



Arbitrarily vertex decomposable suns with few rays

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Abstract

A graph G of order n is called arbitrarily vertex decomposable if for each sequence (n_1, \dots, n_k) of positive integers with $n_1 + \dots + n_k = n$, there exists a partition (V_1, \dots, V_k) of the vertex set of G such that V_i induces a connected subgraph of order n_i , for all $i = 1, \dots, k$. A sun with r rays is a unicyclic graph obtained by adding r hanging edges to r distinct vertices of a cycle. We characterize all arbitrarily vertex decomposable suns with at most three rays. We also provide a list of all on-line arbitrarily vertex decomposable suns with any number of rays.

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1. Introduction

Let $G = (V, E)$ be a graph of order n . A sequence $\tau = (n_1, \dots, n_k)$ of positive integers is called *admissible for G* if $n_1 2 + \dots + n_k = n$. If $\tau = (n_1, \dots, n_k)$ is an admissible sequence for a graph G and there exists a partition (V_1, \dots, V_k) of the vertex set V such that for each $i \in \{1, \dots, k\}$ the subgraph $G[V_i]$ induced by V_i is a connected subgraph of order n_i , then τ is called *G -realizable* or *realizable in G* and the sequence (V_1, \dots, V_k) is said to be a *G -realization of τ* or a *realization of τ in G* . Each set V_i will be called a *τ -part* of a realization of τ in G . A graph G is *arbitrarily vertex decomposable* (avd, for short) if for each admissible sequence τ for G there exists a G -realization of τ .

Arbitrarily vertex decomposable graphs have been investigated by several authors (cf. [1–9]). This notion originated from some applications to computer networks (cp. [1] for details). In general, the problem of deciding whether a given graph is avd is NP-complete [1]. It is not known whether it is NP-complete for the class of all trees. In [1], Barth et al. showed that this problem is polynomial for the class of tripodes. A *tripode* $S(a_1, a_2, a_3)$ is a tree homeomorphic to the star $K_{1,3}$ obtained from $K_{1,3}$ by substituting its edges by paths of orders a_1, a_2 and a_3 , respectively. In particular, the tripode $S(2, a, b)$ is a caterpillar with three hanging vertices, and we will denote it by $\text{Cat}(a, b)$, assuming that $2 \leq a \leq b$.

We will make use of the following characterization of avd caterpillars with three hanging vertices due to Barth et al. [1] and, independently, to Horňák and Woźniak [5]. By $(d)^\lambda$ we denote the sequence (d, \dots, d) , where d appears λ times.

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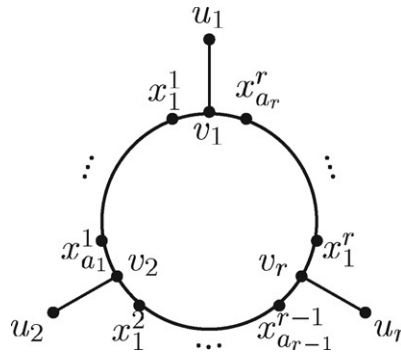


Fig. 1. Sun(a_1, \dots, a_r) with r rays.

Proposition 1 (Barth et al. [1] and Horňák and Woźniak [5]). A caterpillar $\text{Cat}(a, b)$ is arbitrarily vertex decomposable if and only if a and b are coprime. Moreover, each admissible and non-realizable sequence in $\text{Cat}(a, b)$ is of the form $(d)^\lambda$, where $d > 1$ is a common divisor of a and b .

Analogous, but more intricate characterizations of avd caterpillars with more hanging vertices have been recently obtained by Cichacz et al. [3].

The following observation makes it possible to shorten some proofs.

Proposition 2. A graph G is avd if and only if every admissible sequence $\tau = (n_1, \dots, n_k)$ with $n_i \geq 2$ for each $i = 1, \dots, k$, has a realization in G .

