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On-line arbitrarily vertex decomposable suns

Rafał Kalinowski *, Monika Pilśniak, Mariusz Woźniak, Irmina A. Zioło

Department of Applied Mathematics, AGH University of Science and Technology, Cracow, Poland

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ABSTRACT

We give a complete characterization of on-line arbitrarily vertex decomposable graphs in the family of unicycle graphs called suns. A *sun* is a graph with maximum degree three, such that deleting vertices of degree one results in a cycle. This result has already been used in another paper to prove some Ore-type conditions for on-line arbitrarily decomposable graphs.

1. Arbitrary vertex decomposability

We deal with finite, simple and undirected graphs. For any two integers p, q with $p \leq q$, we write $[p, q] = \{p, p+1, \dots, q\}$. By $(d)^k$ we denote a k-element sequence $(d, \dots, d)$.

Let $G = (V, E)$ be a graph of order n. A sequence $\tau = (n_1, \dots, n_k)$ of positive integers is called *admissible for G* if it sums up to n. If $\tau = (n_1, \dots, n_k)$ is an admissible sequence for G and there exists a partition $(V_1, \dots, V_k)$ of the vertex set V such that V_i induces a connected subgraph of order n_i, for every $i \in [1, k]$, then τ is called *realizable in G*, and the partition $(V_1, \dots, V_k)$ is said to be *a realization of τ in G*. A graph G is *arbitrarily vertex decomposable* (we will use the abbreviation *avd*), if each admissible sequence τ has a realization in G.

Arbitrarily vertex decomposable graphs have been investigated in a couple of papers (cp. [1–9]). The notion of avd graphs was first introduced by Barth et al. [1] in connection with the following problem in computer science. Consider a network connecting different computing resources; such a network is modeled by a graph. Suppose there are k different users, where the ith one needs n_i resources in our graph. The subgraph induced by the set of resources attributed to each user should be connected and each resource should be attributed to exactly one user. Thus, we are seeking a realization of the sequence $\tau = (n_1, \dots, n_k)$ in this graph. Hence, such a network should be an avd graph.

A *tripode* $S(a_1, a_2, a_3)$ is a tree homeomorphic to the star $K_{1,3}$ obtained from $K_{1,3}$ by substituting its edges by paths of orders a_1, a_2 and a_3, respectively. In particular, the tripode $S(2, a, b)$ is a caterpillar of order $n = a + b$ with three hanging vertices (leaves), and we denote it by $\mathrm{Cat}(a, b)$, assuming that $2 \leq a \leq b$. Barth et al. [1] showed that determining whether a tripode is avd can be done by a polynomial algorithm.

The first result characterizing some avd caterpillars was found by Barth et al. [1], and independently by Horňák and Woźniak [5]:

Theorem 1. *A caterpillar* $\mathrm{Cat}(a, b)$ *is arbitrarily vertex decomposable if and only if* a *and* b *are coprime. Moreover, each admissible and non-realizable sequence in* $\mathrm{Cat}(a, b)$ *is of the form* $(d)^k$*, where* $d > 1$*, and* $a \equiv b \equiv 0 \pmod d$.

* Corresponding author.
E-mail addresses: kalinows@agh.edu.pl (R. Kalinowski), pilsniak@agh.edu.pl (M. Pilśniak), mwozniak@agh.edu.pl (M. Woźniak), ziolo@agh.edu.pl (I.A. Zioło).

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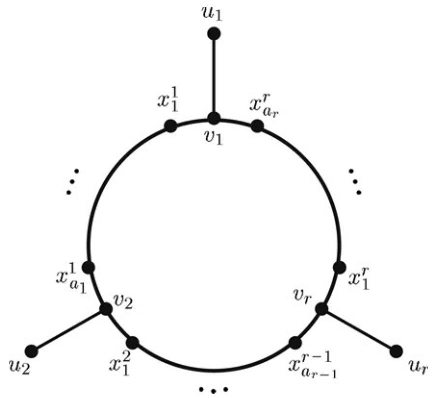

Fig. 1. Sun$(a_1, \ldots, a_r)$ with r rays.

Some results concerning avd caterpillars with more hanging vertices can be found in [2] and [3]. It turns out that the structure of avd caterpillars is not obvious in general. Anyway, Barth and Fournier showed in [2] that, for every $s \geq 3$, there exist avd caterpillars with s hanging vertices.

Observe that the independence number $\alpha(G)$ of any avd graph G of order n cannot be greater than $\lceil n/2 \rceil$. Indeed, if $n = 2k$ is even, then G admits a perfect matching, i.e. a realization of the sequence $(2)^k$, and therefore $\alpha(G) \leq n/2$. If n is odd, then G contains a realization of the sequence $(2, \ldots, 2, 1)$, called a quasi-perfect matching, so $\alpha(G) \leq (n+1)/2$.

It is clear that each path, and therefore each traceable graph, is avd. So, the problems concerning avd graphs can be considered as a generalization of hamiltonian problems. Some interesting Ore-type conditions for avd graphs have been recently found by Marczyk. Two of them follow. Let $\sigma_2(G)$ denote the smallest sum of degrees of two non-adjacent vertices in a graph G.

