# On solving a Lyapunov operator equation for time–delay systems of neutral–type

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**Abstract:** We construct a quadratic Lyapunov functional for a class of neutral time–delay systems. The results simplify and generalize those of [8, 9].

## 1. INTRODUCTION. MOTIVATING EXAMPLE AND FORMULATION OF THE PROBLEM

Consider the nuclear reactor temperature control system depicted in Figure 1.1. The dynamics equations of the

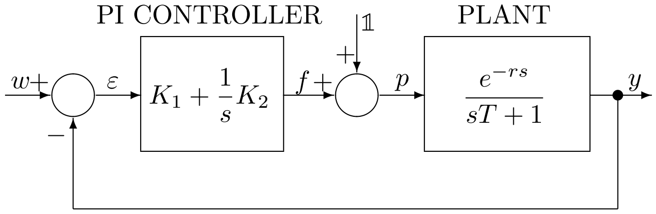


Fig. 1.1. The nuclear reactor temperature control system

system are

$$\left\{ \begin{aligned} T\dot{y}(t) + y(t) &= p(t-r) \\ \mathbb{1}(t) + f(t) &= p(t) \\ K_1\varepsilon(t) + K_2\int_0^t \varepsilon(\tau)\mathrm{d}\tau &= f(t) \\ w - y(t) &= \varepsilon(t) \end{aligned} \right\}, \qquad t > 0 \quad (1.1)$$

where $T = 0.2$, $r = 0.5$ and $K_1$, $K_2$ are parameters, $\mathbb{1}$ denotes the Heaviside step function and $r$, $T$ are fixed positive constants. If we assume that the system is asymptotically stable and until the moment of the appearance of a disturbance it remains in equilibrium, then for $t < 0$

$$\varepsilon = 0, \quad f = K_2\int_0^\infty \varepsilon(t)\mathrm{d}t = w, \quad p = w, \quad y = w \quad (1.2)$$

From (1.1) and (1.2) we get

$$\ddot{\varepsilon}(t) = -\frac{1}{T}\dot{\varepsilon}(t) - \frac{1}{T}\delta(t-r) - \frac{K_1}{T}\dot{\varepsilon}(t-r) - \frac{K_2}{T}\varepsilon(t-r)$$

where $\delta$ denotes Dirac's pseudofunction, together with the initial conditions $\varepsilon(\theta) = 0$, $\dot{\varepsilon}(\theta) = 0$ for $\theta \in [-r, 0]$. Hence, introducing the state variables $z_1(t) = \varepsilon(t+r)$, $z_2(t) = \dot{\varepsilon}(t+r)$ and the notation

$$z_2^0 = a \neq 0,$$
$$a = -\frac{1}{T},\ b = -\frac{K_1}{T} = -5K_1,\ d = -\frac{K_2}{T} = -5K_2$$

we obtain the final version of the dynamics equations

$$\left\{ \begin{aligned} \dot{z}_1(t) &= z_2(t) \\ \dot{z}_2(t) &= az_2(t) + bz_2(t-r) + dz_1(t-r) \\ z_1(\theta) &= 0, \qquad -r \le \theta \le 0 \\ z_2(\theta) &= 0, \qquad -r \le \theta < 0 \\ z_2(0) &= z_2^0 \end{aligned} \right\} \quad (1.3)$$

The problem, originally posed but not solved in [3], is to determine a pair $(b, d)$ minimizing the integral performance index

$$J = \int_0^\infty \varepsilon^2(t)\mathrm{d}t \underbrace{=}_{\varepsilon\equiv 0 \text{ on } [-r,0]} \int_0^\infty \varepsilon^2(t+r)\mathrm{d}t = \int_0^\infty z_1^2(t)\mathrm{d}t. \quad (1.4)$$

The system (1.3) is a special case of the neutral system

$$\left\{ \begin{aligned} \dot{v}(t) &= A_1v(t) + (A_1A_0 + A_2)z(t-r), & t \ge 0 \\ v(t) &= z(t) - A_0z(t-r), & t \ge 0 \\ v(0) &= v_0 & \\ z(\theta) &= \phi(\theta) \quad \text{for almost every } \theta \in [-r, 0] & \end{aligned} \right\} \quad (1.5)$$

where $A_1$, $A_2$, $A_0 \in \mathbf{L}(\mathbb{R}^n)$, $r > 0$, $v_0 \in \mathbb{R}^n$, $\phi$ is a function defined on $(-r, 0)$ with values in $\mathbb{R}^n$. Here

$$n = 2;\ A_1 = \begin{bmatrix} 0 & 1 \\ 0 & a \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 0 \\ d & b \end{bmatrix},\ A_0 = 0 \in \mathbf{L}(\mathbb{R}^2);$$
$$v = z;\ v_0 = \begin{bmatrix} 0 \\ z_2^0 \end{bmatrix},\ \phi \equiv 0\ .$$

Simultaneously (1.4) is a special case of the quadratic integral performance index

$$J\left(\begin{bmatrix} v_0 \\ \phi \end{bmatrix}\right) = \int_0^\infty \begin{bmatrix} v(t) \\ z(t-r) \end{bmatrix}^T \begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix} \begin{bmatrix} v(t) \\ z(t-r) \end{bmatrix} \mathrm{d}t \quad (1.6)$$

with $P$, $Q$, $R \in \mathbf{L}(\mathbb{R}^n)$, $P = P^T$, $R = R^T$, $\begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix} \ge 0$.

