappa)^T}{d\varkappa} - \beta(\varkappa)^T A\right]\Bigg|_{\varkappa = \theta - \sigma} \tag{3.8}$$

into (3.2) we obtain a Fréchet-differentiable functional $V: \mathcal{X} \longrightarrow \mathbb{R}$ for which

$$\dot V(u) = -\left[v^T, \psi(-r)^T\right]\begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix}\begin{bmatrix} v \\ \psi(-r) \end{bmatrix} \quad \forall u = (v,\psi) \in \mathcal{D}(\mathcal{A})$$

If furthermore $\begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix}$ is positive semi-definite then V is also a continuous Lyapunov functional for (2.1) on $\mathcal{D}(\mathcal{A})$.

As we verify the hypotheses of the Proposition 3 solving (3.6) and (3.7) then of great importance is the following. Method for solving the equation (3.6) and the boundary-value problem (3.7):

Substituting

$$\vartheta(\theta) = \beta(-r-\theta)^T, \quad \theta \in [-r,0] \tag{3.9}$$

and using the tensor product of matrices [4] we reduce (3.6) to the equation

$$(C^T \otimes C^T - I \otimes I)\, \mathrm{col}\, \gamma = -\mathrm{col}\, R \tag{3.10}$$

and (3.7) to the initial-value problem

$$\begin{bmatrix} C^T \otimes C^T - I \otimes I & 0 \otimes 0 \\ 0 \otimes 0 & C^T \otimes C^T - I \otimes I \end{bmatrix}\begin{bmatrix} \frac{d}{d\theta}\mathrm{col}\,\beta(\theta) \\ \frac{d}{d\theta}\mathrm{col}\,\vartheta(\theta) \end{bmatrix} = \tag{3.11}$$

$$\begin{bmatrix} -(A^T \otimes I + B^T \otimes C^T) & -I \otimes (AC+B)^T \\ (B+AC)^T \otimes I & I \otimes A^T + C^T \otimes B^T \end{bmatrix}\begin{bmatrix} \mathrm{col}\,\beta(\theta) \\ \mathrm{col}\,\vartheta(\theta) \end{bmatrix}$$

$$\mathrm{col}\left[\beta(0) + \beta(0)^T\right] = -2\mathrm{col}P - 2\mathrm{col}\,\gamma - 2(A^T \otimes I + I \otimes A^T)\mathrm{col}\,\alpha$$

$$\mathrm{col}\,\vartheta(0) = 2\mathrm{col}Q^T + 2\left[(AC+B)^T \otimes I\right]\mathrm{col}\,\alpha + (C^T \otimes I)\mathrm{col}\left[\beta(0)^T\right] +$$

$$+ 2(C^T \otimes I)\mathrm{col}\,\gamma$$

Solving (3.11) and (3.12) with respect to γ, β', ϑ we find (if it is possible) the unknown matrix α from (3.9) (usually taking $\theta = -r$).

Remark. The matrix $(C^T \otimes C^T - I \otimes I)$ is non-singular if and only if $\lambda_i \lambda_j \neq 1 \quad \forall i,j = 1,2,\dots,n$, $\lambda_k \in \lambda(C)$ [4]. If (2.15) holds then this matrix is non-singular.

The above construction of the Lyapunov functional generalizes the results of earlier works [5], [6], [7]. In [7] a similar construction was proposed, but the Lyapunov functional given there was more complicated due to the fact that the initial-value problem (2.1) was interpreted as an abstract one on the Sobolev space $W^{1,2}([-r,0], \mathbb{R}^n)$ i.e. much less generally than in section 2. Our approach leads thus to simpler calculations and extends the range of applicability.

