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Evaluation of quadratic cost functionals for neutral systems: the frequency-domain approach

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As shown in Grabowski (1983), the problem of evaluation of the quadratic performance index for delay-differential systems of neutral type can be effectively solved by the use of Lyapunov functionals. In this paper another solution to that problem, based on the frequency-domain approach, is proposed. With the use of Plancherel's theorem and other tools of complex analysis, this task is reduced to a Riemann–Hilbert problem, an explicit solution to which is found by elementary methods. Our results can be regarded as a generalization of those due to Walton and Marshall (1984, 1987). Two simple examples of application are also provided.

1. Introduction and formulation of the problem

Consider the neutral system

$$\left.\begin{aligned} &\dot{v}(t) = Av(t) + (AC + B)x(t-r), \quad t \geqslant 0 \\ &v(t) = x(t) - Cx(t-r), \qquad\qquad\; t \geqslant 0 \\ &v(0) = v_0 \\ &x(\theta) = \phi(\theta) \quad \text{for almost every } \theta \in [-r, 0] \end{aligned}\right\} \tag{1}$$

where $A, B, C \in \mathscr{L}(\mathbb{R}^n)$, $r > 0$, $v_0 \in \mathbb{R}^n$ and ϕ is a function defined on $[-r, 0]$ with values in $\mathbb{R}^n$.

Note that system (1) is easily obtained from the model of neutral systems commonly used in earlier papers:

$$\left.\begin{aligned} &\dot{x}(t) - C\dot{x}(t-r) = Ax(t) + Bx(t-r), \quad t \geqslant 0 \\ &x(\theta) = \phi(\theta) \quad (\phi \in C^1([-r, 0], \mathbb{R}^n)) \end{aligned}\right\} \tag{1'}$$

by the substitution $v(t) = x(t) - Cx(t-r)$. System (1) is more general than (1′) as it makes sense also for discontinuous initial conditions which evidently follows from the method of steps. For fixed $t \geqslant 0$ we define the function

$$x_t : [-r, 0] \ni \theta \mapsto x_t(\theta) = x(t+\theta) \in \mathbb{R}^n$$

Grabowski (1983) has proved (see also Górecki *et al.* (1987)), by the direct application of the Hille–Phillips–Yosida Theorem that the family of operators

$$\{S(t)\}_{t \geqslant 0}, \quad S(t)(v_0, \phi) = (v(t), x_t)$$

on $M^2 = \mathbb{R}^n \times L^2(-r, 0; \mathbb{R}^n)$ is a C_0-semigroup. M^2 is a Hilbert space with the scalar

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product

$$\langle (v_1, \phi_1), (v_2, \phi_2) \rangle = v_1^{\mathrm{T}} v_2 + \int_{-r}^{0} \phi_1^{\mathrm{T}}(\theta) \phi_2(\theta) \, d\theta, \quad (v_i, \phi_i) \in M^2, \quad i = 1, 2$$

The semigroup $\{S(t)\}_{t \geqslant 0}$ is said to be exponentially stable if there exist constants $M \geqslant 1$, $\mu > 0$ such that

$$\|S(t)\|_{\mathscr{L}(M^2)} \leqslant M \exp(-\mu t) \quad \forall t \geqslant 0 \tag{2}$$

This is equivalent to

$$C \text{ has all eigenvalues inside the open unit circle} \tag{3 a}$$

$$\{s \in \mathbb{C} : \det X(s) = 0\} \subset \{s \in \mathbb{C} : \operatorname{Re} s < 0\} \tag{3 b}$$

where

$$X(s) = sI - s \exp(-sr) C - A - \exp(-sr) B \tag{4}$$

The aim of this paper is to present a method for explicit evaluation of the quadratic performance index

$$J = \int_0^\infty x^{\mathrm{T}}(t) Q x(t) \, dt, \quad Q \in \mathscr{L}(\mathbb{R}^n), \quad Q = Q^{\mathrm{T}} \geqslant 0 \tag{5}$$

The range of applicability of this method is comparable with that obtained by the Lyapunov approach—see Grabowski (1983). The present method can be regarded as a generalization of results recently derived by Walton and Marshall (1984, 1987). Since our results are based on the frequency-domain technique, some properties of the Laplace transform of x will be discussed in the next section.

2. Properties of Laplace transform of x

We start from the following result.

Lemma 1

If (2) holds, then there exist $m_1, m_2 > 0$ such that for every $(v_0, \phi) \in M^2$ the following estimates are valid:

$$\int_{-r}^{\infty} |x(t)| \, dt \leqslant m_1 \|(v_0, \phi)\|_{M^2} \tag{6}$$

$$\int_{-r}^{\infty} |x(t)|^2 \, dt \leqslant m_2 \|(v_0, \phi)\|_{M^2}^2 \tag{7}$$

Here $|\cdot|$ denotes a norm in $\mathbb{R}^n$.

Proof

We prove only (6) since (7) can be proved in the same way:

$$\int_{-r}^{\infty} |x(t)| \, dt = \int_{-r}^{0} |\phi(\theta)| \, d\theta + \int_0^r |x(t)| \, dt + \int_r^{2r} + \int_{2r}^{3r} + \ldots$$

$$= \int_{-r}^{0} [|\phi(\theta)| + |x_r(\theta)| + |x_{2r}(\theta)| + \ldots] \, d\theta$$

$$\leqslant \sqrt{r}[\|\phi\|_{L^2} + \|x_r\|_{L^2} + \|x_{2r}\|_{L^2} + \dots]$$
$$\leqslant \sqrt{r}[1 + M\exp(-\mu r) + M\exp(-2\mu r) + \dots]\|(v_0, \phi)\|_{M^2}$$
$$= \sqrt{r}\left[1 + M\frac{\exp(-\mu r)}{1-\exp(-\mu r)}\right]\|(v_0, \phi)\|_{M^2} \qquad \square$$

An important corollary from (7) is that the integral (5) can be expressed in the unique way as a strongly-continuous quadratic form on M^2. By (7) and the Paley–Wiener Theorem, the Laplace transform $\hat{x}$ of x belongs to the Hardy space $H^2(\Pi_+, \mathbb{C}^n)$, i.e. to the class of functions which are analytic on $\Pi_+ = \{s \in \mathbb{C} : \text{Re}\, s > 0\}$ and satisfy

$$\sup_{\sigma > 0} \int_{-\infty}^{+\infty} |\hat{x}(\sigma + i\omega)|^2 \, d\omega < \infty$$

On the other hand by (6) and some elementary estimations, x has the Fourier transform (in the sense of L^1-theory), which is exactly the restriction of $\hat{x}$ to the imaginary axis and $\hat{x}$ is continuous and bounded on $\bar{\Pi}_+ = \{s \in \mathbb{C} : \text{Re}\, s \geqslant 0\}$. Owing to these properties of $\hat{x}$ and by Plancherel's theorem (Parseval's formula) we have

$$J = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{x}^{\mathrm{T}}(-i\omega) Q \hat{x}(i\omega)\, d\omega = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \hat{x}^{\mathrm{T}}(-s) Q \hat{x}(s)\, ds \tag{8}$$

As a result of application of the Laplace transformation to (1) one obtains

$$s\hat{v}(s) - v_0 = A\hat{v}(s) + (AC + B)\exp(-sr)\hat{x}(s) + (AC + B)\exp(-sr)\phi_F(s) \tag{9 a}$$
$$\hat{v}(s) = \hat{x}(s) - \exp(-sr) C\hat{x}(s) - \exp(-sr) C\phi_F(s) \tag{9 b}$$

where $\hat{v}$ denotes the Laplace transform of v, while ϕ_F is the finite Laplace transform of ϕ, i.e. $\phi_F(s) = \int_{-r}^{0} \exp(-s\theta)\phi(\theta)\, d\theta$. ϕ_F is an entire function of exponential type. Now, from (9) we get

$$\hat{x}(s) = X^{-1}(s) p(s) \tag{10 a}$$
$$p(s) = v_0 + (B + sC)\exp(-sr)\phi_F(s) \tag{10 b}$$
$$s\hat{x}(s) = (I - \exp(-sr)C)^{-1}[v_0 + A\hat{v}(s) + (AC + B)\exp(-sr)\hat{x}(s) + (AC + B + sC)\exp(-sr)\phi_F(s)] \tag{10 c}$$

This in turn allows us to establish some additional properties of $\hat{x}$. Namely, we can conclude from (10 *a*), (10 *b*) that $\hat{x}$ is analytic in the half-plane Re $s > -\mu$ and $s\hat{x}(s)$ is bounded on $\bar{\Pi}_+$, provided that $s\exp(-sr)\phi_F(s)$ is bounded on $\bar{\Pi}_+$. The following result is important in this context.