This can be seen by taking

$$P = cc^T, \qquad c^T = [\,1\ 0\,], \qquad Q = R = 0 \in \mathbf{L}(\mathbb{R}^2)\ .$$

Motivated by the above example, we pose the following problem.

*Problem 1.1. Evaluate the quadratic performance index (1.6) over trajectories of (1.5).*

This problem has been solved in [5, 6, 7, 8] using the method of *Lyapunov functionals/Lyapunov operator equa-*

*tion.* An entire presentation could be found in [9]; see also [11]. The construction of a Lyapunov functional given therein generalizes the method proposed by Castelan and Infante [2]. A characteristic feature of the latter is the reduction of the whole construction to solving a *functional differential equation* a solution to which is being found in terms of a *finite set of its eigensolutions.*

*Non–Lyapunov attempts* to solving Problem 1.1 has been proposed in [10], [11] and the literature therein.

The aim of this paper is to simplify the Lyapunov approach by eliminating the eigenanalysis from previous presentations. For that some more advances properties of the *tensor (Kronecker) product of matrices* will be applied [1].

## 2. BASIC FACTS ON ABSTRACT OBSERVED SYSTEMS. LYAPUNOV OPERATOR EQUATION

Throughout this section H will stand for a Hilbert state space with scalar product $\langle\cdot,\cdot\rangle_{\mathrm{H}}$.

*Definition 2.1. A family* $\{S(t)\}_{t\geq 0}\subset \mathbf{L}(H)$ *is said to be a* $C_0$–semigroup *on H if:* **(i)** $S(0)=I$, $S(t+\tau)=S(t)S(\tau)$ *for all* $t,\tau\geq 0$, **(ii)** $\lim\limits_{t\to 0+} S(t)x = x$ *for all* $x\in H$.

*The semigroup* $\{S(t)\}_{t\geq 0}$ *is* asymptotically stable *(***AS***) if* $S(t)x_0\to 0$ *(strongly) as* $t\to\infty$ *for any* $x_0\in H$*; it is* exponentially stable *(***EXS***) if* $\|S(t)\|_{\mathbf{L}(H)}$ *as* $t\to\infty$ *or, equivalently,*

$$\exists M\geq 1\ \exists\alpha>0:\quad \|S(t)\|_{\mathbf{L}(H)}\leq Me^{-\alpha t}\quad \forall t\geq 0\ .$$

*Finally, the linear operator*

$$\mathcal{A}x := \lim_{t\to 0+}\frac{1}{t}\left[S(t)x-x\right],$$
$$D(\mathcal{A}) = \left\{x\in H:\ \exists\lim_{t\to 0+}\frac{1}{t}\left[S(t)x-x\right]\right\}$$

*is called the* infinitesimal generator *of the semigroup* $\{S(t)\}_{t\geq 0}$ *on H.*

*Theorem 2.1.* (Lummer–Phillips). *A linear operator* $\mathcal{A}:(D(\mathcal{A})\subset H)\longrightarrow H$ *satisfying the assumptions:*

- **(i)** *there exists* $\lambda_0>0$ *such that* $\mathcal{R}(\lambda I-\mathcal{A})=H$ *for all* $\lambda>\lambda_0$,
- **(ii)** *there exist* $\omega\in\mathbb{R}$ *and an* equivalent scalar product $\langle\cdot,\cdot\rangle_e$ *in H such that* $\mathcal{A}$ *is* $\omega$–dissipative with respect to $\langle\cdot,\cdot\rangle_e$, *i.e.,*

$$\langle\mathcal{A}x,x\rangle_e+\langle x,\mathcal{A}x\rangle_e\leq 2\omega\|x\|_e^2\qquad\forall x\in D(\mathcal{A})\ ,$$

*is the infinitesimal generator of (or simply generates) a* $C_0$*–semigroup* $\{S(t)\}_{t\geq 0}$ *on H for which*

$$\|S(t)x\|_e\leq e^{\omega t}\|x\|_e\qquad\forall t\geq 0\quad\forall x\in H\ . \tag{2.1}$$

Let us consider an *abstract observation* system on H,

$$\left\{\begin{array}{rl}\dot{x}(t)=\mathcal{A}x(t), & t\geq 0\\ x(0)=x_0 & \\ y=\mathcal{C}x & \end{array}\right\} \tag{2.2}$$

with $\mathcal{A}:(D(\mathcal{A})\subset \mathrm{H})\longrightarrow\mathrm{H}$ generating a linear $\mathrm{C}_0$–semigroup $\{S(t)\}_{t\geq 0}$ on H and the $\mathcal{A}$*–bounded* output operator $\mathcal{C}:(D(\mathcal{C})\subset\mathrm{H})\to\mathrm{Y}$, Y is a Hilbert space with the scalar product $\langle\cdot,\cdot\rangle_{\mathrm{Y}}$, i.e., an operator satisying: $D(\mathcal{A})\subset D(\mathcal{C})$ and there exists $\gamma>0$ such that

$$\|\mathcal{C}x\|_{\mathrm{Y}}\leq\gamma(\|x\|_{\mathrm{H}}+\|\mathcal{A}x\|_{\mathrm{H}})\qquad\forall x\in D(\mathcal{A})$$

For each fixed $x_0\in D(\mathcal{A})$, the function (output trajectory) $[0,\infty)\ni t\longmapsto\mathcal{C}S(t)x_0\in\mathrm{Y}$ is continuous with the Laplace transform: $\mathcal{C}(sI-\mathcal{A})^{-1}x_0$, $s\in\rho(\mathcal{A})$ [1] .