Theorem 2 (*Marczyk [9]*). *Let G be a connected graph of order n. If $\sigma_2(G) \geq n - 2$ then G is arbitrarily vertex decomposable, unless $\alpha(G) = \frac{n+2}{2}$.*

Theorem 3 (*Marczyk [9]*). *Let G be a 2-connected graph of order n. If $\alpha(G) \leq \lceil n/2 \rceil$ and $\sigma_2(G) \geq n - 3$ then G is arbitrarily vertex decomposable.*

These Ore-type results of Marczyk were obtained by use of some characterizations (Theorems 4 and 5 below) of avd graphs among a family of graphs with a large dominating cycle, called suns. This spurs on and justifies investigations of arbitrary vertex decomposability within this family of graphs.

A *sun with r rays* is a graph of order $n \geq 2r$ with r hanging vertices $u_1, \ldots, u_r$ whose deletion yields a cycle C_{n-r}, and each vertex v_i on C_{n-r} adjacent to u_i is of degree three. If the sequence of vertices v_i is situated on the cycle C_{n-r} in such a way that there are exactly $a_i \geq 0$ vertices of degree two between v_i and v_{i+1} ($i \in [1, r]$, and the indices are to be read modulo r), then such a sun is denoted by Sun$(a_1, \ldots, a_r)$, and it is unique up to isomorphism (cp. Fig. 1).

Clearly, every sun with one ray is avd since it is traceable. For suns with two or three rays the authors of the present paper proved the following results in [8].

Theorem 4. Sun(a, b) *with two rays is arbitrarily vertex decomposable if and only if either its order n is odd, or both a and b are even. Moreover, if Sun (a, b) is not avd then $(2)^{n/2}$ is the unique admissible and non-realizable sequence.*

Theorem 5. Sun(a, b, c) *with three rays is arbitrarily vertex decomposable if and only if none of the following three conditions is fulfilled:*

(1) *at most one of the numbers a, b, c is even,*
(2) $a \equiv b \equiv c \equiv 0 \pmod 3$,
(3) $a \equiv b \equiv c \equiv 2 \pmod 3$.

2. On-line arbitrary vertex decomposability

The notion of an on-line arbitrarily vertex decomposable graph has been introduced by Horňák, Tuza and Woźniak in [4]. This version of the problem is even more natural when applied to the problem in computer networks mentioned in Section 1.

The definition of on-line avd graphs is quite natural, but we now formulate it for completeness. We are given a graph G of order n. Imagine the following decomposition procedure consisting of k stages, where k is a random variable attaining integer values from $[1, n]$. In the ith stage, where $i = 1, \ldots, k$, a positive integer n_i arrives and we have to choose a connected subgraph G_i of G of order n_i that is vertex-disjoint from all subgraphs chosen in the previous stages (without a possibility of

Table 1
Values a, b such that Cat (a, b) is on-line avd.

a	b
2	$\equiv 1 \pmod 2$
3	$\equiv 1, 2 \pmod 3$
4	$\equiv 1 \pmod 2$
5	6, 7, 9, 11, 14, 19
6	$\equiv 1, 5 \pmod 6$
7	8, 9, 11, 13, 15
8	11, 19
9	11
10	11
11	12

Table 2
Values a, b such that Sun (a, b) is on-line avd.

a	b
0	Arbitrary
1	$\equiv 0 \pmod 2$
2	$\not\equiv 3 \pmod 6$, 3, 9, 21
3	$\equiv 0 \pmod 2$
4	$\equiv 2, 4 \pmod 6$, $[4, 19] \setminus \{15\}$
5	$\equiv 2, 4 \pmod 6$, 6, 18
6	6, 7, 8, 10, 11, 12, 14, 16
7	8, 10, 12, 14, 16
8	8, 9, 10, 11, 12
9	10, 12

changing our choice in the future). More precisely, if a graph G_j of order n_j has been already chosen in the jth stage, for all $j \leq i - 1$, and $n_i \in [1, n - \sum_{j=1}^{i-1} n_j]$, then G_i has to be chosen as a connected subgraph of $G - \bigcup_{j=1}^{i-1} G_j$. If the decomposition procedure can be accomplished for any sequence of positive integers $\tau = (n_1, \ldots, n_k)$ summing up to n, then G is said to be *on-line arbitrarily vertex decomposable* (*on-line avd*, for short).

Obviously, every on-line avd graph is also avd, and every necessary condition for avd graphs is also necessary for on-line avd ones.

The definition immediately implies the following two observations.

Proposition 6 (*Horňák, Tuza and Woźniak [4]*). *For any graph G of order n, the following two conditions are equivalent:*

(i) *G is on-line avd;*
(ii) *for each $n_1 \in [1, n - 1]$ there exists a connected subgraph G_1 of order n_1 such that the graph $G - G_1$ is on-line avd.*

Proposition 7. *If G is on-line avd then adding any new edge results in an on-line avd graph.* ■

It seems that the characterization of avd trees is very difficult. The situation is different in the case of on-line avd trees. The theorem below provides their complete list.