Proof. Obviously, it suffices to justify the following assertion: if every admissible sequence with elements greater than 1 has a G -realization, then every admissible sequence also does. Suppose this is not the case, and let $\tau = (n_1, \dots, n_k)$ with $n_1 \geq \dots \geq n_k = 1$ be a non-realizable sequence with the least possible number of 1s. By assumption, the sequence $\tau' = (n_1, \dots, n_{k-2}, n_{k-1} + n_k)$ has a G -realization (V_1, \dots, V_{k-1}) since τ' has less elements equal to 1 than τ . Now, choose any vertex $v \in V_{k-1}$ that is not a cut-vertex of $G[V_{k-1}]$ to obtain a realization $(V_1, \dots, V_{k-1} \setminus \{v\}, \{v\})$ of τ in G . □

A sun with r rays is a graph of order $n \geq 2r$ with r hanging vertices u_1, \dots, u_r , whose deletion yields a cycle C_{n-r} , and each vertex v_i adjacent to u_i is of degree three. Each hanging edge $u_i v_i$ is called a ray. If the sequence of vertices v_i is situated on the cycle C_{n-r} in such a way that there are exactly $a_i \geq 0$ vertices, each of degree two, between v_i and v_{i+1} , $i = 1, \dots, r$ (the indices taken modulo r), then this sun is denoted by $\text{Sun}(a_1, \dots, a_r)$, and is unique up to isomorphism (cp. Fig. 1). Note that the order of $\text{Sun}(a_1, \dots, a_r)$ equals $n = 2r + a_1 + \dots + a_r$ and the unique cycle is dominating.

A perfect matching in a graph $G = (V, E)$ of even order n can be defined as a partition $(V_1, \dots, V_{n/2})$ of V such that V_i is a set of two adjacent vertices, $i = 1, \dots, n/2$. Thus, a perfect matching is a realization of the sequence $(2)^{n/2}$. A perfect quasi-matching in a graph of odd order n is realization of the sequence $(1, (2)^{(n-1)/2})$, i.e. a partition $(V_1, \dots, V_{(n+1)/2})$ of V such that V_1 is a single vertex and all other sets V_i contain two adjacent vertices. It follows that every avd graph G has either a perfect matching or a perfect quasi-matching, hence the independence number $\alpha(G)$ has to be at most $\lceil n/2 \rceil$.

Note that every traceable graph is avd since every path is avd. It may be well to add that just recently Theorems 7 and 8 of Section 3 of the present paper have been already used by Marczyk to prove the following Ore-type conditions for a graph to be avd.

Theorem 3 (Marczyk [8]). If $\deg(x) + \deg(y) \geq n - 2$ for every pair of non-adjacent vertices x, y of a graph G of order n , then G is arbitrarily vertex decomposable unless $\alpha(G) = n/2 + 1$.

Theorem 4 (Marczyk [9]). Let G be a 2-connected graph of order $n \geq 11$ having either a perfect matching or a perfect quasi-matching. If $\deg(x) + \deg(y) \geq n - 4$ for every pair of non-adjacent vertices x, y , then G is arbitrarily vertex decomposable.

Section 2 concerns realizations of so-called l -good sequences in suns with any number of rays, and it enables to simplify considerably proofs in Section 3, where we characterize all avd suns with at most three rays. In the last Section 4, we give a complete list of all on-line avd suns with any number of rays.

Given an admissible sequence $\tau = (n_1, \dots, n_k)$ for a graph G of order n , we will use the following convention to describe a realization (V_1, \dots, V_k) of τ in G . Namely, we choose an ordering $s = (v_1, \dots, v_n)$ of the vertex set of G . Next, we define the τ -parts according to the sequence s , that is $V_1 = \{v_1, \dots, v_{n_1}\}$, $V_2 = \{v_{n_1+1}, \dots, v_{n_1+n_2}\}$, and so on.

2. Realizations of l -good sequences

Let $\tau = (n_1, \dots, n_k)$ be an admissible sequence for a graph G of order n . An element n_i of τ is called *good* if either $n_i = 1$ or n_i is even. For $l \geq 0$, the sequence τ is called l -good if either τ contains at least l good elements, or all its elements are good (when $k < l$). The following observation is analogous to Proposition 2.

Proposition 5. *Let G be a graph of even order n . Then every l -good sequence is G -realizable if and only if every l -good sequence with all elements greater than 1 is G -realizable.*

Proof. Clearly, it suffices to prove the “if” part. Suppose there exists an l -good sequence with some elements equal to 1, which has no realization in G . Let $\tau = (n_1, \dots, n_k)$ be such a sequence with the least number of 1’s. Assume $n_1 \geq \dots \geq n_k = 1$.

If there exists an element n_i that is not good in τ , then the sequence $\tau' = (n'_1, \dots, n'_{k-1})$, with $n'_i = n_i + 1$ and $n'_j = n_j$ for all $j \neq i$, has the same number of good elements and less number of 1s. Hence τ' has a G -realization (V_1, \dots, V_{k-1}) . Similarly as in the proof of Proposition 2, choose any vertex $w \in V_i$ that is not a cut-vertex of $G[V_i]$, define $V_k = \{w\}$ and re-define $V_i = V_i \setminus \{w\}$, $1 \leq i \leq k-1$, to obtain a realization of τ in G , contrary to the definition of τ .