*Definition 2.2. The observation operator* $\mathcal{C}$ *is called* admissible *if there exists* $\beta_\infty>0$ *such that*

$$\int_0^\infty\|\mathcal{C}S(t)x_0\|_Y^2\,dt\leq\beta_\infty\|x_0\|_H^2\qquad\forall x_0\in D(\mathcal{A}) \tag{2.3}$$

*i.e., the* observability map

$$\Psi:(D(\mathcal{A})\subset H)\ni x_0\longmapsto\mathcal{C}S(\cdot)x_0\in L^2(0,\infty;Y)$$

*is (densely) defined on* $D(\mathcal{A})$ *and bounded.*

*Theorem 2.2.* $\mathcal{C}$ *is admissible iff there exists* $\mathcal{H}=\mathcal{H}^*\in\mathbf{L}(H)$, $\mathcal{H}\geq 0$, *and* $\mathcal{H}$ *satisfies the* Lyapunov operator equation

$$\langle\mathcal{A}x,\mathcal{H}z\rangle_H+\langle x,\mathcal{H}\mathcal{A}z\rangle_H=-\langle\mathcal{C}x,\mathcal{C}z\rangle_Y\quad\forall x,z\in D(\mathcal{A}) \tag{2.4}$$

*If* $\mathcal{C}$ *be admissible then* $\mathcal{H}=\Psi^*\overline{\Psi}$, *called the* observability Gramian *is the unique solution of (2.4), provided that the semigroup* $\{S(t)\}_{t\geq 0}$ *is* **AS**; *here* $\overline{\Psi}$ *denotes the extension of* $\Psi$ *to an operator from* $\mathbf{L}(H,L^2(0,\infty;Y))$.

## 3. SOLUTION OF THE PROBLEM 1.1

We shall give a solution to Problem 1.1 employing Theorem 2.2.

***Step 1.*** In the state space $\mathrm{H}=\mathbb{M}^2=\mathbb{R}^n\oplus\mathrm{L}^2(-r,0;\mathbb{R}^n)$ we can write (1.5) as an abstract initial value problem, a part of (2.2) with

$$x_0=\begin{bmatrix}v_0\\ \phi\end{bmatrix}$$

$$\mathcal{A}x=\mathcal{A}\begin{bmatrix}v\\ \psi\end{bmatrix}=\begin{bmatrix}A_1v+(A_1A_0+A_2)\psi(-r)\\ \psi'\end{bmatrix},\ D(\mathcal{A})=\left\{\begin{bmatrix}v\\ \psi\end{bmatrix}\in\mathbb{R}^n\oplus\mathrm{W}^{1,2}(-r,0;\mathbb{R}^n),\ v=\psi(0)-A_0\psi(-r)\right\}.$$

It can be proved using Theorem 2.1 with an equivalent scalar product (see [9, pp. 47–48] for details)

$$\left\langle\begin{bmatrix}v_1\\ \psi_1\end{bmatrix},\begin{bmatrix}v_2\\ \psi_2\end{bmatrix}\right\rangle_e := v_1^Tv_2+\int_{-r}^0\psi_1^T(\theta)\left[I-\frac{\theta}{r}A_0^TA_0\right]\psi_2(\theta)\mathrm{d}\theta$$

that $\mathcal{A}$ generates a linear $\mathrm{C}_0$–semigroup $\{S(t)\}_{t\geq 0}$ on H,

$$S(t)\begin{bmatrix}v_0\\ \phi\end{bmatrix}=\begin{bmatrix}v(t)\\ z_t\end{bmatrix},\qquad t\geq 0$$

where for a fixed $t\geq 0$, $z_t$ denotes the *Krasovskii–Hale segmentation,* $z_t:\ [-r,0]\ni\theta\longmapsto z_t(\theta)=z(t+\theta)\in\mathbb{R}^n$.

***Step 2.*** In what follows we shall assume that this semigroup is **EXS** which holds [4, Lemma 6.2.1, p. 151] iff the spectrum of $A_0$ is in an open unit disk $\mathbb{D}$, i.e.,

$$\sigma(A_0)\subset\mathbb{D} \tag{3.1}$$

[1] For each $x_0\in$ H this is the Laplace transform of a Laplace–transformable, Y–valued distribution with support in $[0,\infty)$.

and all roots of the *characteristic quasipolynomial* $\lambda \longmapsto \det[\lambda I - \lambda e^{-r\lambda}A_0 - A_1 - e^{-r\lambda}A_2]$, which is an entire function, have negative real parts, i.e.,

$$\left\{\lambda \in \mathbb{C} :\ \det[\lambda I - \lambda e^{-r\lambda}A_0 - A_1 - e^{-r\lambda}A_2] = 0\right\} \subset \mathbb{C}^- . \tag{3.2}$$