Lemma 2

The function $s \mapsto s\exp(-sr)\phi_F(s)$ is bounded on $\bar{\Pi}_+$, if $\phi \in \text{BV}([-r, 0], \mathbb{R}^n)$ where the last symbol denotes the space of functions defined on $[-r, 0]$ with values in $\mathbb{R}^n$, having bounded variation on $[-r, 0]$.

Proof

Since $\phi \in \mathrm{BV}([-r,0],\mathbb{R}^n)$ the Riemann–Stieltjes integral $\int_{-r}^{0} \exp(-s(\theta+r))\, d\phi(\theta)$ exists, and moreover an integration by parts yields

$$s\exp(-sr)\,\phi_F(s) = \int_{-r}^{0} \frac{d}{d\theta}[-\exp(-s(\theta+r))]\,\phi(\theta)\,d\theta$$

$$= -\exp(-s(\theta+r))\,\phi(\theta)|_{-r}^{0} + \int_{-r}^{0} \exp(-s(\theta+r))\,d\phi(\theta)$$

This implies

$$|s\exp(-sr)\,\phi_F(s)| \leqslant |\phi(0)| + |\phi(-r)| + \operatorname*{var}_{-r\leqslant\theta\leqslant 0} |\phi(\theta)| < \infty \qquad \square$$

Remark

Lemma 2 can also be deduced from some theorems concerning the rate of decreasing of the coefficients in Fourier series expansion of ϕ—see Zygmund (1959, Theorem 4.12, p. 48) or Hardy and Rogosinski (1956, Theorem 37), since by the Phragmén–Lindelöf principle (Markushevich 1983, p. 278), $\bar{\Pi}_+$ can be replaced by the imaginary axis.

3. Reduction to the Riemann–Hilbert problem

Riemann–Hilbert problem

Find a function $y:\mathbb{C}\to\mathbb{C}$ such that

(RH 1) y is analytic and single-valued in an open set containing $\bar{\Pi}_+$ except for a finite number of simple poles,

(RH 2) $\overline{y(i\omega)} = y(-i\omega) \quad \forall\, \omega\in\mathbb{R}, \quad i\omega$ is not a pole of y,

(RH 3) $\hat{x}^{\mathrm{T}}(-i\omega)Q\hat{x}(i\omega) = 2\mathrm{Re}\, y(i\omega)$ for almost every $\omega\in\mathbb{R}$,

(RH 4) there exists a finite limit

$$\lim_{R\to\infty} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} y(R\exp(i\beta))\exp(i\beta)\,R\,d\beta$$

In the standard R–H problem with the contour being the imaginary axis, the poles of y are excluded and we require a different type of asymptotic behaviour of y for large $|s|$. In the next section we solve the above R–H problem without using Plemelj's formulae and the Hilbert transform—see Noble (1958). The following lemma allows us to reduce the problem of evaluation of integral (8) to the solution of the above R–H problem.

Lemma 3

Suppose that y is a solution to the R–H problem formulated above. Then,

$$J = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \hat{x}^{\mathrm{T}}(-s)Q\hat{x}(s)\,ds$$

$$= -\sum_{k} \operatorname*{Res}_{\substack{s=i\omega_k,\\ \omega_k\in\mathbb{R}}} y(s) - 2\sum_{s_l\in\Pi_+} \operatorname*{Res}_{s=s_l} y(s) + \frac{1}{\pi}\lim_{R\to\infty}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} y(R\exp(i\beta))\exp(i\beta)\,R\,d\beta \qquad (11)$$

Proof

By (RH 2) the poles of y are located symmetrically on the imaginary axis and Re $y(i\omega)$, Im $y(i\omega)$ are even and odd functions of ω, respectively. Thus by (8) and (RH 3)

$$J = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \hat{x}^{\mathrm{T}}(-s)Q\hat{x}(s)\,ds = \frac{1}{\pi i}\,\text{V.p.}\int_{-i\infty}^{+i\infty} y(s)\,ds$$

where V.p. denotes the principal value of the integral. Now we notice that for the contour depicted in Fig. 1 we have

$$\begin{aligned} J &= \frac{1}{\pi i}\,\text{V.p.}\int_{-i\infty}^{+i\infty} y(s)\,ds \\ &= \frac{1}{\pi i}\lim_{\substack{\rho\searrow 0\\ R\nearrow\infty}}\left[\int_{C_{R,\rho}} y(s)\,ds - \sum_k \int_{C_{R,\rho}\cap\partial K(i\omega_k,\rho)} y(s)\,ds - \int_{C_{R,\rho}\cap\partial K(0,R)} y(s)\,ds\right]\end{aligned}$$

where $K(s_0, R_0)$ is a circle centred at $s_0 \in \mathbb{C}$ with radius R_0. By (RH 4) and the Residue Theorem

$$\begin{aligned} J = \frac{1}{\pi i}\Bigg\{ &-2\pi i \sum_{s_l\in\Pi_+}\operatorname*{Res}_{s=s_l} y(s) \\ &- \sum_k \lim_{\rho\searrow 0}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} y(i\omega_k + \rho\exp(i\alpha))\rho i\exp(i\alpha)\,d\alpha \\ &- \lim_{R\nearrow\infty}\int_{\frac{1}{2}\pi}^{-\frac{1}{2}\pi} y(R\exp(i\beta))R\exp(i\beta)i\,d\beta\Bigg\}\end{aligned}$$

By (RH 1), y can be represented in the neighbourhood of $s = i\omega_k$ in the form

$$y(s) = \frac{1}{s - i\omega_k}\operatorname*{Res}_{s=i\omega_k} y(s) + \Delta_k(s)$$

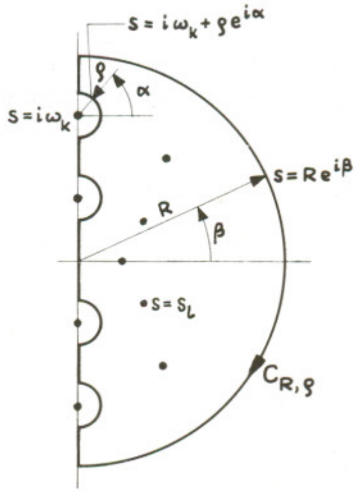

Figure 1. Contour $C_{R,\rho}$ used in the proof of (11).

where Δ_k is an analytic function. Hence,

$$\lim_{\rho\searrow 0}\left[\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} y(i\omega_k+\rho\exp(i\alpha))\,i\rho\exp(i\alpha)\,d\alpha\right]$$

$$=\lim_{\rho\searrow 0}\left[\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{i\rho\exp(i\alpha)}{\rho\exp(i\alpha)}\operatorname*{Res}_{s=i\omega_k} y(s)\,d\alpha+\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\Delta_k(i\omega_k+\rho\exp(i\alpha))\,i\rho\exp(i\alpha)\,d\alpha\right]$$

$$=i\pi\operatorname*{Res}_{s=i\omega_k} y(s)$$

and finally (11) holds. □

4. Explicit solution to the R–H problem

It is well known—see Lancaster (1969) that the operation 'col',

$$Q=\begin{bmatrix} q_1^{\mathrm{T}}\\ q_2^{\mathrm{T}}\\ \vdots\\ q_n^{\mathrm{T}}\end{bmatrix}\Rightarrow \operatorname{col} Q=\begin{bmatrix} q_1\\ q_2\\ \vdots\\ q_n\end{bmatrix}$$

and the tensor (Kronecker) product of matrices

$$U=[u_{kl}]_{\substack{k=1,\dots,p\\ l=1,\dots,q}}\Rightarrow U\otimes W=[u_{kl}\quad W]_{\substack{k=1,\dots,p\\ l=1,\dots,q}}$$

are connected by the relationship

$$\operatorname{col}(U\quad Z\quad W)=(U\otimes W^{\mathrm{T}})\operatorname{col} Z \tag{12}$$

This enables us to represent the integrand of integral (8) in the form

$$\hat{x}^{\mathrm{T}}(-s)Q\hat{x}(s)=\operatorname{col}^{\mathrm{T}} Q[X^{-1}(-s)\otimes X^{-1}(s)][p(-s)\otimes p(s)] \tag{13}$$