4. EXAMPLE

Consider the system of automatic control presented fig. 1. The dynamics of this system is described by the equations

$$\varepsilon(t) = -y(t-r) + w, \quad w = \mathrm{const.} \tag{4.1}$$

$$K_0\left[\delta(t) + K_1\varepsilon(t) + K_2\dot\varepsilon(t) + K_3\int_0^t \varepsilon(\tau)d\tau\right] = qy(t) + T_0\dot y(t) \tag{4.2}$$

$\delta(t)$ denotes the Dirac pseudofunction

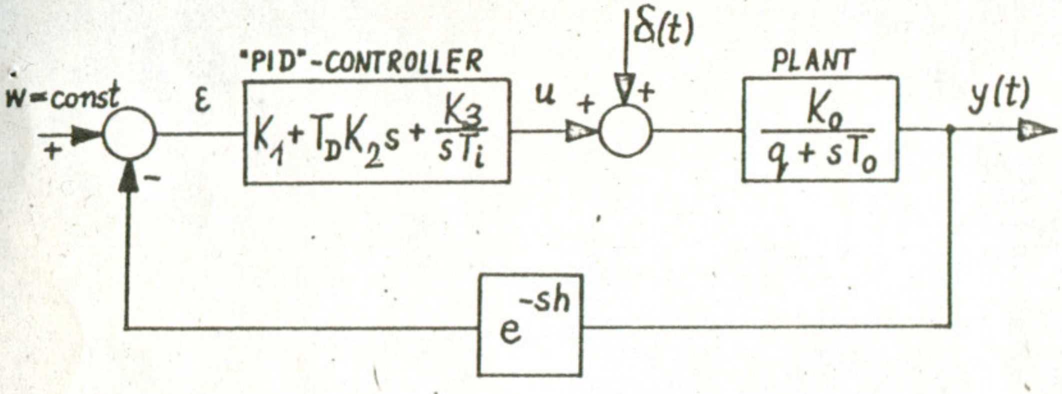

Fig. 1

Usually q=0 or q=1. If the system is asymptotically stable then we have:

$$\lim_{t\to\infty}\varepsilon(t)=0,\quad \lim_{t\to\infty}y(t)=w,\quad \int_0^\infty \varepsilon(t)dt=\frac{qw}{K_0K_3} \tag{4.3}$$

Introducing the state variables in the form

$$\begin{cases} x_1(t)=y(t)-w \\ x_2(t)=\dfrac{K_0K_3}{T_0}\displaystyle\int_0^t \varepsilon(\tau)d\tau-\frac{qw}{T_0}\end{cases} \tag{4.4}$$

we obtain the system of equations

$$\frac{d}{dt}\left\{\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}-\begin{bmatrix}\frac{-K_0K_2}{T_0} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t-r)\\x_2(t-r)\end{bmatrix}\right\}=\begin{bmatrix}\frac{-q}{T_0} & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}+ \tag{4.5}$$

$$+\begin{bmatrix}\frac{-K_0K_1}{T_0} & 0\\ \frac{-K_0K_3}{T_0} & 0\end{bmatrix}\begin{bmatrix}x_1(t-r)\\x_2(t-r)\end{bmatrix}+\begin{bmatrix}\frac{K_0}{T_0}\delta(t)\\ 0\end{bmatrix}$$

Assume that for $t<0$ the system was in equilibrium, then from (4.5) we obtain an initial-value problem of the following general form, which will be considered later on

$$\frac{d}{dt}\left\{\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}-\begin{bmatrix}c & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t-r)\\x_2(t-r)\end{bmatrix}\right\}=\begin{bmatrix}a & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}+ \tag{4.6}$$

$$+\begin{bmatrix}b & 0\\ d & 0\end{bmatrix}\begin{bmatrix}x_1(t-r)\\x_2(t-r)\end{bmatrix},\quad x_1(\theta)=\begin{cases}0, & \theta\in[-r,0)\\ x_0, & \theta=0\end{cases},\quad x_2(\theta)=0,\ \theta\in[-r,0]$$

In our problem

$$c=\frac{-K_0K_2}{T_0},\quad a=\frac{-q}{T_0},\quad b=\frac{-K_0K_1}{T_0},\quad d=\frac{-K_0K_3}{T_0},\quad x_0=\frac{K_0}{T_0} \tag{4.7}$$

The problem (4.6) has the form (2.1) with

$$A=\begin{bmatrix}a & 1\\ 0 & 0\end{bmatrix},\quad B=\begin{bmatrix}b & 0\\ d & 0\end{bmatrix},\quad C=\begin{bmatrix}c & 0\\ 0 & 0\end{bmatrix},\quad \phi(\theta)^T=[0,0],\quad v^T=[x_0,0]$$