*Step 3.* A linear observation operator $\mathcal{C} : \mathrm{H} \longrightarrow \mathrm{Y}$, $\mathrm{Y} = \mathbb{R}^{2n}$,

$$\mathcal{C}\begin{bmatrix} v \\ \psi \end{bmatrix} = \begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} v \\ \psi(-r) \end{bmatrix} ,$$

corresponding to the integrand in (1.6) is an $\mathcal{A}$–bounded operator. This clearly follows from the identity

$$\psi(-r) = \left[(A_1+A_2)^{-1} - \int_{-r}^{0} (A_1+A_2)^{-1}A_2(\cdot)\mathrm{d}\theta\right] \mathcal{A}\begin{bmatrix} v \\ \psi \end{bmatrix}, \quad \begin{bmatrix} v \\ \psi \end{bmatrix} \in D(\mathcal{A})$$

where thanks to (3.2): $\det(A_1 + A_2) \neq 0$. Since the semigroup $\{S(t)\}_{t\geq 0}$ is **EXS** we have

$$\int_0^\infty \|z(t-r)\|^2_{\mathbb{R}^n}\,\mathrm{d}t = \sum_{k=0}^{\infty}\int_{kr}^{(k+1)r} \|z(t-r)\|^2_{\mathbb{R}^n}\,\mathrm{d}t =$$
$$= \sum_{k=0}^{\infty}\int_{-r}^{0} \|z(kr+\theta)\|^2_{\mathbb{R}^n}\,\mathrm{d}\theta = \sum_{k=0}^{\infty}\int_{-r}^{0} \|z_{kr}(\theta)\|^2_{\mathbb{R}^n}\,\mathrm{d}\theta \leq$$
$$\leq M^2\,\|x_0\|^2_{\mathrm{H}} \sum_{k=0}^{\infty} e^{-2\mu k r} =$$
$$= M^2\,\|x_0\|^2_{\mathrm{H}} \frac{1}{1-e^{-2\mu r}} \qquad \forall x_0 \in \mathrm{H} .$$

Employing the Rayleigh inequality we get for all $x_0 \in D(\mathcal{A})$

$$\left\|\mathcal{C}S(\cdot)\begin{bmatrix} v_0 \\ \phi \end{bmatrix}\right\|^2_{\mathrm{L}^2(0,\infty;\mathbb{R}^{2n})} \leq$$
$$\leq \lambda_{\max}\left(\begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix}\right)\left[\frac{1}{2\mu} + \frac{1}{1-e^{-2\mu r}}\right] M^2\,\|x_0\|^2_{\mathrm{H}}$$

and thus (2.3) holds, i.e., $\mathcal{C}$ is admissible.

*Step 4.* If follows from Theorem 2.2 that

$$J(x_0) = \langle x_0, \mathcal{H}x_0\rangle_{\mathrm{H}} = \left\|\overline{\Psi}x_0\right\|^2_{\mathrm{L}^2(0,\infty;\mathrm{Y})} \qquad \forall x_0 \in \mathrm{H}$$

where $\mathcal{H}$ is a unique bounded self–adjoint nonnegative solution to the Lyapunov operator equation (2.4) which here reads as

$$\langle x_1, \mathcal{H}\mathcal{A}x_2\rangle_{\mathrm{H}} + \langle x_1, \mathcal{H}\mathcal{A}x_2\rangle_{\mathrm{H}} =$$
$$= -\begin{bmatrix} v_1 \\ \psi_1(-r) \end{bmatrix}^T \begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix} \begin{bmatrix} v_2 \\ \psi_2(-r) \end{bmatrix}, x_1, x_2 \in D(\mathcal{A}). \tag{3.3}$$

The solution of (3.3) will be sought in the form

$$\mathcal{H}\begin{bmatrix} v \\ \psi \end{bmatrix} = \begin{bmatrix} \alpha v + \displaystyle\int_{-r}^{0} \beta(\theta)\psi(\theta)\mathrm{d}\theta \\ \beta^T(\cdot)v + \displaystyle\int_{-r}^{0} \delta(\cdot,\sigma)\psi(\sigma)\mathrm{d}\sigma + \gamma\psi \end{bmatrix}$$

with $\alpha, \gamma \in \mathbf{L}(\mathbb{R}^n)$, $\alpha = \alpha^T$, $\gamma = \gamma^T$,

$$\delta(\theta,\sigma) = \left\{\begin{array}{ll} \Phi(\theta-\sigma), & \theta < \sigma \\ \Phi^T(\sigma-\theta), & \theta > \sigma \end{array}\right\} = \delta^T(\sigma,\theta) , \tag{3.4}$$

and $\Phi, \beta \in \mathrm{C}^\infty([-r,0], \mathbf{L}(\mathbb{R}^n))$. The *matrix kernel* function (3.4) may have a discontinuity along the diagonal $\theta = \sigma$ of the square $[-r,0]\times[-r,0]$, or equivalently, $\Phi(0)$ may not be a symmetric matrix. The matrix $\gamma$ is a (unique) solution of the discrete Lyapunov matrix equation

$$A_0^T\gamma A_0 - \gamma = -R$$

and

$$\Phi(\theta) = \frac{\mathrm{d}\beta^T(\theta)}{\mathrm{d}\theta} - \beta^T(\theta)A_1 = A_2^T\beta(-r-\theta) - A_0^T\frac{\mathrm{d}\beta(-r-\theta)}{\mathrm{d}\theta}$$