Lemma 4

The following identity holds:

$$X^{-1}(-s)\otimes X^{-1}(s)=S^{-1}(s)[-I\otimes I-(\exp(-sr)I)\otimes(B+sC)X^{-1}(s)$$
$$-(B-sC)X^{-1}(-s)\otimes(\exp(sr)I)]$$
$$\forall\, s\in\mathbb{C},\ \det S(s)\neq 0 \tag{14}$$

where

$$S(s)=(sI+A)\otimes(sI-A)+(B-sC)\otimes(B+sC) \tag{15}$$

Proof

Directly from (4)

$$X(-s)\otimes X(s)+X(-s)\otimes(-sI+A)+(sI+A)\otimes X(s)=S(s)$$

Premultiplying this by $S^{-1}(s)$ and postmultiplying by $X^{-1}(-s)\otimes X^{-1}(s)$ we get the identity

$$X^{-1}(-s)\otimes X^{-1}(s)=S^{-1}(s)[I\otimes I+I\otimes(-sI+A)X^{-1}(s)+(sI+A)X^{-1}(-s)\otimes I]$$

Hence (14) easily follows, since from (4)

$$-sI + A = -X(s) - \exp(-sr)(B + sC) \qquad \square$$

Taking (14) and (10 *a*, *b*) in (13) into account we have

$$\begin{aligned}\hat{x}^{\mathrm{T}}(-s)Q\hat{x}(s) &= -\mathrm{col}^{\mathrm{T}}\, QS^{-1}(s)[I \otimes I + (\exp(-sr)I) \otimes (B + sC)X^{-1}(s) \\ &\qquad + (B - sC)X^{-1}(-s) \otimes (\exp(sr)I)][p(-s) \otimes p(s)] \\ &= -\mathrm{col}^{\mathrm{T}}\, QS^{-1}(s)\{p(-s) \otimes p(s) + [\exp(-sr)p(-s)] \\ &\qquad \otimes [(B + sC)\hat{x}(s)] + [(B - sC)\hat{x}(-s)] \\ &\qquad \otimes [\exp(sr)p(s)]\}\end{aligned} \tag{16}$$

$p(-s) \otimes p(s)$ may be expressed as

$$\begin{aligned}p(-s) \otimes p(s) &= v_0 \otimes v_0 + [(B - sC)\exp(sr)\phi_F(-s)] \otimes v_0 \\ &\quad + v_0 \otimes [(B + sC)\exp(-sr)\phi_F(s)] \\ &\quad + [(B - sC) \otimes (B + sC)][\phi_F(-s) \otimes \phi_F(s)]\end{aligned} \tag{17}$$

$\phi_F(-s) \otimes \phi_F(s)$ may be written, in turn, in the form

$$\begin{aligned}\phi_F(-s) \otimes \phi_F(s) &= \int_{-r}^{0}\int_{-r}^{0} \exp(s(\theta - \xi))[\phi(\theta) \otimes \phi(\xi)]\, d\theta\, d\xi \\ &= \int_{-r}^{0}\int_{\theta}^{0} \exp(s(\theta - \xi))[\phi(\theta) \otimes \phi(\xi)]\, d\theta\, d\xi \\ &\quad + \int_{-r}^{0}\int_{-r}^{\theta} \exp(s(\theta - \xi))[\phi(\theta) \otimes \phi(\xi)]\, d\theta\, d\xi\end{aligned} \tag{18}$$

Lemma 5

If $\phi \in \mathbf{BV}([-r, 0], \mathbb{R}^n)$ then the entire function

$$s \mapsto -s^2 \int_{-r}^{0}\int_{\theta}^{0} \exp(s(\theta - \xi))[\phi(\theta) \otimes \phi(\xi)]\, d\theta\, d\xi + s\int_{-r}^{0} \phi(\theta) \otimes \phi(\theta)\, d\theta$$

is bounded on $\bar{\Pi}_+$.

Proof

Arguing as in the proof of Lemma 2 we establish the existence of the integral

$$\int_{\theta}^{0} d[\exp(-s\xi)]\phi(\xi) = \phi(0) - \exp(-s\theta)\phi(\theta) - \int_{\theta}^{0} \exp(-s\xi)\, d\phi(\theta)$$

Hence,

$$\begin{aligned}&-s^2 \int_{-r}^{0}\int_{\theta}^{0} \exp(s(\theta - \xi))\phi(\theta) \otimes \phi(\xi)\, d\theta\, d\xi + s\int_{-r}^{0} \phi(\theta) \otimes \phi(\theta)\, d\theta \\ &\quad = \int_{-r}^{0} d[\exp(s\theta)]\phi(\theta) \otimes \int_{\theta}^{0} d[\exp(-s\xi)]\phi(\xi) + s\int_{-r}^{0} \phi(\theta) \otimes \phi(\theta)\, d\theta \\ &\quad = \int_{-r}^{0} d[\exp(s\theta)]\phi(\theta) \otimes \left[\phi(0) - \int_{\theta}^{0} \exp(-s\xi)\, d\phi(\xi)\right]\end{aligned}$$

If $\phi \in \mathrm{BV}([-r,0],\mathbb{R}^n)$ then also $\left(\theta \mapsto \int_\theta^0 \exp(-s\xi)\, d\phi(\xi)\right) \in \mathrm{BV}([-r,0],\mathbb{R}^n)$ and consequently

$$\left(\theta \mapsto \phi(\theta) \otimes \left[-\int_\theta^0 \exp(-s\xi)\, d\phi(\xi) + \phi(0)\right]\right) \in \mathrm{BV}([-r,0],\mathbb{R}^{n^2})$$

Thus, the investigated expression is equal to

$$-\exp(-sr)\phi(-r) \otimes \left[\phi(0) - \int_{-r}^0 \exp(-s\theta)\, d\phi(\theta)\right] - \int_{-r}^0 \exp(s\theta) \times d\left\{\phi(\theta) \otimes \left[\phi(0) - \int_\theta^0 \exp(-s\xi)\, d\phi(\xi)\right]\right\} + \phi(0) \otimes \phi(0)$$

All components of this sum are bounded on $\bar{\Pi}_+$ if $\phi \in \mathrm{BV}([-r,0],\mathbb{R}^n)$. □

Formulae (16), (17), (18) suggest that the function y, defined as follows:

$$\begin{aligned} y(s) = -\mathrm{col}^{\mathrm{T}}\, QS^{-1}(s)\Big\{\frac{1}{2} v_0 \otimes v_0 + v_0 \otimes [(B+sC)\exp(-sr)\phi_F(s)] \\ + [(B-sC)\otimes(B+sC)] \int_{-r}^0 \int_\theta^0 \exp(s(\theta-\xi)) \\ \times [\phi(\theta)\otimes\phi(\xi)]\, d\theta\, d\xi \\ + [\exp(-sr)p(-s)] \otimes [(B+sC)\otimes \hat{x}(s)]\Big\} \end{aligned} \tag{19}$$

is a candidate for a solution to the posed R–H problem. Directly from (19) we find that y is analytic and single-valued on an open set containing $\bar{\Pi}_+$ except for a finite number of poles. Every pole of y in $\bar{\Pi}_+$ is a pole of $S^{-1}(\cdot)$, but not conversely, since some factors of the polynomial det $S(s)$ may coincide with the factors into which the numerator of the meromorphic function y can be decomposed. To satisfy (RH 1) we assume that the poles of y are simple.

By elementary considerations, we find directly from (19) that $\overline{y(-\bar{s})} = y(-s)$ $\forall\, s \in \mathbb{C}$, s is not a pole of y. But this implies (RH 2). (RH 3) follows from the following result.