Suppose all elements of τ are good, whence $n_{k-1} = 1$ since n is even. The sequence $\tau' = (n_1, \dots, n_{k-2}, 2)$ is also l -good and has less number of 1’s. Therefore τ' has a realization (V_1, \dots, V_{k-1}) in G . Then $(V_1, \dots, V_{k-2}, \{w_1\}, \{w_2\})$, where $\{w_1, w_2\} = V_{k-1}$, is obviously a G -realization of τ . \square

Theorem 6. *Every $(r-2)$ -good sequence is realizable in $\text{Sun}(a_1, \dots, a_r)$ if and only if at most one of the numbers a_1, \dots, a_r is odd.*

Proof. Let n denote the order of a graph $G = \text{Sun}(a_1, \dots, a_r)$.

Necessity. If n is even, then the sequence $(2)^{n/2}$ is $(r-2)$ -good. Hence each number a_i has to be even since each ray $u_i v_i$ has to create a τ -part. For odd n , the sequence $((2)^{(n-3)/2}, 3)$ is $(r-2)$ -good since $r \leq n/2$. It follows easily that exactly one of the numbers a_1, \dots, a_r is odd.

Sufficiency. First, we consider the case when n is even. Thus all numbers a_1, \dots, a_r are even. Let $\tau = (n_1, \dots, n_k)$ be an $(r-2)$ -good sequence. By Proposition 5, we may assume that $n_i \geq 2$ for all i . Define a realization (V_1, \dots, V_k) of τ in $\text{Sun}(a_1, \dots, a_r)$ according to the following sequence of vertices

$$s = (v_1, u_1, x_1^1, \dots, x_{a_1}^1, v_2, u_2, x_1^2, \dots, v_r, u_r, x_1^r, \dots, x_{a_r}^r).$$

This clearly gives a realization of τ in G if all elements of τ are even. Hence, assume n_1, \dots, n_{k_1} are odd, for some $k_1 \geq 1$, and n_{k_1+1}, \dots, n_k are even. Obviously, k_1 is even since n is even. Suppose that this construction does not give a realization of τ in G , and let i_0 denote the smallest i such that the subgraph $G[V_i]$ is disconnected. It follows that $v_{j_0} \in V_{i_0-1}$ while $u_{j_0} \in V_{i_0}$, for some j_0 with $2 \leq j_0 \leq r$. The integer n_{i_0-1} is odd since the number of elements following v_{j_0} in s is odd. We distinguish two cases.

Case A: $j_0 \in \{3, \dots, r\}$ or $a_r = 0$. We modify the ordering of elements in τ by moving the last even element of τ just before n_{i_0-1} to obtain a new sequence

$$\tau = (n_1, \dots, n_{i_0-2}, n_k, n_{i_0-1}, \dots, n_{k-1}).$$

We define new τ -parts according to s . It is easily seen that, for each $j = 1, \dots, j_0$, both vertices u_j, v_j belong to the same τ -part.

We then find the first disconnected subgraph $G[V_{i_0}]$ and repeat the above modification of the sequence τ by moving its last even element before the element n_{i_0-1} . Then we partition the set of vertices of G according to the modified τ .

If $j_0 \geq 3$, then the number of necessary modifications is not greater than the least possible number $r - 2$ of even elements of τ . Hence finally, we obtain a G -realization of τ . If $a_r = 0$ then it is easy to see that we also need at most $r - 2$ modifications of τ to obtain a realization of τ in G since $n_i \geq 2$ for all i .

Case B: $j_0 = 2$ and $a_r \geq 2$. We start our procedure again with another ordering $\tau = (n_1, \dots, n_{k_1-1}, n_{k_1+1}, n_k, n_{k_1})$ and we partition the set $V(G)$ according to the following new sequence

$$s^1 = (x_{a_r}^r, v_1, u_1, x_1^1, \dots, x_{a_1}^1, v_2, u_2, x_1^2, \dots, v_r, u_r, x_1^r, \dots, x_{a_r-1}^r).$$

Thus, each of the first two rays u_1v_1 and u_2v_2 belong to certain connected subgraphs induced by τ -parts of G . Then we proceed in the same way as in Case A. Observe that at each step, the first disconnected subgraph corresponds to an odd element of the current sequence. Therefore, moving at most $r - 2$ even elements in τ yields a realization of τ in G .