The constant matrix $\alpha$ and the matrix–valued function $\beta$ satisfy the boundary–value problem

$$\left\{\begin{array}{l} \dfrac{\mathrm{d}}{\mathrm{d}\theta}\left[\beta(\theta) + \beta^T(-r-\theta)A_0\right] = A_1^T\beta(\theta) + \beta^T(-r-\theta)A_2 \\ A_1^T\alpha + \alpha A_1 + \beta^T(0) + \beta(0) + \gamma = -P \\ \gamma A_0 + \alpha(A_1A_0 + A_2) + \beta(0)A_0 - \beta(-r) = -Q \ . \end{array}\right\} \tag{3.5}$$

*Step 5.* In this paragraph we give full solution of (3.5) in terms of the Kronecker (tensor) product of matrices. Less advanced properties of the Kronecker product are presented in basic courses on matrix algebra and gathered in [1], therefore we shall employ them without notifying – only more advanced properties of the Kronecker product will be explained.

By substituting

$$\vartheta(\theta) = \beta^T(-r-\theta), \qquad -r \leq \theta \leq 0 \tag{3.6}$$

one can reduce the first equation of (3.5) to the system

$$\left\{\begin{array}{rcl} \dfrac{\mathrm{d}}{\mathrm{d}\theta}[\beta(\theta) + \vartheta(\theta)A_0] &=& A_1^T\beta(\theta) + \vartheta(\theta)A_2 \\ \dfrac{\mathrm{d}}{\mathrm{d}\theta}[A_0^T\beta(\theta) + \vartheta(\theta)] &=& -A_2^T\beta(\theta) - \vartheta(\theta)A_1 \end{array}\right\} . \tag{3.7}$$

Employing the *Kronecker product of matrices*, we find

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\begin{bmatrix} I\otimes I & I\otimes A_0^T \\ A_0^T\otimes I & I\otimes I \end{bmatrix}\begin{bmatrix} \mathrm{col}\,\beta \\ \mathrm{col}\,\vartheta \end{bmatrix} =$$
$$= \begin{bmatrix} A_1^T\otimes I & I\otimes A_2^T \\ -A_2^T\otimes I & -I\otimes A_1^T \end{bmatrix}\begin{bmatrix} \mathrm{col}\,\beta \\ \mathrm{col}\,\vartheta \end{bmatrix}$$

where $\otimes$ stands for the Kronecker product of matrices while $\mathrm{col}\,\beta$, $\mathrm{col}\,\vartheta$ are $n^2$–dimensional vectors having rows composed of the rows of matrices $\beta$ and $\vartheta$, respectively.

*Lemma 3.1.* (Schur).

$$\det G_1 \neq 0 \Rightarrow \det\begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix} = \det G_1 \det\left[G_4 - G_3G_1^{-1}G_2\right] .$$

*Proof.* Since $\det G_1 \neq 0$ then

$$\begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}\begin{bmatrix} I & -G_1^{-1}G_2 \\ 0 & I \end{bmatrix} = \begin{bmatrix} G_1 & 0 \\ G_3 & G_4 - G_3G_1^{-1}G_2 \end{bmatrix} .$$

□

By Lemma 3.1 and (3.1) we have

$$\det\begin{bmatrix} I\otimes I & I\otimes A_0^T \\ A_0^T\otimes I & I\otimes I\end{bmatrix} = \det\left(I\otimes I - A_0^T\otimes A_0^T\right)\neq 0 \tag{3.8}$$

and, moreover,

$$\begin{bmatrix} I\otimes I & I\otimes A_0^T \\ A_0^T\otimes I & I\otimes I\end{bmatrix}^{-1} =$$

$$= \begin{bmatrix} \left(I\otimes I - A_0^T\otimes A_0^T\right)^{-1} & 0\otimes 0 \\ 0\otimes 0 & \left(I\otimes I - A_0^T\otimes A_0^T\right)^{-1}\end{bmatrix}\cdot$$

$$\cdot\begin{bmatrix} I\otimes I & -I\otimes A_0^T \\ -A_0^T\otimes I & I\otimes I\end{bmatrix} .$$

Consequently, (3.7) may equivalently be written as

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\begin{bmatrix}\operatorname{col}\beta \\ \operatorname{col}\vartheta\end{bmatrix} = \underbrace{\begin{bmatrix}\mathbf{a_{11}} & \mathbf{a_{12}} \\ \mathbf{a_{21}} & \mathbf{a_{22}}\end{bmatrix}}_{:=\mathbf{A}}\begin{bmatrix}\operatorname{col}\beta \\ \operatorname{col}\vartheta\end{bmatrix} \tag{3.9}$$

where

$$\mathbf{a_{11}} := \left(I\otimes I - A_0^T\otimes A_0^T\right)^{-1}\left(A_1^T\otimes I + A_2^T\otimes A_0^T\right) ,$$

$$\mathbf{a_{12}} := \left(I\otimes I - A_0^T\otimes A_0^T\right)^{-1}\left[I\otimes\left(A_1A_0+A_2\right)^T\right] ,$$

$$\mathbf{a_{21}} := \left(I\otimes I - A_0^T\otimes A_0^T\right)^{-1}\left[-\left(A_1A_0+A_2\right)^T\otimes I\right] ,$$

$$\mathbf{a_{22}} := \left(I\otimes I - A_0^T\otimes A_0^T\right)^{-1}\left(-I\otimes A_1^T - A_0^T\otimes A_2^T\right) .$$