Lemma 6

$$\hat{x}^{\mathrm{T}}(-s)Q\hat{x}(s) = y(s) + y(-s) \quad \forall\, s \in \mathbb{C},\ s \text{ is not a pole of } y. \tag{20}$$

Proof

By virtue of (16), (17), (18), (19) it suffices to prove that the functions

$$\begin{aligned} y_1(s) = -\mathrm{col}^{\mathrm{T}}\, QS^{-1}(s) \int_{-r}^0 \int_\theta^0 \exp(s(\theta-\xi)) \\ \times \{[(B-sC)\phi(\theta)] \otimes [(B+sC)\phi(\xi)]\}\, d\theta\, d\xi \end{aligned} \tag{21}$$

$$y_2(s) = -\mathrm{col}^{\mathrm{T}}\, QS^{-1}(s)\{v_0 \otimes [(B+sC)\exp(-sr)\phi_F(s)]\} \tag{22}$$

$$y_3(s) = -\mathrm{col}^{\mathrm{T}}\, QS^{-1}(s)\{[\exp(-sr)p(-s)] \otimes [(B+sC)\hat{x}(s)]\} \tag{23}$$

satisfy the following relationships:

$$y_1(-s) = -\operatorname{col}^{\mathrm{T}} QS^{-1}(s) \int_{-r}^{0} \int_{-r}^{\theta} \exp(s(\theta-\xi)) \times \{[(B-sC)\phi(\theta)] \otimes [(B+sC)\phi(\xi)]\}\, d\theta\, d\xi \tag{21'}$$

$$y_2(-s) = -\operatorname{col}^{\mathrm{T}} QS^{-1}(s)\{[(B-sC)\exp(sr)\phi_F(-s)] \otimes v_0\} \tag{22'}$$

$$y_3(-s) = -\operatorname{col}^{\mathrm{T}} QS^{-1}(s)\{[(B-sC)\hat{x}(-s)] \otimes [\exp(sr)p(s)]\} \tag{23'}$$

First, before proving (21′), (22′), (23′) we show that the matrix $R(s)$ defined by

$$\operatorname{col}^{\mathrm{T}} R(s) = -\operatorname{col}^{\mathrm{T}} QS^{-1}(s) \tag{24}$$

satisfies the identity

$$R^{\mathrm{T}}(-s) = R(s) \tag{25}$$

Consider the matrix equation

$$(sI + A^{\mathrm{T}})R(s)(sI - A) + (B^{\mathrm{T}} - sC^{\mathrm{T}})R(s)(B + sC) = -Q \tag{26}$$

By virtue of (12), (26) is equivalent to the equation

$$-\operatorname{col} Q = S^{\mathrm{T}}(s) \operatorname{col} R(s)$$

and therefore

$$\operatorname{col}^{\mathrm{T}} R(s) = -\operatorname{col}^{\mathrm{T}} QS^{-1}(s) \tag{27}$$

determines the unique solution to (26). On the other hand, transposing (26) and substituting s by $-s$ we find that if $R(s)$ is a solution of (26) then so is $R^{\mathrm{T}}(-s)$. Hence (25) follows from the uniqueness.

To prove (21′) observe that by (21), (27), (12), (25) and the Fubini–Tonelli theorem we have

$$\begin{aligned} y_1(-s) &= \operatorname{col}^{\mathrm{T}} R(-s) \int_{-r}^{0} \int_{\theta}^{0} \exp(-s(\theta-\xi)) \\ &\quad \times [(B+sC)\phi(\theta) \otimes (B-sC)\phi(\xi)]\, d\theta\, d\xi \\ &= \int_{-r}^{0} \int_{\theta}^{0} \exp(-s(\theta-\xi))\phi^{\mathrm{T}}(\theta)(B^{\mathrm{T}}+sC^{\mathrm{T}})R(-s)(B-sC)\phi(\xi)\, d\theta\, d\xi \\ &= \int_{-r}^{0} \int_{-r}^{\xi} \exp(-s(\theta-\xi))\phi^{\mathrm{T}}(\xi)(B^{\mathrm{T}}-sC^{\mathrm{T}})R(s)(B+sC)\phi(\theta)\, d\xi\, d\theta \end{aligned}$$

The change of variables $(\theta, \xi) \mapsto (\xi, \theta)$ and application of (27), (12) gives

$$\begin{aligned} y_1(-s) &= \int_{-r}^{0} \int_{-r}^{\theta} \exp(s(\theta-\xi))\phi^{\mathrm{T}}(\theta)(B^{\mathrm{T}}-sC^{\mathrm{T}})R(s)(B+sC)\phi(\xi)\, d\theta\, d\xi \\ &= \operatorname{col}^{\mathrm{T}} R(s) \int_{-r}^{0} \int_{-r}^{\theta} \exp(s(\theta-\xi)) \\ &\quad \times \{[(B-sC)\phi(\theta)] \otimes [(B+sC)\phi(\xi)]\}\, d\theta\, d\xi \\ &= -\operatorname{col}^{\mathrm{T}} QS^{-1}(s) \int_{-r}^{0} \int_{-r}^{\theta} \exp(s(\theta-\xi)) \\ &\quad \times \{[(B-sC)\phi(\theta)] \otimes [(B+sC)\phi(\xi)]\}\, d\theta\, d\xi \end{aligned}$$

and so (21′) holds.

The proof of (22′) is simpler—we get from (22), (27), (12), (25)

$$\begin{aligned} y_2(-s) &= \operatorname{col}^{\mathrm{T}} R(-s)[v_0 \otimes (B - sC)\exp(sr)\phi_F(-s)] \\ &= v_0^{\mathrm{T}} R(-s)(B - sC)\exp(sr)\phi_F(-s) \\ &= \phi_F^{\mathrm{T}}(-s)\exp(sr)(B^{\mathrm{T}} - sC^{\mathrm{T}})R^{\mathrm{T}}(-s)v_0 \\ &= \phi_F^{\mathrm{T}}(-s)\exp(sr)(B^{\mathrm{T}} - sC^{\mathrm{T}})R(s)v_0 \\ &= \operatorname{col}^{\mathrm{T}} R(s)[(B - sC)\exp(sr)\phi_F(-s) \otimes v_0] \\ &= -\operatorname{col}^{\mathrm{T}} QS^{-1}(s)[(B - sC)\exp(sr)\phi_F(-s) \otimes v_0] \end{aligned}$$

To prove (23′) we proceed similarly. Indeed, (23) yields by virtue of (25), (12), (27)

$$\begin{aligned} y_3(-s) &= \operatorname{col}^{\mathrm{T}} R(-s)[\exp(sr)p(s) \otimes (B - sC)\hat{x}(s)] \\ &= p^{\mathrm{T}}(s)\exp(sr)R(-s)(B - sC)\hat{x}(-s) \\ &= \hat{x}^{\mathrm{T}}(-s)(B^{\mathrm{T}} - sC^{\mathrm{T}})R(s)\exp(sr)p(s) \\ &= -\operatorname{col}^{\mathrm{T}} QS^{-1}(s)[(B - sC)\hat{x}(-s) \otimes \exp(sr)p(s)] \end{aligned}$$ □

To verify whether (RH 4) holds we consider two cases:

(i) $C = 0$

In this case all expressions inside the brackets { } in (19) are bounded on $\bar{\Pi}_+$. Since for large $|s|$, $S^{-1}(s)$ behaves as $s^{-2}(I \otimes I - C \otimes C)^{-1}$ then by elementary estimates we get

$$\lim_{R\to\infty} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} y(R\exp(i\beta))R\exp(i\beta)\, d\beta = 0$$

(ii) $C \neq 0$, $\phi \in \mathrm{BV}([-r, 0], \mathbb{R}^n)$

We rewrite (19) in a slightly modified form:

$$\begin{aligned} y(s) = &-\operatorname{col}^{\mathrm{T}} QS^{-1}(s)\left[\{\ \} + s\int_{-r}^{0} C\phi(\theta) \otimes C\phi(\theta)\, d\theta\right] \\ &+ \operatorname{col}^{\mathrm{T}} QS^{-1}(s)s(C \otimes C)\int_{-r}^{0} \phi(\theta) \otimes \phi(\theta)\, d\theta \end{aligned}$$

By Lemmas 2 and 5 the expression in the square brackets in the first component is bounded on $\bar{\Pi}_+$ and arguing as in (i) one obtains with the aid of (12)

$$\begin{aligned} &\frac{1}{\pi}\lim_{R\to\infty}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} y(R\exp(i\beta))R\exp(i\beta)\, d\beta \\ &\qquad = \operatorname{col}^{\mathrm{T}} Q(I \otimes I - C \otimes C)^{-1}(C \otimes C)\int_{-r}^{0} \phi(\theta) \otimes \phi(\theta)\, d\theta \\ &\qquad = \int_{-r}^{0} \phi^{\mathrm{T}}(\theta)\gamma\phi(\theta)\, d\theta \end{aligned} \tag{28}$$

where $\gamma \in \mathscr{L}(\mathbb{R}^n)$, $\gamma = \gamma^{\mathrm{T}} \geqslant 0$ is the unique solution of a discrete Lyapunov equation

$$C^{\mathrm{T}}\gamma C - \gamma = -C^{\mathrm{T}}QC \tag{29}$$

We have thus proved that under very mild restrictions, (19) provides a solution to the posed R–H problem.

On the basis of Lemma 3 we thus can formulate our main result.

Theorem 1

Suppose that all poles of y defined by (19) are simple. Let also $\phi \in \mathrm{BV}([-r, 0], \mathbb{R}^n)$ if $C \neq 0$. Then the integral (8) is given by formulae (11), (19), (28) and (29).