Now we shall express explicitly the integral

$$J=\int_0^\infty [y(t)-y_\infty]^2\,dt=\int_0^\infty \varepsilon^2(t)dt=\int_0^\infty x_1^2(t)dt \tag{4.8}$$

by the parameters a,b,c,d. In accordance with the theory presented in the section 2 this integral exists if $|c|<1$ (see (2.15)) and the zeros of the function $s\longmapsto e^{sr}(s^2-as)-s^2c-sb-d$ has negative real parts (see (2.16)). The set of points in the space of parameters where the second requirement is fulfilled can be determined from Pontryagin's theorem [8,9] and is considered elsewhere [3].

The observed quantity is x_1, so in (2.18) we put $H_0=[1,0]$, $H_1=[0,0]$. In accordance with (2.19) the abstract output equation has a form $\ell u=v_1+c\,\varphi_1(-r)$. As it follows from the results presented in the sections 1 and 2, we should find a Lyapunov functional V for which $\dot{V}(u) = -\left[v_1 + c\,\psi_1(-r)\right]^2$, $u = (v, \psi) \in \mathcal{D}(\mathcal{A})$. For this purpose we substitute in (3.1)

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} c^2 & 0 \\ 0 & 0 \end{bmatrix}$$

Notice that it is not necessary to determine all elements of the matrix α, actually we only need to know the element α_{11}, because for our particular initial conditions we have

$$J = \alpha_{11} x_0^2 \qquad (4.9)$$

The equation (3.10) yields

$$\gamma = \begin{bmatrix} \frac{c^2}{1-c^2} & 0 \\ 0 & 0 \end{bmatrix} \qquad (4.10)$$

Taking into account (4.10) we can rewrite (3.11) in the form

$$\frac{d}{d\theta}\begin{bmatrix} \beta_{11}(0) \\ \beta_{12}(0) \\ \beta_{21}(0) \\ \beta_{22}(0) \\ \vartheta_{11}(0) \\ \vartheta_{12}(0) \\ \vartheta_{21}(0) \\ \vartheta_{22}(0) \end{bmatrix} = \begin{bmatrix} \frac{a+bc}{1-c^2} & 0 & \frac{cd}{1-c^2} & 0 & \frac{b+ac}{1-c^2} & \frac{d}{1-c^2} & 0 & 0 \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & b+ac & d \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{b+ac}{1-c^2} & 0 & -\frac{d}{1-c^2} & 0 & -\frac{a+cb}{1-c^2} & -\frac{cd}{1-c^2} & 0 & 0 \\ 0 & -b-ac & 0 & -d & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \beta_{11}(\theta) \\ \beta_{12}(\theta) \\ \beta_{21}(\theta) \\ \beta_{22}(\theta) \\ \vartheta_{11}(\theta) \\ \vartheta_{12}(\theta) \\ \vartheta_{21}(\theta) \\ \vartheta_{22}(\theta) \end{bmatrix} \qquad (4.11a)$$

with initial conditions

$$\begin{cases} \beta_{11}(0) = \frac{-1}{1-c^2} - 2a\,\alpha_{11} \\ \beta_{12}(0) + \beta_{21}(0) = -2\,\alpha_{11} - 2a\,\alpha_{12} \\ \beta_{22}(0) = -2\,\alpha_{12} \\ \vartheta_{11}(0) = \frac{2c}{1-c^2} + 2(ac+b)\,\alpha_{11} + 2d\,\alpha_{12} + c\,\beta_{11}(0) \\ \vartheta_{12}(0) = 2(ac+b)\,\alpha_{12} + 2d\,\alpha_{22} + c\,\beta_{21}(0) \\ \vartheta_{21}(0) = 0 \\ \vartheta_{22}(0) = 0 \end{cases} \qquad (4.11b)$$