The above arguments show that a solution of the first equation of (3.5) is

$$\begin{aligned}\operatorname{col}\beta(\theta) &= \epsilon_{11}(\theta)\operatorname{col}\beta(0) + \epsilon_{12}(\theta)\operatorname{col}\vartheta(0)\underbrace{=}_{(3.6)} \\ &= \epsilon_{11}(\theta)\operatorname{col}\beta(0) + \epsilon_{12}(\theta)\underbrace{\operatorname{col}\beta^T(-r)}_{=\operatorname{col}\vartheta(0)} = \\ &= \epsilon_{11}(\theta)\operatorname{col}\beta(0) + \epsilon_{12}(\theta)\mathbf{U}\operatorname{col}\beta(-r)\end{aligned} \tag{3.10}$$

where $\epsilon_{11}(\theta)$ and $\epsilon_{12}(\theta)$ denote entries of the fundamental matrix of (3.9)

$$e^{\theta\mathbf{A}} = \begin{bmatrix}\epsilon_{11}(\theta) & \epsilon_{12}(\theta) \\ \epsilon_{21}(\theta) & \epsilon_{22}(\theta)\end{bmatrix} ,$$

and $\mathbf{U}$ is the *permutation matrix* [1, p. 772, Formula (4)]

$$\mathbf{U} := \sum_{i=1}^{n}\sum_{j=1}^{n} e_i e_j^T\otimes e_j e_i^T ,$$

where $\{e_i\}_{i=1}^{i=n}$ stands for the Cartesian orthonormal basis.

In terms of the Kronecker products, the second equation of (3.5) reads as

$$\left(A_1^T\otimes I + I\otimes A_1^T\right)\operatorname{col}\alpha + (I+\mathbf{U})\operatorname{col}\beta(0) = -\operatorname{col}\gamma - \operatorname{col}P \tag{3.11}$$

whilst the third equation of (3.5) takes the form

$$\operatorname{col}\beta(-r) = \operatorname{col}Q + \operatorname{col}(\gamma A_0) + \left[I\otimes\left(A_1A_0+A_2\right)^T\right]\operatorname{col}\alpha + \left[I\otimes A_0^T\right]\operatorname{col}\beta(0) . \tag{3.12}$$

This enables us to eliminate $\operatorname{col}\beta(-r)$ from

$$\left[I - \epsilon_{12}(-r)\mathbf{U}\right]\operatorname{col}\beta(-r) = \epsilon_{11}(-r)\operatorname{col}\beta(0)$$

which is being obtained from (3.10) at $\theta = -r$. Hence we get

$$\begin{aligned}&\left[I - \epsilon_{12}(-r)\mathbf{U}\right]\{\operatorname{col}Q + \operatorname{col}(\gamma A_0) + \\ &+ \left[I\otimes\left(A_1A_0+A_2\right)^T\right]\operatorname{col}\alpha + \left[I\otimes A_0^T\right]\operatorname{col}\beta(0)\} \\ &= \epsilon_{11}(-r)\operatorname{col}\beta(0) .\end{aligned} \tag{3.13}$$

(3.11) jointly with (3.13) can be written in vector form

$$\begin{bmatrix}\left(A_1^T\otimes I + I\otimes A_1^T\right) \\ \left[I - \epsilon_{12}(-r)\mathbf{U}\right]\left[I\otimes\left(A_1A_0+A_2\right)^T\right]\end{bmatrix}\cdot$$

$$\begin{bmatrix} I+\mathbf{U} \\ \left[I - \epsilon_{12}(-r)\mathbf{U}\right]\left[I\otimes A_0^T\right] - \epsilon_{11}(-r)\end{bmatrix}\cdot\begin{bmatrix}\operatorname{col}\alpha \\ \operatorname{col}\beta(0)\end{bmatrix} =$$

$$= \begin{bmatrix} -\operatorname{col}\gamma - \operatorname{col}P \\ -\left[I - \epsilon_{12}(-r)\mathbf{U}\right]\left[\operatorname{col}Q + \operatorname{col}(\gamma A_0)\right]\end{bmatrix} \tag{3.14}$$

Having (3.14) solved, one knows $\operatorname{col}\alpha$ and $\operatorname{col}\beta(0)$. Moreover, since $A_1^T\otimes I + I\otimes A_1^T$ is a finite–dimensional Lyapunov operator, written in terms of the Kronecker product, and $I+\mathbf{U}$ maps $\operatorname{col}\beta(0)$ into $\operatorname{col}\left[\beta(0)+\beta^T(0)\right]$ then $\operatorname{col}\alpha$ corresponds to a symmetric matrix $\alpha$. Next, using (3.12), one can determine $\operatorname{col}\beta(-r)$. Finally, $\beta(\theta)$ can be obtained from (3.10). The whole procedure yields a closed–form solution, provided that (3.14) has a solution, so the following theorem is fundamental.