Now, suppose that n is odd. Without loss of generality we may assume that a_1 is the unique odd number among a_1, \dots, a_r , the sequence $\tau = (n_1, \dots, n_k)$ is $(r - 2)$ -good, n_1, \dots, n_{k_1} are odd numbers, for some $k_1 \geq 1$, and all other elements of τ are even.

If $1 \in \{n_i \mid i = 1, \dots, k_1\}$, say $n_1 = 1$, then we take a hanging vertex u_1 as a τ -part assigned to n_1 . Deletion of u_1 yields a sun G' with $r - 1$ rays. The sequence (n_2, \dots, n_k) is $(r - 3)$ -good and thus realizable in G' since the order of G' is even.

Otherwise, we start defining τ -parts of $\text{Sun}(a_1, \dots, a_r)$ according to the sequence s . Clearly, $G[V_1]$ is connected since n_1 is odd. Then it is not difficult to see that exactly the same method as the one used previously for even n provides a realization of τ in G . \square

The following example shows that the number $r - 2$ in the above theorem cannot be smaller. Consider a sun with r rays $G = \text{Sun}(3m_1, 3m_2, 0, \dots, 0)$, for some non-negative integers m_1, m_2 , and a sequence $\tau = ((2)^{r-3}, (3)^{m_1+m_2+2})$. It is easy to check that τ is $(r - 3)$ -good but has no realization in G .

3. Avd suns with at most three rays

Every sun with one ray is avd since it is traceable.

Theorem 7. *A sun with two rays $\text{Sun}(a, b)$ is arbitrarily vertex decomposable if and only if at most one of the numbers a and b is odd. Moreover, $\text{Sun}(a, b)$ of order n is not avd if and only if $(2)^{n/2}$ is the unique admissible and non-realizable sequence.*

Proof. The first claim follows immediately from Theorem 6 for $r = 2$ since every admissible sequence is 0-good.

To prove the “only if” part of the second claim of the theorem (the “if” part is obvious) suppose that $\text{Sun}(a, b)$ is not avd and (n_1, \dots, n_k) is an admissible and non-realizable sequence. Hence both a and b are odd, and the order n is even. If we choose a vertex of degree three and delete a non-hanging edge incident to it, then we obtain a caterpillar, either $\text{Cat}(a + 1, b + 3)$ or $\text{Cat}(a + 3, b + 1)$. Clearly, the sequence $\tau = (n_1, \dots, n_k)$ cannot be realized in any of these two trees. Hence, by Proposition 1, this sequence is of the form $\tau = (d)^k$ with d being a common divisor of four numbers $a + 1, a + 3, b + 1, b + 3$. It follows that $d = 2$. \square

Theorem 8. *A sun with three rays $\text{Sun}(a, b, c)$ is not arbitrarily vertex decomposable if and only if at least one of the following three conditions is fulfilled:*

- (1) *at least two of the numbers a, b, c are odd,*
- (2) $a \equiv b \equiv c \equiv 0 \pmod{3}$,
- (3) $a \equiv b \equiv c \equiv 2 \pmod{3}$.

Proof. *Sufficiency.* If all three numbers a, b, c are odd, then $n \geq 9$ is odd, and it is easy to see that the sequence $(3, (2)^{(n-3)/2})$ is admissible and not realizable in $\text{Sun}(a, b, c)$. If exactly two numbers among a, b, c are odd, then n is even and the sequence $(2)^{n/2}$ cannot be realized. If either condition (2) or (3) is fulfilled, then it is not difficult to see that the sequence $(3)^{n/3}$ is non-realizable.

Necessity. We will show that if condition (1) is not fulfilled and $\text{Sun}(a, b, c)$ is not avd, then either (2) or (3) holds. Suppose that at most one of the numbers a, b, c is odd. Let $\tau = (n_1, \dots, n_k)$ be an admissible and non-increasing

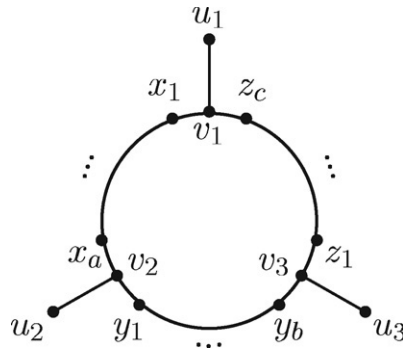


Fig. 2. $\text{Sun}(a, b, c)$ with three rays.

sequence. If τ contains a good element, then τ is realizable in $\text{Sun}(a, b, c)$, by Theorem 6. Therefore assume that all elements of τ are odd and greater than two, i.e. $n_1 \geq \dots \geq n_k \geq 3$.