From this we obtain $\vartheta_{21}(\theta) = 0$, $\vartheta_{22}(\theta) = 0$ $\forall \theta \in [-r, 0]$, from (3.9) also $\beta_{12}(\theta) = 0$, $\beta_{22}(\theta) = 0$ $\forall \theta \in [-r, 0]$. Due to this we can reduce the system (4.11) to the form:

$$\frac{d}{d\theta}\begin{bmatrix} \beta_{11}(\theta) \\ \beta_{21}(\theta) \\ \vartheta_{11}(\theta) \\ \vartheta_{12}(\theta) \end{bmatrix} = \begin{bmatrix} \frac{a+bc}{1-c^2} & \frac{cd}{1-c^2} & \frac{b+ac}{1-c^2} & \frac{d}{1-c^2} \\ 1 & 0 & 0 & 0 \\ -\frac{b+ac}{1-c^2} & -\frac{d}{1-c^2} & -\frac{a+cb}{1-c^2} & \frac{-cd}{1-c^2} \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \beta_{11}(\theta) \\ \beta_{21}(\theta) \\ \vartheta_{11}(\theta) \\ \vartheta_{12}(\theta) \end{bmatrix} \qquad (4.12a)$$

with initial conditions

$$\begin{bmatrix} \beta_{11}(0) \\ \beta_{21}(0) \\ \vartheta_{11}(0) \\ \vartheta_{12}(0) \end{bmatrix} = \begin{bmatrix} -2a\,\alpha_{11} - \frac{1}{1-c^2} \\ -2\,\alpha_{11} \\ \frac{c}{1-c^2} + 2b\,\alpha_{11} \\ 2d\,\alpha_{22} - 2c\,\alpha_{11} \end{bmatrix} \qquad (4.12b)$$

moreover

$$\alpha_{12} = 0 \qquad (4.13)$$

The characteristic polynomial of the system (4.12) is biquadratic

$$\lambda^4 + \lambda^2 \left\{ \left(\frac{b+ac}{1-c^2}\right)^2 - \left(\frac{a+bc}{1-c^2}\right)^2 - \frac{2cd}{1-c^2} \right\} - \frac{d^2}{1-c^2} \qquad (4.14)$$

Introduce the following notation

$$k_1 = \frac{b^2-a^2-2cd}{1-c^2}, \quad k_2 = \frac{d^2}{1-c^2} \qquad (4.15)$$

$$G^2 = \frac{1}{2}\left[k_1 + \sqrt{k_1^2 + 4k_2}\right], \quad H^2 = \frac{1}{2}\left[-k_1 + \sqrt{k_1^2 + 4k_2}\right] \qquad (4.16)$$

In this notations (4.14) can be written in a form

$$(\lambda^2 + G^2)(\lambda^2 - H^2) \qquad (4.17)$$

hence the fundamental matrix of (4.12) has a form

$$e^{\theta T} = (\cos G\theta)\, Z_1 + (\sin G\theta)\, Z_2 + (\cosh H\theta)\, Z_3 + (\sinh H\theta)\, Z_4 \qquad (4.18a)$$

where T-the matrix of the system (4.12), and

$$\begin{bmatrix} Z_1 \\ Z_3 \end{bmatrix} = \frac{1}{G^2+H^2}\begin{bmatrix} H^2 I - T^2 \\ G^2 I + T^2 \end{bmatrix}; \quad \begin{bmatrix} Z_2 \\ Z_4 \end{bmatrix} = \frac{1}{GH(G^2+H^2)}\begin{bmatrix} H^3 I - HT^3 \\ G^3 I + GT^3 \end{bmatrix} \qquad (4.18b)$$

We determine the unknown elements α_{11}, α_{22} using the equality

$$\begin{bmatrix} \vartheta_{11}(-r) \\ \vartheta_{12}(-r) \end{bmatrix} = \left[\text{the last two rows of the matrix } e^{-rT}\right]_{2\times 4} \cdot \qquad (4.19)$$

$$\cdot \left[\text{a vector of initial conditions (4.12b)}\right]_{4\times 1} = \begin{bmatrix} \beta_{11}(0) \\ \beta_{21}(0) \end{bmatrix},$$

arising from (3.9). After tedious but elementary calculations, it turns out that the system (4.19) has a form