*Theorem 3.1. If the semigroup* $\{S(t)\}_{t\geq 0}$ *is* **EXS**, *or equivalently, (3.1) and (3.2) hold, then the linear algebraic non–homogeneous system (3.14) has a unique solution.*

*Proof.* It is enough to show that the matrix of the linear algebraic non–homogeneous system (3.14) is nonsingular. Suppose that this is not the case. Then the associated linear algebraic homogeneous system has a nonzero solution with $\alpha$ necessarily symmetric. This implies that for $P = Q = R = 0$ (observe that, by (3.1), $R = 0 \iff \gamma = 0$) there exists, a generally nonzero pair $(\operatorname{col}\alpha, \operatorname{col}\beta(0))$, which solves (3.14) with the null RHS. The corresponding $\operatorname{col}\beta(-r)$ and $\operatorname{col}\beta(\theta)$ are being determined by (3.12) and (3.10), respectively. Consequently there exists $0\neq\mathcal{H}=\mathcal{H}^*\in\mathbf{L}(\mathrm{H})$ such that

$$\langle\mathcal{A}x,\mathcal{H}x\rangle_{\mathrm{H}} + \langle x,\mathcal{H}\mathcal{A}x\rangle_{\mathrm{H}} = 0 \qquad \forall x\in D(\mathcal{A}) .$$

However, this contradicts **EXS**. Indeed, inserting $x = S(t)x_0$ where $x_0\in D(\mathcal{A})$ is arbitrary, we conclude that

$$\begin{aligned}0 &= \langle\mathcal{A}S(t)x_0,\mathcal{H}S(t)x_0\rangle_{\mathrm{H}} + \langle S(t)x_0,\mathcal{H}\mathcal{A}S(t)x_0\rangle_{\mathrm{H}} = \\ &= \frac{\mathrm{d}}{dt}\langle S(t)x_0,\mathcal{H}S(t)x_0\rangle_{\mathrm{H}} \iff \langle S(t)x_0,\mathcal{H}S(t)x_0\rangle_{\mathrm{H}} = \\ &= \langle x_0,\mathcal{H}x_0\rangle_{\mathrm{H}} \quad \forall t\geq 0,\ \forall x\in D(\mathcal{A}) .\end{aligned}$$

Since $\mathcal{H}\in\mathbf{L}(\mathrm{H})$ and $D(\mathcal{A})$ is dense in H the last equality holds for any $x_0\in\mathrm{H}$, and by **EXS**:

$$\langle x_0,\mathcal{H}x_0\rangle_{\mathrm{H}} = 0 \qquad \forall x_0\in\mathrm{H} .$$

But this means that for any $x_0\in\mathrm{H}$ there exists $y_0\in R(\mathcal{H})$, namely $y_0 = \mathcal{H}x_0$, such that $\langle x_0, y_0\rangle_{\mathrm{H}} = 0$. Hence, any $x_0\in\mathrm{H}$ is perpendicular to the linear subspace $R(\mathcal{H})$. Thus $R(\mathcal{H}) = \{0\}$ and $\mathcal{H} = 0$. □

## 4. DISCUSSION

Observe that

$$\lambda I-\mathbf{A}=\begin{bmatrix}\lambda I\otimes I & 0\otimes 0\\ 0\otimes 0 & \lambda I\otimes I\end{bmatrix}-\begin{bmatrix}I\otimes I & I\otimes A_0^T\\ A_0^T\otimes I & I\otimes I\end{bmatrix}^{-1}\cdot$$
$$\cdot\begin{bmatrix}A_1^T\otimes I & I\otimes A_2^T\\ -A_2^T\otimes I & -I\otimes A_1^T\end{bmatrix}=\begin{bmatrix}I\otimes I & I\otimes A_0^T\\ A_0^T\otimes I & I\otimes I\end{bmatrix}^{-1}\cdot$$
$$\cdot\begin{bmatrix}(\lambda I-A_1^T)\otimes I & I\otimes(\lambda A_0^T-A_2^T)\\ (\lambda A_0^T+A_2^T)\otimes I & I\otimes(\lambda I+A_1^T)\end{bmatrix},$$

whence, up to the reciprocal of the determinant (3.8), the characteristic polynomial of $\mathbf{A}$ equals, by Lemma 3.1,

$$\det\begin{bmatrix}(\lambda I-A_1^T)\otimes I & I\otimes(\lambda A_0^T-A_2^T)\\ (\lambda A_0^T+A_2^T)\otimes I & I\otimes(\lambda I+A_1^T)\end{bmatrix}=$$
$$=\det[(\lambda I-A_1^T)\otimes I]\det\{[I\otimes(\lambda I+A_1^T)]$$
$$-\left[(\lambda A_0^T+A_2^T)\otimes I\right]\underbrace{[(\lambda I-A_1^T)\otimes I]^{-1}}_{:=I\otimes(\lambda I-A_1^T)^{-1}}[I\otimes(\lambda A_0^T-A_2^T)]\}.$$

But the matrices $(\lambda A_0^T+A_2^T)\otimes I$, $I\otimes(\lambda I-A_1^T)^{-1}$ commute, consequently, up to a constant multiplier, the characteristic equation of $\mathbf{A}$ is

$$\det\left[(\lambda I-A_1^T)\otimes(\lambda I+A_1^T)+(A_2^T+\lambda A_0^T)\otimes(A_2^T-\lambda A_0^T)\right].\tag{4.1}$$