Without loss of generality, we may assume that $0 \leq a \leq b \leq c$, the graph $G = \text{Sun}(a, b, c)$ has three hanging vertices u_1, u_2, u_3 adjacent to v_1, v_2, v_3 , respectively, and the subgraph $G - \{u_1, u_2, u_3\}$ is a cycle with $n - 3$ vertices appearing in the following order: $v_1, x_1, \dots, x_a, v_2, y_1, \dots, y_b, v_3, z_1, \dots, z_c$ (cp. Fig. 2).

Arrange all the vertices of G into the following sequence

$$s = (v_1, u_1, x_1, \dots, x_a, v_2, u_2, y_1, \dots, y_b, v_3, u_3, z_1, \dots, z_c).$$

Define the τ -parts $V_i, i = 1, \dots, k$, according to the sequence s . The induced subgraphs $G[V_i]$ will be connected for all i , unless one of the following Cases A or B appears.

Case A: There exists an i such that $v_2 \in V_i$ while $u_2 \in V_{i+1}$.

In this case, we modify the ordering of vertices of G and consider a new sequence

$$s^1 = (z_c, v_1, u_1, x_1, \dots, x_a, v_2, u_2, y_1, \dots, y_b, v_3, u_3, z_1, \dots, z_{c-1}).$$

The sets $V_j, j = 1, \dots, k$ are now defined according to s^1 . Observe that this construction yields a τ -realization in G , unless either $c = 0$, or v_3 is the last vertex of V_j for some j , i.e. $n_1 + \dots + n_j = a + b + 6$.

If $c = 0$, then $a = b = 0$ since $a \leq b \leq c$. Hence condition (2) is fulfilled, and we are done.

If $n_1 + \dots + n_j = a + b + 6$, for some j , then we take another sequence

$$s^2 = (z_{c-1}, z_c, v_1, u_1, x_1, \dots, x_a, v_2, u_2, y_1, \dots, y_b, v_3, u_3, z_1, \dots, z_{c-2}),$$

and define the τ -parts according to s^2 . This would not give a realization of τ in G only if either $c = 1$ or $n_1 = 3$.

If $c = 1$, then a and b have to be even, so $a = b = 0$, and $n = 7$. But for a graph of order 7, and there does not exist an admissible sequence of odd elements greater than two.

When $n_1 = 3$, then $n_1 = n_2 = \dots = n_k = 3$, as τ is non-increasing. Hence $a \equiv 0 \pmod{3}$, by the definition of Case A. Also, $b \equiv c \equiv 0 \pmod{3}$ since $n_1 + \dots + n_j = a + b + 6$ and $n = 3k$. Thus, again condition (2) is satisfied.

Case B: The vertices v_2 and u_2 both belong to the same τ -part, but there exists an i such that $v_3 \in V_i$ while $u_3 \notin V_i$, i.e. $n_1 + \dots + n_i = a + b + 5$.

It follows that $c \geq 2$. Here, we define the sets V_1, \dots, V_k according to the sequence s^1 . This will not give a realization of τ in G , only if $n_1 + \dots + n_j = a + 4$ for some j with $1 \leq j < k$. In the latter case, if we re-define the sets V_1, \dots, V_k according to s^2 , then we obtain a G -realization of τ unless either $b = 0$ or $n_1 = 3$.

This is not possible that $b = 0$, since then $a = 0$ and $n_1 + \dots + n_j = 4$, while all elements of τ are odd and greater than two.

If $n_1 = 3$, then $n_2 = \dots = n_k = 3$. Consequently, $a + 4 = 3j, a + b + 5 = 3i$ and $a + b + c + 6 = 3k$. It follows that $a \equiv b \equiv c \equiv 2 \pmod{3}$, that is condition (3) holds. \square

The following corollary follows immediately from the above proof.