$$\left[\begin{array}{c} bG^2\cos Gr + (cG^3-dG)\sin Gr + bH^2\cosh Hr - (cH^3+dH)\sinh Hr \\ \hline (d-G^2c)\cos Gr + bG\sin Gr - (d+cH^2)\cosh Hr + bH\sinh Hr + \end{array}\right.$$

$$\begin{array}{c|c} +aG^2 + aH^2 & \dfrac{bd^2}{1-c^2}\cos Gr + \dfrac{cd^2G^2-d^3}{G(1-c^2)}\sin Gr - \dfrac{bd^2}{1-c^2}\cosh Hr + \\ \hline +G^2 + H^2 & (dH^2 - \dfrac{cd^2}{1-c^2})\cos Gr + \dfrac{bd^2}{G(1-c^2)}\sin Gr + (dG^2 + \dfrac{cd^2}{1-c^2})\cosh Hr \end{array}$$

$$\left.\begin{array}{c} + \dfrac{d^3+cH^2d^2}{H(1-c^2)}\sinh Hr \\ \hline - \dfrac{bd^2}{H(1-c^2)}\sinh Hr \end{array}\right] \begin{bmatrix} \alpha_{11} \\ \alpha_{22} \end{bmatrix} = \frac{1}{2(1-c^2)} \left[\begin{array}{c} -G^2-H^2+(d-cG^2)\cos Gr + \\ \hline -b\cos Gr + (\dfrac{d-cG^2}{G})\sin Gr + \end{array}\right.$$

$$\left.\begin{array}{c} + bH\sinh Hr + bG\sin Gr - (d+cH^2)\cosh Hr \\ \hline + b\cosh Hr - \dfrac{d+cH^2}{H}\sinh Hr \end{array}\right]. \qquad (4.20)$$

The element α_{11} is determined from (4.20):

$$\alpha_{11} = \frac{\left[-(G^2+H^2)+(d-cG^2)\cos Gr + bG\sin Gr - (d+cH^2)\cosh Hr + \right.}{2(1-c^2)\left\{\left[bG^2\cos Gr + (cG^3-dG)\sin Gr + bH^2\cosh Hr - \right.\right.}$$

$$\frac{\left. + bH\sinh Hr\right]\left[(H^2 - \dfrac{cd}{1-c^2})\cos Gr + \dfrac{bd}{G(1-c^2)}\sin Gr + (G^2 + \dfrac{cd}{1-c^2})\cosh Hr - \right.}{\left. - (cH^3+dH)\sinh Hr + aG^2 + aH^2\right]\left[(H^2 - \dfrac{cd}{1-c^2})\cos Gr + \dfrac{bd}{G(1-c^2)}\sin Gr + \right.}$$

$$- \frac{bd}{H(1-c^2)} \sinh Hr \Big] - \Big[-b\cos Gr + \frac{d-cG^2}{G} \sin Gr + b\cosh Hr -$$

$$+ (G^2 + \frac{cd}{1-c^2})\cosh Hr - \frac{bd}{H(1-c^2)} \sinh Hr \Big] - \Big[(d-cG^2)\cos Gr + bG\sin Gr -$$

$$- \frac{d+cH^2}{H} \sinh Hr \Big]\Big[\frac{bd}{1-c^2} \cos Gr + \frac{cdG^2-d^2}{G(1-c^2)} \sin Gr - \frac{bd}{1-c^2} \cosh Hr +$$

$$-(d+cH^2)\cosh Hr + bH\sinh Hr + G^2+H^2 \Big]\Big[\frac{bd}{1-c^2}\cos Gr + \frac{d^2+cdH^2}{H(1-c^2)} \sin Gr -$$

$$\frac{cdG^2-d^2}{G(1-c^2)}$$

$$+ \frac{d^2+cdH^2}{H(1-c^2)} \sinh Hr \Big]$$

$$\Big[- \frac{bd}{1-c^2} \cosh Hr + \frac{d^2+cdH^2}{H(1-c^2)} \sinh Hr \Big]\Big\} \qquad (4.21)$$

The formulas (4.9), (4.21) give the desired expression for the performance index J in terms of the parameters of the system.