Thus

$$e^{\lambda\theta}\begin{bmatrix}L\\ M\end{bmatrix}\tag{4.2}$$

is an eigensolution of (3.7) where $\lambda$ is a root of (4.1), and matrices $L$, $M\in\mathbf{L}(\mathbb{C}^{n^2})$ satisfy the system

$$\left\{\begin{array}{rl}\lambda L+\lambda MA_0= & A_1^TL+MA_2\\ \lambda A_0^TL+\lambda M= & -A_2^TL-MA_1\end{array}\right\}.\tag{4.3}$$

By multiplying the equations of (4.3) by $(-1)$, transposing and reordering them, one can see that if (4.2) is an eigensolution then $e^{-\lambda\theta}\begin{bmatrix}M^T\\ L^T\end{bmatrix}$ is an eigensolution too. In fact, copying the arguments of [2] and shortly reported in [12, pp. 15 - 16], one can prove that $\lambda$ and $-\lambda$ have the same geometric and algebraic multiplicities. Applying the Kronecker product to (4.3) and to its transmutation just described, we see that $\begin{bmatrix}\text{col } L\\ \text{col } M\end{bmatrix}$, $\begin{bmatrix}\mathbf{U}\text{col } M\\ \mathbf{U}\text{col } L\end{bmatrix}$ belong to $\ker(\lambda I-\mathbf{A})$, i.e., they are eigenvectors of $\mathbf{A}$, and thus $e^{\lambda\theta}\begin{bmatrix}\text{col } L\\ \text{col } M\end{bmatrix}$, $e^{-\lambda\theta}\begin{bmatrix}\mathbf{U}\text{col } M\\ \mathbf{U}\text{col } L\end{bmatrix}$ are eigensolutions of (3.9).

Now assume, in addition, that all eigenvalues of (3.7) have *linear elementary divisors*. Then the corresponding eigenvectors form a basis in $\mathbb{C}^{n^2}$ and a general solution of (3.7) is

$$\begin{bmatrix}\beta(\theta)\\ \vartheta(\theta)\end{bmatrix}=\sum_{i=1}^{n^2}\left\{\kappa_ie^{\lambda_i\theta}\begin{bmatrix}L_i\\ M_i\end{bmatrix}+\mu_ie^{-\lambda_i\theta}\begin{bmatrix}M_i^T\\ L_i^T\end{bmatrix}\right\}.$$

This solution satisfies the functional equation (3.6) iff $\mu_i=\kappa_ie^{-\lambda_ir}$, so

$$\beta(\theta)=\sum_{i=1}^{n^2}\kappa_i\left[e^{\lambda_i\theta}L_i+e^{-\lambda_i(r+\theta)}M_i^T\right]\iff$$
$$\text{col}\beta(\theta)=\sum_{i=1}^{n^2}\kappa_i\left[e^{\lambda_i\theta}\text{col } L_i+e^{-\lambda_i(r+\theta)}\mathbf{U}\text{col } M_i\right]\tag{4.4}$$

is a general solution of the first equation of (3.5). Substituting (4.4) into the second and third equation of (3.5) yields

$$\gamma+A_1^T\alpha+\alpha A_1+\sum_{i=1}^{n^2}\kappa_i\left[L_i+L_i^T+e^{-\lambda_ir}(M_i+M_i^T)\right]=$$
$$=-P$$
$$\gamma A_0+\alpha(A_1A_0+A_2)+\sum_{i=1}^{n^2}\kappa_i\left[e^{-\lambda_ir}(M_i^TA_0-L_i)+\right.$$
$$\left.+(L_iA_0-M_i^T)\right]=-Q\ .$$

Applying the Kronecker product of matrices to the last system one obtains

$$\left[\begin{array}{l}A_1^T\otimes I+I\otimes A_1^T\\ I\otimes(A_1A_0+A_2)^T\\ \\ \underbrace{\text{col }[L_i+L_i^T+e^{-\lambda_ir}(M_i+M_i^T)]}_{n^2\text{ vectors }(i=1,2,\ldots,n^2)}\\ \underbrace{\text{col }[e^{-\lambda_ir}(M_i^TA_0-L_i)+L_iA_0-M_i^T]}_{n^2\text{ vectors }(i=1,2,\ldots,n^2)}\end{array}\right]\begin{bmatrix}\text{col }\alpha\\ \kappa_1\\ \kappa_2\\ \kappa_3\\ \vdots\\ \kappa_{n^2}\end{bmatrix}=$$
$$=\begin{bmatrix}-\text{col }\gamma-\text{col } P\\ -\text{col } Q-\text{col }(\gamma A_0)\end{bmatrix}.\tag{4.5}$$

This system has been derived in [7, p. 103], [8] and [9, Section 3.2]. Eliminating, with an aid of (4.4), $\text{col}\beta(0)$ from (3.14) we conclude that (4.5) and (3.14) are consistent, so (3.14) is a generalization of (4.5).

The symmetry of matrices $\alpha$, $\gamma$, $P$ causes that both (4.5) and its generalization (3.14) contain $\frac{n(n-1)}{2}$ redundant equations which can be canceled.

For a variety of initial conditions the evaluation of the performance index does not require the knowledge of all elements and/or entries of the quadruple $\alpha$, $\beta(\theta)$, $\delta(\theta,\sigma)$, $\gamma$, e.g., for $x_0=\begin{bmatrix}v_0\\ \mathbf{0}\end{bmatrix}$ it suffices to determine only the matrix $\alpha$.

We end with a remark that the Kronecker product is easily available in contemporary software for symbolic computations, e.g., in MAPLE 12™.

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