5. Discussion of results

(i) The method used in the proof of Lemma 6 permits us to write y in the form

$$
\begin{aligned}
y(s) &= -\frac{1}{2}\operatorname{col}^{\mathrm{T}} QS^{-1}(s)(v_0 \otimes v_0) + y_2(s) + y_1(s) + y_3(s) \\
&= \frac{1}{2} v_0^{\mathrm{T}} R(s) v_0 + v_0^{\mathrm{T}} R(s)(B + sC) \exp(-sr)\, \phi_F(s) \\
&\quad + \int_{-r}^{0} \int_{\theta}^{0} \exp(s(\theta - \xi))\, \phi^{\mathrm{T}}(\theta)(B^{\mathrm{T}} - sC^{\mathrm{T}}) R(s)(B + sC)\phi(\xi)\, d\theta\, d\xi \\
&\quad + \exp(-sr)\, p^{\mathrm{T}}(-s) R(s)(B + sC)\hat{x}(s)
\end{aligned}
$$

Hence, by (10) we obtain

$$
\begin{aligned}
y(s) = {} & v_0^{\mathrm{T}}\left[\frac{1}{2} R(s) + R(s) \exp(-sr)(B + sC) X^{-1}(s)\right] v_0 + \int_{-r}^{0} v_0^{\mathrm{T}} \\
& \times [R(s)(B + sC) \exp(-s(r + \theta)) \\
& \quad + \exp(-sr) R(s)(B + sC) X^{-1}(s)(B + sC) \exp(-s(r + \theta))]\phi(\theta)\, d\theta \\
& + \int_{-r}^{0} \phi^{\mathrm{T}}(\theta) \exp(s\theta)(B^{\mathrm{T}} - sC^{\mathrm{T}}) R(s)(B + sC) X^{-1}(s) v_0\, d\theta \\
& + \int_{-r}^{0} \int_{\theta}^{0} \exp(s(\theta - \xi))\, \phi^{\mathrm{T}}(\theta)(B^{\mathrm{T}} - sC^{\mathrm{T}}) R(s)(B + sC)\phi(\xi)\, d\theta\, d\xi \\
& + \int_{-r}^{0} \int_{-r}^{0} \exp(s(\theta - r - \xi))\, \phi^{\mathrm{T}}(\theta)(B^{\mathrm{T}} - sC^{\mathrm{T}}) R(s)(B + sC) X^{-1}(s) \\
& \times (B + sC)\phi(\xi)\, d\theta\, d\xi \qquad (30)
\end{aligned}
$$

Let us introduce the following notation:

$$
\begin{aligned}
W_1 = {} & -\sum_{k}\left[\operatorname*{Res}_{s = i\omega_k} R(s)\right]\left[\frac{1}{2} I + \exp(-i\omega_k r)(B + i\omega_k C) X^{-1}(i\omega_k)\right] \\
& - 2 \sum_{s_l \in \Pi_+}\left[\operatorname*{Res}_{s = s_l} R(s)\right]\left[\frac{1}{2} I + \exp(-s_l r)(B + s_l C) X^{-1}(s_l)\right] \qquad (31\,a)
\end{aligned}
$$

$$
\begin{aligned}
W_2(\theta) = {} & -\sum_{k}\left[\operatorname*{Res}_{s = i\omega_k} R(s)\right] \exp(i\omega_k(r + \theta)) \\
& \times [I + \exp(-i\omega_k r)(B + i\omega_k C) X^{-1}(i\omega_k)](B + i\omega_k C)
\end{aligned}
$$

$$-2\sum_{s_l\in\Pi_+}\left[\operatorname*{Res}_{s=s_l} R(s)\right]\exp(-s_l(r+\theta))$$
$$\times[I+\exp(-s_l r)(B+s_l C)X^{-1}(s_l)](B+s_l C) \tag{31 b}$$

$$W_3(\theta)=-\sum_k(B^{\mathrm T}-i\omega_k C^{\mathrm T})\left[\operatorname*{Res}_{s=i\omega_k} R(s)\right]\exp(i\omega_k\theta)(B+i\omega_k C)X^{-1}(i\omega_k)$$
$$-2\sum_{s_l\in\Pi_+}(B^{\mathrm T}-s_l C^{\mathrm T})\left[\operatorname*{Res}_{s=s_l} R(s)\right]\exp(s_l\theta)(B+s_l C)X^{-1}(s_l) \tag{31 c}$$

$$W_4(\theta,\xi)=-\sum_k(B^{\mathrm T}-i\omega_k C^{\mathrm T})\left[\operatorname*{Res}_{s=i\omega_k} R(s)\right](B+i\omega_k C)\exp(i\omega_k(\theta-\xi))$$
$$-2\sum_{s_l\in\Pi_+}(B^{\mathrm T}-s_l C^{\mathrm T})\exp(s_l(\theta-\xi))\left[\operatorname*{Res}_{s=s_l} R(s)\right](B+s_l C) \tag{31 d}$$

$$W_5(\theta,\xi)=-\sum_k(B^{\mathrm T}-i\omega_k C^{\mathrm T})\left[\operatorname*{Res}_{s=i\omega_k} R(s)\right](B+i\omega_k C)X^{-1}(i\omega_k)$$
$$\times(B+i\omega_k C)\exp(i\omega_k(\theta-r-\xi))$$
$$-2\sum_{s_l\in\Pi_+}\exp(s_l(\theta-r-\xi))(B^{\mathrm T}-s_l C^{\mathrm T})\left[\operatorname*{Res}_{s=s_l} R(s)\right]$$
$$\times(B+s_l C)X^{-1}(s_l)(B+s_l C) \tag{31 e}$$

By Theorem 1 and Lebesgue's theorem on the limit passage under the integral sign we may introduce the residues under the integral signs which leads to the following form of the performance index (5):

$$J(v_0,\phi)=v_0^{\mathrm T}W_1v_0+\int_{-r}^{0}v_0^{\mathrm T}W_2(\theta)\phi(\theta)\,d\theta+\int_{-r}^{0}\phi^{\mathrm T}(\theta)W_3(\theta)v_0\,d\theta$$
$$+\int_{-r}^{0}\int_{\theta}^{0}\phi^{\mathrm T}(\theta)W_4(\theta,\xi)\phi(\xi)\,d\theta\,d\xi$$
$$+\int_{-r}^{0}\int_{-r}^{0}\phi^{\mathrm T}(\theta)W_5(\theta,\xi)\phi(\xi)\,d\theta\,d\xi+\int_{-r}^{0}\phi^{\mathrm T}(\theta)\gamma\phi(\theta)\,d\theta$$
$$\text{for every }(v_0,\phi)\in\mathbb{R}^n\times \mathrm{BV}([-r,0],\mathbb{R}^n) \tag{32}$$

Both J and the functional in the right-hand side of (32) are strongly continuous on M^2. On the other hand the set $\mathbb{R}^n\times\mathrm{BV}([-r,0],\mathbb{R}^n)$ is dense in M^2 (since it contains the domain of infinitesimal generator of the semigroup $\{S(t)\}_{t\geqslant 0}$, which is known to be dense in M^2—see Grabowski (1983)). Hence the standard density and continuity arguments yield the following result.

Theorem 2

Under the assumptions that all poles of y defined by (19) are simple, (31) and (32) express the value of the integral (5) for arbitrary initial conditions $(v_0,\phi)\in M^2$.

This result will be illustrated by an example of a one-dimensional delay-differential system, i.e. in the case where $A, B, C\in\mathbb{R}$, $C=0$, $Q=1$, $v_0=x_0\in\mathbb{R}$. Using the Lyapunov approach (see Grabowski 1983 for details) we get

$$J(x_0, \phi) = \alpha x_0^2 + \int_{-r}^{0} x_0 \beta(\theta)\phi(\theta)\, d\theta + \int_{-r}^{0}\int_{\theta}^{0} \phi(\theta)\delta(\theta, \xi)\phi(\xi)\, d\theta\, d\xi$$

where

$$\alpha = \frac{B \sin Kr - K}{2K(A + B\cos Kr)}, \quad K^2 = B^2 - A^2$$

$$\beta(\theta) = \frac{-B[B\sin K\theta + K\cos K(r+\theta) + A\sin K(r+\theta)]}{K(A + B\cos Kr)}$$

$$\delta(\theta, \xi) = \frac{B^2[B\sin K(\theta - \xi + r) + A\sin K(\theta - \xi) - K\cos K(\theta - \xi)]}{K(A + B\cos Kr)}$$

On the other hand, after taking into account that

$$X^{-1}(s) = \frac{1}{s - A - \exp(-sr)B}, \quad -R(s) = S^{-1}(s) = \frac{1}{s^2 + K^2}$$

$$\operatorname*{Res}_{s=iK} R(s) = \frac{i}{2K}, \quad \operatorname*{Res}_{s=-iK} R(s) = -\frac{i}{2K}$$

we obtain with the aid of formulae (31), (32)

$$\alpha = \frac{A\sin Kr - K\cos Kr}{2K(B + A\cos Kr + K\sin Kr)} = W_1$$

$$\beta(\theta) = W_2(\theta) + W_3(\theta)$$

$$= \frac{-B[B\sin K(r+\theta) + K\cos K\theta + A\sin K\theta]}{K(B + A\cos Kr + K\sin Kr)}$$

$$\delta(\theta, \xi) = W_4(\theta, \xi) + W_5(\theta, \xi) + W_5(\xi, \theta)$$

$$= \frac{B^2[B\sin K(\theta - \xi) + A\sin K(\theta - \xi + r) - K\cos K(\theta - \xi + r)]}{K(B + A\cos Kr + K\sin Kr)}$$

It is not difficult to establish the equivalence of the above results.