If $d \to 0$ then $H \to 0$, $G \to k = \sqrt{\frac{b^2 - a^2}{1-c^2}}$ and

$$\alpha_{11} \to \tilde{\alpha} = \frac{-1 - c\cos kr + \frac{b}{k}\sin kr}{2(1-c^2)\left[b\cos kr + ck\sin kr + a\right]} \qquad (4.22)$$

This limit corresponds to the transformation of a "PID"-controller into a "PD"-controller. In turn, assumption $c = 0$ in (4.22) leads to a "P"-controller. In the last case our result agrees with the earlier results of Repin [5] and Górecki H. and Popek [10]. It should be emphasized that taking $d \to 0$ we simultaneously drop out the second equation in (4.6) ($x_2(t) \equiv 0$).

If $r \to 0$ then $\alpha_{11} \to [2(-a-b)(1-c)]^{-1}$, which is in accordance with the direct computations for the system without delay.

Numerical examples of parametric optimization.

1. $a = -1$, $b = 0$, $c = 0$, $d = \frac{-1}{T_1}$, $r = 1$

All zeros of the quasipolynomial have negative real parts if and only if $d^* < d < 0$, d^* - the greatest negative solution of the equation

$$d = \frac{1 - \sqrt{1 + 4d^2}}{2\cos\sqrt{\frac{-1 + \sqrt{1+4d^2}}{2}}} \qquad [11]$$

Numerical calculations give $d^* = -1.1349147$. Accordingly to (4.21) we have

$$\alpha_{11} = \frac{\left[-G^2-H^2 + d\cos G - d\cosh H\right]\left[H^2\cos G + G^2\cosh H\right] -}{2\Big\{\left[-dG\sin G - dH\sinh H - G^2-H^2\right]\left[H^2\cos G + G^2\cosh H\right] - \left[d\cos G - d\cosh H + G^2+H^2\right]\left[\frac{d^2}{G}\sin G + \frac{d^2}{H}\sinh H\right]\Big\}}$$

$$- \left[\frac{d}{G}\sin G - \frac{d}{H}\sinh H\right]\left[- \frac{d^2}{G}\sin G + \frac{d^2}{H}\sinh H\right], \text{ where}$$

$$G^2 = \frac{-1 + \sqrt{1 + 4d^2}}{2}, \quad H^2 = \frac{1 + \sqrt{1 + 4d^2}}{2} \qquad (4.23)$$

The relation between d and α_{11} is shown in the table 1.

Table 1

d	$\alpha_{11} = \frac{1}{x_0^2} J$
- 1.1349147	+ ∞
- 1.1	11.992139
- 1.0	3.1999364
- 0,9	1.8935834
- 0.8	1.3676259
- 0.7	1.0837608
- 0.6	0.90621833
- 0.5	0.78476538
- 0.4	0.696509
- 0.3	0.62952466
- 0.2	0.5769895
- 0.1	0.53471810
0.0	0.50000000

The calculations indicate that the optimum of the performance index (4.8) is achieved on the greatest admissible value of T_1.

2. The cases: $a = 0$, $b = -1.5$, $r = 1$, $d = 0$, c-a parameter and $a = -0.5$, $b = -1.5$, $r = 1$, $d = 0$, c-the parameter, correspond to a "PD"-controller.

In the table 2, $\tilde{\alpha}$ is tabularized as the function of c in the whole stability region. The optima of the performance index (4.8) exist. The analysis of their location in the region of stablity in the space of parameters leads to the conclusion that the optimal transient behaviour of x is of oscillatory type.

Table 2

1-st case		2-nd case	
c	$\frac{J}{x_0^2}$	c	$\frac{J}{x_0^2}$
-0.79950799	+∞	-0,85829616	+∞
-0.75	6.659014	-0.85	22.519904
-0.70	3.7791525	-0.80	3.6327104
-0.60	2.4324478	-0.70	1.6475572
-0.50	2.1012043	-0.60	1.2217841
-0.442121	2.063127	-0.50	1.0629388
-0.40	2.0832485	-0.40	1.0072749
-0.30	2.2883107	-0.36055	1.0030266
-0.20	2.8051252	-0.35	1.0033182
-0.10	4.0912174	-0.30	1.0124567
0.00	9.4127512	-0.20	1.0691183
0.0673334	+∞	-0.10	1.1860149
		0.00	1.3950406
		0.10	1.7838417
		0.20	2.6490302
		0.30	5.9060129
		0.373696	+∞