Corollary 9. *Sun(a, b, c) of order n is not arbitrarily vertex decomposable if and only if either $(2)^{n/2}$, or $(3)^{n/3}$, or else $(3, (2)^{(n-3)/2})$ is an admissible and non-realizable sequence. \square*

Table 1
Values a, b such that $\text{Cat}(a, b)$ is on-line avd

a	b
2	$\equiv 1 \pmod{2}$
3	$\equiv 1, 2 \pmod{3}$
4	$\equiv 1 \pmod{2}$
5	6, 7, 9, 11, 14, 19
6	$\equiv 1, 5 \pmod{6}$
7	8, 9, 11, 13, 15
8	11, 19
9	11
10	11
11	12

Table 2
Values a, b such that $\text{Sun}(a, b)$ is on-line avd

a	b
0	Arbitrary
1	$\equiv 0 \pmod{2}$
2	$\not\equiv 3 \pmod{6}, 3, 9, 21$
3	$\equiv 0 \pmod{2}$
4	$\equiv 2, 4 \pmod{6}, [4, 19] \setminus \{15\}$
5	$\equiv 2, 4 \pmod{6}, 6, 18$
6	6, 7, 8, 10, 11, 12, 14, 16
7	8, 10, 12, 14, 16
8	8, 9, 10, 11, 12
9	10, 12

It has to be noted that, as opposed to [Theorem 7](#) for suns with two rays, [Corollary 9](#) does not characterize all possible admissible and non-realizable sequences for suns with three rays. For instance, if all numbers a, b, c are odd and $a \leq b \leq c$, then it is not difficult to see that, for every odd d with $1 \leq d \leq a + 4$, each sequence of the form $\tau = (d, (2)^{(n-d)/2})$ is admissible and non-realizable in $\text{Sun}(a, b, c)$.

4. All on-line avd suns

The notion of an on-line arbitrarily vertex decomposable graph has been introduced by [Horňák, Tuza and Woźniak](#) in [\[4\]](#).

Let G be a graph of order n . Imagine the following decomposition procedure consisting of k stages, where k is a random variable attaining integer values from 1 to n . In the i -th stage, where $i = 1, \dots, k$, a positive integer n_i arrives and we have to choose a connected subgraph G_i of G of order n_i that is vertex-disjoint from all subgraphs G_1, \dots, G_{i-1} chosen in the previous stages (without a possibility of changing the choice in the future). More precisely, if a graph G_j of order n_j has been already chosen in a stage j , for all $j \leq i - 1$, and $1 \leq n_i \leq n - (n_1 + \dots + n_{i-1})$, then G_i has to be chosen as a connected subgraph of $G - (G_1 \cup \dots \cup G_{i-1})$. If the decomposition procedure can be accomplished for any (random) sequence of positive integers $\tau = (n_1, \dots, n_k)$ adding up to the order n of G , then G is said to be *on-line arbitrarily vertex decomposable* (on-line avd, for short).

It seems that the characterization of avd trees is very difficult. The situation is different in the case of on-line avd trees. The theorem below (being the main result of [\[4\]](#)) provides their complete list.

Theorem 10 ([Horňák et al. \[4\]](#)). *A tree T is on-line avd if and only if either T is a path or T is a caterpillar $\text{Cat}(a, b)$ with a and b given in [Table 1](#) or T is the tripod $S(3, 5, 7)$.*

Similarly as in the case of trees, the complete characterization of avd suns seems to be very difficult but the on-line version is much easier. The theorem below provides the complete list of on-line avd suns.

Table 3

Values a, b, c such that $\text{Sun}(a, b, c)$ is on-line avd

a	b	c
0	0	$\equiv 1, 2 \pmod{3}$
0	1	$\equiv 0 \pmod{2}$
0	2	$\equiv 2, 4 \pmod{6}, 3, 6, 7, 11, 18, 19$
0	3	$\equiv 2, 4 \pmod{6}$
0	4	4, 5, 6, 8, 10, 11, 12, 14, 16
0	5	6, 8, 16
0	6	8, 10
0	7	8, 10
0	8	8, 9
1	2	$\equiv 2, 4 \pmod{6}, 6, 18$
2	3	4, 8, 16

Theorem 11. *A sun with one ray is always avd.*

A sun with two rays $\text{Sun}(a, b)$ is on-line avd if and only if a and b take values given in Table 2.

A sun with three rays $\text{Sun}(a, b, c)$ is on-line avd if and only if a and b take values given in Table 3.

A sun with four rays is on-line avd if and only if it is of the form $\text{Sun}(0, 0, 1, d)$ with $d \equiv 2, 4 \pmod{6}$.

A sun with five or more rays is never on-line avd.

The proof of this theorem is arduous and too long to be presented in this paper. We refer the reader to [7].

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