(ii) A modification of the method presented above may be successfully applied to the non-homogeneous neutral-differential system

$$\left.\begin{aligned} &\dot{v}(t) = Av(t) + (AC + B)x(t - r) + f(t), \quad t \geqslant 0 \\ &v(t) = x(t) - Cx(t - r), \qquad t \geqslant 0 \\ &v(0) = v_0 \\ &x(\theta) = \phi(\theta) \quad \text{a.e. on } [-r, 0] \end{aligned}\right\} \tag{33}$$

where A, B, C, v_0, ϕ are as in (1), while $f \in L^2(0, \infty; \mathbb{R}^n) \cap L^1(0, \infty; \mathbb{R}^n)$ and is such that $\hat{f}$, the Laplace transform of f, is analytic in the neighbourhood of the poles of $S^{-1}(\cdot)$. For system (33) the formulae (10) modify in such a way that now

$$p(s) = v_0 + \exp(-sr)(B + sC)\phi_F(s) + \hat{f}(s) \tag{34}$$

This modification implies in turn that

$$\hat{x}(s) = \hat{x}_H(s) + X^{-1}(s)\hat{f}(s)$$

where $\hat{x}$ is the Laplace transform of the solution x of (33) and $\hat{x}_H$ is the Laplace transform of the solution x_H of (1). Since under (2), the fundamental matrix of (1) decays exponentially as $t \to \infty$ and $f \in L^1 \cap L^2$, then by elementary properties of the convolution (Desoer and Vidyasagar 1975), their convolution belongs to $L^1 \cap L^2 \cap L^\infty$, is uniformly continuous, tends to zero as $t \to \infty$ and has the Laplace transform which belongs to $H^2(\Pi_+, \mathbb{C}^n)$, is continuous and bounded on $\bar{\Pi}_+$ and analytic in the neighbourhood of poles of $S^{-1}(\cdot)$. Hence, all properties of $\hat{x}$, which are essential for the solution of R–H problem in the homogeneous case, will be preserved also for system (33).

Now we replace p by (34) in (16) and moreover,

$$\begin{aligned} p(-s) \otimes p(s) = {} & [v_0 + \exp(sr)(B - sC)\phi_F(-s)] \\ & \underset{①}{\otimes} [v_0 + \exp(-sr)(B + sC)\phi_F(s)] + v_0 \underset{②}{\otimes} \hat{f}(s) \\ & + \exp(sr)(B - sC)\phi_F(-s) \underset{③}{\otimes} \hat{f}(s) + \hat{f}(-s) \underset{④}{\otimes} \hat{f}(s) \\ & + \hat{f}(-s) \underset{⑤}{\otimes} v_0 + \hat{f}(-s) \underset{⑥}{\otimes} [\exp(sr)(B + sC)\phi_F(s)] \end{aligned} \tag{35}$$

The first component in (35) resolves into the sum $y(s) + y(-s)$, where y is determined by (19). The second and fifth terms, we leave unchanged. The fourth component has the following resolution:

$$\begin{aligned} \hat{f}(-s) \otimes \hat{f}(s) &= \int_0^\infty \int_0^\infty \exp(s(\tau - t)) f(\tau) \otimes f(t)\, d\tau\, dt \\ &= \int_0^\infty \int_0^t \exp(s(\tau - t)) f(\tau) \otimes f(t)\, d\tau\, dt \\ &\quad + \int_0^\infty \int_t^\infty \exp(s(\tau - t)) f(\tau) \otimes f(t)\, d\tau\, dt \end{aligned} \tag{36}$$

If we now introduce the notation

$$\begin{aligned} y_4(s) &= -\operatorname{col}^{\mathrm{T}} QS^{-1}(s) \int_0^\infty \int_0^t \exp(s(\tau - t)) f(\tau) \otimes f(t)\, d\tau\, dt \\ &= \int_0^\infty \int_0^t \exp(s(\tau - t)) f^{\mathrm{T}}(\tau) R(s) f(t)\, d\tau\, dt \end{aligned} \tag{37}$$

then by (12), (24), (25) and the Fubini–Tonelli theorem

$$y_4(-s) = -\operatorname{col}^{\mathrm{T}} QS^{-1}(s) \int_0^\infty \int_t^\infty \exp(s(\tau - t)) f(\tau) \otimes f(t)\, d\tau\, dt$$

The sum of the third and sixth terms in (35) can be represented as

$$\begin{aligned} & \exp(sr)(B - sC)\phi_F(-s) \otimes \hat{f}(s) + \hat{f}(-s) \otimes \exp(-sr)(B + sC)\phi_F(s) \\ &\quad = (B - sC)\phi_F(-s) \otimes \exp(sr)\hat{f}(s) + \exp(-sr)\hat{f}(-s) \otimes (B + sC)\phi_F(s) \\ &\quad = (B - sC)\phi_F(-s) \otimes \int_0^\infty \exp(-s(t - r)) f(t)\, dt \end{aligned}$$

$$+\int_0^{\infty} \exp(s(\tau-r))f(\tau)\,d\tau \otimes (B+sC)\phi_F(s)$$

$$=(B-sC)\phi_F(-s)\otimes \hat{f}_r(s)+\hat{f}_r(-s)\otimes (B+sC)\phi_F(s)$$

$$+\int_{-r}^{0}\int_{-r}^{0}(B-sC)\exp(s(\theta-\xi))\phi(\theta)\otimes f(r+\xi)\,d\theta\,d\xi$$

$$+\int_{-r}^{0}\int_{-r}^{0}\exp(s(\xi-\theta))f(r+\xi)\otimes (B+sC)\phi(\theta)\,d\xi\,d\theta \tag{38}$$

where $\hat{f}_r$ is the Laplace transform of f_r, $f_r(t)=f(t+r)$, $t\geqslant 0$. Define y_5 as

$$y_5(s)=-\mathrm{col}^{\mathrm{T}}\, QS^{-1}(s)[(B-sC)\phi_F(-s)\otimes \hat{f}_r(s)]$$

$$+\int_{-r}^{0}\int_{\theta}^{0}(B-sC)\exp(s(\theta-\xi))\phi(\theta)\otimes f(r+\xi)\,d\theta\,d\xi$$

$$+\int_{-r}^{0}\int_{-r}^{\theta}\exp(s(\xi-\theta))f(r+\xi)\otimes (B+sC)\phi(\theta)\,d\xi\,d\theta$$

Notice that the part of (38) which is neglected in the definition of y_5, multiplied from the left on $-\mathrm{col}^{\mathrm{T}}\, QS^{-1}(s)$, is equal to $y_5(-s)$.

Since y_4, y_5, $s\mapsto -\mathrm{col}^{\mathrm{T}}\, QS^{-1}(s)[v_0\otimes \hat{f}(s)]$ are bounded on $\bar{\Pi}_+$, provided that $\phi\in \mathrm{BV}([-r,0],\mathbb{R}^n)$ and they have the properties required in Lemma 3, therefore adding

$$y_4(s)+y_5(s)-\mathrm{col}^{\mathrm{T}}\, QS^{-1}(s)[v_0\otimes \hat{f}(s)]$$

to the right-hand side of (19), we are able to evaluate (5) for system (33). The evaluation of integral (5) obtained in this way is valid for every $(v_0,\phi)\in \mathbb{R}^n\times \mathrm{BV}([-r,0],\mathbb{R}^n)$, but the procedure described in (i) permits us to extend its validity onto the whole space M^2.