5. A SHORT DISCUSSION OF THE OBTAINED RESULTS

To conclude, let us consider the virtues of the Lyapunov approach in parametric optimization in comparision with the direct approach. Direct application of the iterative numerical optimization methods ot our optimization problem has several shortcomings. First of all, the evaluation ot the performance index (2.20) requires the integration of the state equations over an infinite time interval which may cause serious numerical difficulties. Secondly, there is no reliable method of telling whether the point at which the performance index is evaluated belongs to the domain.

Another drawback is the difficulty in numerical evaluation of the gradient which may exclude the rapidly convergent gradient optimization methods.

The Lyapunov approach enables us to avoid some of those difficulties. In simple cases, eg. the system (4.6) with a proportional action controller and a first order inertial plant with delay, it makes it possible to express explicitly the performance index by the parameters, which gives the optimal solution in a simple analytic form. If there are many optimization parameters, such analytic solutions become very complicated and practically are of little use. In such cases numerical optimization is preferable. Due to the Lyapunov approach the integration over an infinite interval is replaced by integration over a time interval of length r. More precisely, the two-boundary problem (3.7) must be solved which leads to a rather high dimensional (n^2) system of ordinary differential equations. Still, there may be difficulties in recognizing if the point belongs to the domain of the performance index, but it is easier to detect the boundary of the domain. Lastly, it is noteworthy that the Lyapunov approach makes it easier to determine the gradient of the performance index.

It should be stressed that there still remain some unsolved questions, the main of which is the existence of the Lyapunov functional discussed in section 1. Consequently we do not know the range of applicability of the Lyapunov functional approach to neutral systems, presented in section 3. A forthcoming paper will be devoted to this problem.

REFERENCES

1. Walker J.A.: Dynamical Systems and Evolution Equations. Plenum Press. N.York-London 1980.
2. Walker J.A.: On the application of Lyapunov's direct method to linear dynamical systems. J.Math.Anal.Appl.1976,53.2.187-220.
3. Górecki H.,Grabowski P., Fuksa S., Korytowski A.: Analysis and Synthesis of Time-Delay Systems.
4. Lancaster P.: Theory of Matrices. Ac.Press.N.York-London 1969.
5. Repin Yu.: Quadratic Lyapunov functionals for delay systems. Prikladnaja Matiematika i Miechanika. 1965.29.3.564-566.
6. Infante E.F., Castelan W.B.: A Liapunov functional for a matrix difference-differential equation with one delay. J.Diff.Eqs.1978.29. 3.439-451.
7. Castelan W.B., Infante E.F.: A Liapunov functional for a matrix neutral difference-differential equation with one delay. J.Math.Anal.Appl. 1979.71.1.105-130.

8. Pontryagin L.: On zeros of some transcendental functions. Izv.Akad. Nauk.SSSR.Ser.Mat.1942.6.115-134. English translation in AMS Transl. 1955.1(2).95-110.

9. Hale J.K.: Theory of Functional Differential Equations. Springer. N.York-Heidelberg-Berlin 1977.

10. Górecki H., Popek L.: Control of the systems with time delay. Preprints of 3rd IFAC Symposium. Control of Distributed Parameter Systems.29 June - 2 July 1982.Ed.J.P.Babary, L.Le Letty.XXII.9.

11. Bhatt S.J., Hsu C.S.: Stability criteria for 2-nd order dynamical systems with time-lag. Trans.ASME.J.Appl.Mech.1966.E-33.1.113-118.

12. Grabowski P.: Operatory akkretywne w teorii układów semidynamicznych. Dissertation.AGH.Kraków 1978/1979.

13. Yosida K.: Functional Analysis. Springer. N.York-Heidelberg-Berlin. 5-th edition. 1978.

14. Henry D.: Linear autonomous neutral functional equations. J.Diff. Eqs.1974.15. 2. 106-128.

Consultee:

Doc.dr hab.inż. Stanisław Raczyński