6. Examples of application

(i) Consider the system of nuclear reactor temperature control, depicted in Fig. 2. Assume after Dorf (1980) that

$$G_1(s)=K_1+\frac{1}{s}K_2,\ K_1,K_2 \text{ are parameters;}\quad G_2(s)=\frac{\exp(-sr)}{sT+1}$$

$$r=0{\cdot}5[s],\quad T=0{\cdot}2[s]$$

and suppose that the disturbance, represented by the Heaviside step function $\mathbb{1}$, enters between the plant and the controller. Then, the dynamics equations are

$$T\dot{y}(t)+y(t)=p(t-r),\quad t>0 \tag{39 a}$$

$$p(t)=z(t)+u(t)=z_0\mathbb{1}(t)+u(t),\quad t>0 \tag{39 b}$$

$$K_1\varepsilon(t)+K_2\int_0^t \varepsilon(\tau)\,d\tau=u(t),\quad t>0 \tag{39 c}$$

$$w-y(t)=\varepsilon(t),\quad t>0 \tag{39 d}$$

If we assume that the system is asymptotically stable and until the moment of the

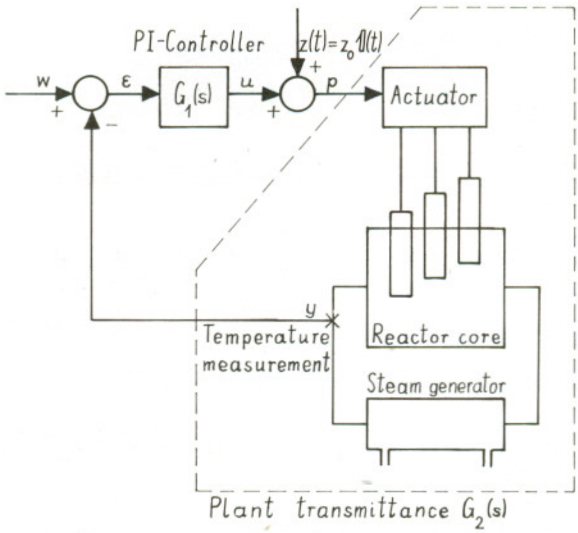

Figure 2. Scheme of nuclear reactor temperature control system.

appearance of a disturbance it remains in equilibrium, then for $t<0$

$$\varepsilon=0, \quad u=K_2\int_0^\infty \varepsilon(t)\,dt=w, \quad p=w, \quad y=w \tag{40}$$

From (39) and (40) we get

$$\ddot{\varepsilon}(t)=-\frac{1}{T}\dot{\varepsilon}(t)-\frac{z_0}{T}\delta(t-r)-\frac{K_1}{T}\dot{\varepsilon}(t-r)-\frac{K_2}{T}\varepsilon(t-r)$$

where δ denotes Dirac's pseudofunction, together with the initial conditions

$$\varepsilon(\theta)=0, \quad \dot{\varepsilon}(\theta)=0 \quad \text{for } \theta\in[-r,0]$$

Hence, introducing the state variables $x_1(t)=\varepsilon(t+r), x_2(t)=\dot{\varepsilon}(t+r)$ and the notation

$$x_0=z_0a\neq 0; \quad a=-\frac{1}{T}=-5; \quad b=-\frac{K_1}{T}=-5K_1; \quad d=-\frac{K_2}{T}=-5K_2$$

we obtain the final version of the dynamics equations:

$$\left.\begin{aligned} &\dot{x}_1(t)=x_2(t)\\ &\dot{x}_2(t)=ax_2(t)+bx_2(t-r)+dx_1(t-r)\\ &x_1(\theta)=0, \quad \theta\in[-r,0]\\ &x_2(\theta)=0 \text{ for } \theta\in[-r,0) \text{ and } x_2(0)=x_0 \end{aligned}\right\} \tag{41}$$

Of course (41) is a special case of (1) with

$$A=\begin{bmatrix}0 & 1\\ 0 & a\end{bmatrix}; \quad B=\begin{bmatrix}0 & 0\\ d & b\end{bmatrix}; \quad v_0=\begin{bmatrix}0\\ x_0\end{bmatrix}; \quad C=0\in\mathscr{L}(\mathbb{R}^2); \quad v=x$$

and $\phi\equiv 0$. Dorf (1980) posed the problem of evaluating a pair (b,d) minimizing for fixed x_0, the integral performance index:

$$J = \int_0^\infty \varepsilon^2(t)\,dt = \int_0^\infty \varepsilon^2(t+r)\,dt = \int_0^\infty x_1^2(t)\,dt\ (\varepsilon \equiv 0 \text{ on } [-r, 0]) \tag{42}$$

This corresponds to the matrix $Q = cc^{\mathrm{T}}$, $c^{\mathrm{T}} = [1, 0]$ in (5) and according to Theorem 2 we have

$$J = v_0^{\mathrm{T}} W_1 v_0 \tag{43}$$

where W_1 is given by (31 *a*). Thus for evaluation of (43) we need to determine $v_0^{\mathrm{T}} R(s)[\frac{1}{2}I + \exp(-sr)BX^{-1}(s)]v_0$. By (27) $\mathrm{col}^{\mathrm{T}} R(s) = -\mathrm{col}^{\mathrm{T}} QS^{-1}(s) = -(c \otimes c)^{\mathrm{T}} S^{-1}(s)$, where $S(s) = (sI + A) \otimes (sI + A) + B \otimes B$ and elementary calculations yield

$$R(s) = \begin{bmatrix} \dfrac{-s^2 - b^2 + a^2}{s^4 + s^2(b^2 - a^2) - d^2} & \dfrac{-bd + s(s^2 + b^2 - a^2)}{[s^4 + s^2(b^2 - a^2) - d^2]s(s+a)} \\ \dfrac{-bd - s(s^2 + b^2 - a^2)}{[s^4 + s^2(b^2 - a^2) - d^2]s(s-a)} & \dfrac{1}{s^4 + s^2(b^2 - a^2) - d^2} \end{bmatrix}$$
$$= R^{\mathrm{T}}(-s),$$

$$\frac{1}{2}I + \exp(-sr)BX^{-1}(s)$$
$$= \frac{1}{2[(s^2 - as) - (bs + d)\exp(-sr)]} \times \begin{bmatrix} s^2 - as - (bs + d)\exp(-sr) & 0 \\ 2sd\exp(-sr) - 2ad\exp(-sr) & s^2 - as + (bs + d)\exp(-sr) \end{bmatrix}$$

Hence,

$$v_0^{\mathrm{T}} R(s)[\tfrac{1}{2}I + \exp(-sr)BX^{-1}(s)]v_0 = x_0^2 Y(s)$$

$$Y(s) = \frac{(s^2 - as) + (bs + d)\exp(-sr)}{2[s^4 + s^2(b^2 - a^2) - d^2][(s^2 - as) - (bs + d)\exp(-sr)]}$$

The function Y has three poles in $\bar{\Pi}_+ : iK, -iK, \lambda$, where

$$K = \sqrt{\frac{b^2 - a^2 + \sqrt{(b^2 - a^2)^2 + 4d^2}}{2}}, \quad \lambda = \sqrt{\frac{a^2 - b^2 + \sqrt{(b^2 - a^2)^2 + 4d^2}}{2}}$$

and from (43) we find

$$\frac{1}{x_0^2} J = -\operatorname*{Res}_{s=iK} Y(s) - \operatorname*{Res}_{s=-iK} Y(s) - 2\operatorname*{Res}_{s=\lambda} Y(s)$$
$$= \frac{(\lambda^2 - a\lambda) + (b\lambda + d)\exp(-\lambda r)}{2\lambda(\lambda^2 + K^2)[\lambda^2 - a\lambda - (b\lambda + d)\exp(-\lambda r)]}$$
$$+ \frac{(ad - bK^2)\cos Kr + (dK + abK)\sin Kr}{(\lambda^2 + K^2)\{K^4 + K^2(a^2 + b^2) + d^2 + 2(dK^2 + abK^2)\cos Kr + 2(bK^3 - adK)\sin Kr\}} \tag{44}$$

The level curves of the performance index $(1/x_0^2)J$, as a function of (b, d), are depicted in Fig. 3.

(ii) Consider the simple boundary-controlled distributed system depicted in Fig. 4. The dynamics of this system is governed by the equations

$$L\frac{\partial i(x,t)}{\partial t} = -\frac{\partial u(x,t)}{\partial x} - Ri(x,t) \quad t \geqslant 0, \quad 0 \leqslant x \leqslant 1 \tag{45}$$

$$C\frac{\partial u(x,t)}{\partial t} = -\frac{\partial i(x,t)}{\partial x} - Gu(x,t) \quad t \geqslant 0, \quad 0 \leqslant x \leqslant 1 \tag{46}$$

$$i(1,t)R_0 = u(1,t) \quad t \geqslant 0 \tag{47}$$

$$u(0,t) = w(t) - Ku(1,t) \quad t \geqslant 0 \tag{48}$$

Górecki (private communication, July 1987) posed the problem of the evaluation of the integral performance index

$$J = \int_0^\infty \left[u(1,t) - \lim_{t\to\infty} u(1,t) \right]^2 dt \tag{49}$$

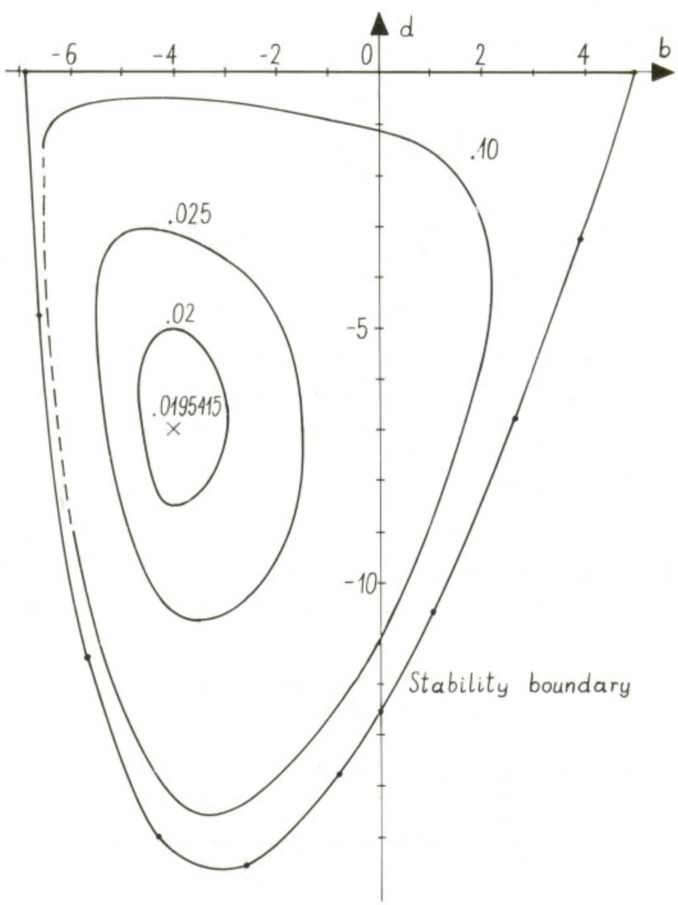

Figure 3. Level curves of $(1/x_0^2)J$.

under the assumptions

$$i(x, 0) = 0; \quad u(x, 0) = 0, \quad 0 \leqslant x \leqslant 1 \tag{50 a}$$

$$w(t) = w_0 \mathbb{1}(t) \tag{50 b}$$

$$\frac{R}{L} = \frac{G}{C} = \alpha \tag{50 c}$$

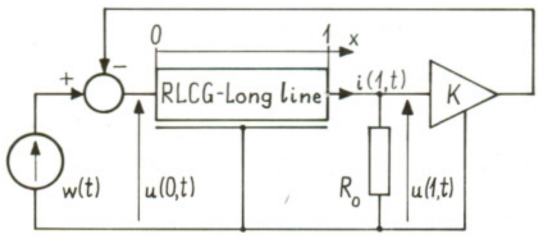

Figure 4. Control system with RLCG-long line.

We show now that the above problem can be solved with the aid of Theorem 2. The general solution of the system (45), (46) is

$$i(x, t) = \exp(-\alpha t) \frac{\Phi(x - vt) - \Psi(x + vt)}{2z} \tag{51 a}$$

$$u(x, t) = \exp(-\alpha t) \frac{\Phi(x - vt) + \Psi(x + vt)}{2} \tag{51 b}$$

where Φ, Ψ are arbitrary sufficiently smooth functions, and

$$v = \frac{1}{\sqrt{(LC)}} = \frac{1}{r}, \quad z = \sqrt{\frac{L}{C}} \tag{52}$$

v is the velocity of wave propagation and z is the wave impedance of a line. Substituting (51) into (47), (48) we get the system of functional equations

$$\Psi(1 + vt) = \kappa \Phi(1 - vt) \tag{53}$$

$$\exp(-\alpha t) \frac{\Phi(-vt) + \Psi(vt)}{2} = w(t) - K \exp(-\alpha t) \frac{\Phi(1 - vt) + \Psi(1 + vt)}{2} \tag{54}$$

where $\kappa = (R_0 - z)/(R_0 + z)$ is the reflection coefficient.

Replacing t by $t - r$ in (53), taking this into account in (54) and introducing the new variable

$$\xi(t) = \exp(-\alpha t) \frac{\Phi(-vt)}{2} \tag{55}$$

we reduce (53), (54) to one difference equation

$$\xi(t) + \frac{\kappa}{\rho^2} \xi(t - 2r) = w(t) - \frac{K(1 + \kappa)}{\rho} \xi(t - r) \tag{56}$$

where

$$\rho = \exp(\alpha / v). \tag{57}$$

Choosing

$$x_1(t) = \xi(t-r) - \xi_\infty \tag{58 a}$$

$$x_2(t) = \xi(t) - \xi_\infty \tag{58 b}$$

where

$$\xi_\infty = \frac{w_0}{1 + \kappa/\rho^2 + K(1+\kappa)/\rho} \tag{59}$$

as the state variables we finally obtain

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{\kappa}{\rho^2} & -\dfrac{K(1+\kappa)}{\rho} \end{bmatrix} \begin{bmatrix} x_1(t-r) \\ x_2(t-r) \end{bmatrix}, \quad t \geqslant 0 \tag{60}$$

together with initial conditions

$$\begin{bmatrix} x_1(\theta) \\ x_2(\theta) \end{bmatrix} = \begin{bmatrix} -\xi_\infty \\ -\xi_\infty \end{bmatrix}, \quad \theta \in [-r, 0] \tag{61}$$

The performance index (49) can be written as

$$J = \int_0^\infty \frac{(1+\kappa)^2}{\rho^2} x_2^2(t-r)\, dt \tag{62}$$

Taking

$$C = \begin{bmatrix} 0 & 1 \\ -\dfrac{\kappa}{\rho^2} & -\dfrac{K(1+\kappa)}{\rho} \end{bmatrix}; \quad A = -I, \quad B = C, \quad v_0 = 0 \tag{63}$$

in (1), one readily finds that the exponential stability conditions reduce to (3 *a*) only, and in the expanded form they are

$$1 + \frac{\kappa}{\rho^2} + \frac{K(1+\kappa)}{\rho} > 0 \tag{64 a}$$

$$1 - \frac{\kappa}{\rho^2} > 0 \tag{64 b}$$

$$1 - \frac{K(1+\kappa)}{\rho} + \frac{\kappa}{\rho^2} > 0 \tag{64 c}$$

From now on we assume that the inequalities (64) hold. It is very easy to establish that the function y, defined by (19) is analytic (!) on $\bar{\Pi}_+$ and thus by Theorem 2

$$\int_0^\infty x^{\mathrm{T}}(t) Q x(t)\, dt = \int_{-r}^0 \phi^{\mathrm{T}}(\theta) \gamma \phi(\theta)\, d\theta$$

where $\gamma = \gamma^{\mathrm{T}} \geqslant 0$ is the unique solution of the matrix equation (29). Hence,

$$\int_0^\infty x^{\mathrm{T}}(t-r) Q x(t-r)\, dt = \int_{-r}^0 \phi^{\mathrm{T}}(\theta) H \phi(\theta)\, d\theta \tag{65}$$

where $H = Q + \gamma$ is the solution of

$$C^{\mathrm{T}}HC - H = -Q \tag{66}$$

To get J from (65), (66) we therefore take

$$Q = \frac{(1+\kappa)^2}{\rho^2} cc^{\mathrm{T}}, \quad c^{\mathrm{T}} = [0, 1], \quad \phi^{\mathrm{T}}(\theta) = -\xi_{\infty}[1, 1]$$

Elementary calculations yield

$$J = \frac{w_0^2(1+\kappa)^2\left[\left(\rho + \frac{\kappa}{\rho}\right)\left(\rho^2 + \frac{\kappa^2}{\rho^2}\right) + 2K\kappa(1+\kappa)\right]}{v\left[\rho + \frac{\kappa}{\rho} + K(1+\kappa)\right]^3\left[\rho + \frac{\kappa}{\rho} - K(1+\kappa)\right]\left(\rho - \frac{\kappa}{\rho}\right)} \tag{67}$$

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