Theorem 8 (*Horňák, Tuza and Woźniak [4]*). *A tree T is on-line avd if and only if T is either a path, or the tripode $S(3, 5, 7)$, or else a caterpillar* Cat (a, b) *with a and b given in Table 1.*

Note in particular, that there does not exist an on-line avd tree with more than three hanging vertices.

3. On-line arbitrary vertex decomposable suns

Similarly as in the case of trees, the complete characterization of avd suns seems to be very difficult, but in the case of on-line avd suns the problem turns out to be much easier. The theorem below, already announced in [8], provides a complete list of on-line avd suns.

Theorem 9. 1. *A sun with one ray is always avd.*
2. *A sun with two rays* Sun (a, b) *is on-line avd if and only if a and b take values given in Table 2.*
3. *A sun with three rays* Sun (a, b, c) *is on-line avd if and only if a and b take values given in Table 3.*
4. *A sun with four rays is on-line avd if and only if it is isomorphic to* Sun $(0, 0, 1, d)$, *where* $d \equiv 2, 4 \pmod 6$.
5. *A sun with five or more rays is never on-line avd.*

Before the proof, let us formulate some Ore-type results obtained by the first author in [7] as a consequence of Theorem 9.

Table 3
Values a, b, c such that Sun (a, b, c) is on-line avd.

a	b	c
0	0	$\equiv 1, 2 \pmod 3$
0	1	$\equiv 0 \pmod 2$
0	2	$\equiv 2, 4 \pmod 6$, 3, 6, 7, 11, 18, 19
0	3	$\equiv 2, 4 \pmod 6$
0	4	4, 5, 6, 8, 10, 11, 12, 14, 16
0	5	6, 8,16
0	6	8, 10
0	7	8, 10
0	8	8, 9
1	2	$\equiv 2, 4 \pmod 6$, 6, 18
2	3	4, 8, 16

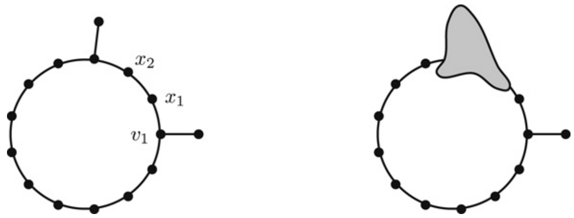

Fig. 2. Sun $(2, 9)$ and the caterpillar Cat $(2, 10)$ as the subgraph First $(3, x_2)$.

Theorem 10. *Let G be a connected graph with $\sigma_2(G) \geq n - 2$. If n is odd, then G is on-line arbitrarily vertex decomposable. If n is even and $\alpha(G) \neq \frac{n+2}{2}$, then G is traceable.*

Theorem 11. *Let G be a 2-connected graph with $\alpha(G) \leq \lceil \frac{n}{2} \rceil$ and $\sigma_2(G) \geq n - 3$. If n is odd, then G is on-line arbitrarily vertex decomposable. If n is even, then G is traceable.*

Proof of Theorem 9. The main tool of the proof is Proposition 6. Given n_1, we have either to define a connected subgraph G_1 of order n_1, such that the graph $G' = G - G_1$ is on-line avd, or to show that it is impossible for any choice of G_1. Obviously, the graph G' must be connected.

The action of defining G_1 will be described in the following way. We assume that the *natural orientation* of the unique cycle of the sun $G = \text{Sun}(a_1, \ldots, a_r)$ is the orientation from v_1 through $v_2, \ldots, v_r$ in this direction. Then, First (n_1, w) will denote a graph First $(n_1, w) = G - G_1$ where G_1 is the subgraph of G induced by n_1 consecutive vertices of G with the vertex w as the first one, where *consecutive* is understood with respect to the natural orientation of the cycle with the convention that the leaf u_i is just after v_i. An example is given in Fig. 2.

A sun with one ray is traceable. So, a fortiori, it is on-line avd.

Let Sun(a, b) be **a sun with two rays**. Without loss of generality we may assume that $a \leq b$. Denote the vertices of the unique cycle of Sun (a, b) as $v_1, x_1, \ldots, x_a, v_2, y_1, \ldots, y_b$ and consider the orientation of this cycle according to the above sequence. We may also assume that $n_1 \geq 2$ since deleting a leaf yields a sun with one ray. The proof will be divided into cases corresponding to the rows in Table 2.

Case a = 0. Then $b \geq 1$. As Sun $(0, b)$ is traceable, the value of b does not matter.

Case a = 1. If Sun$(1, b)$ is on-line avd then, by Theorem 4, b has to be even. On the other hand, Sun $(1, b)$ can be obtained from the caterpillar Cat $(2, b + 3)$ by adding an edge. So, our assertion is implied by Proposition 7 and Theorem 8.

Case a = 2. Suppose that Sun $(2, b)$ is on-line avd. If $n_1 = 2$ or $n_1 \geq 4$, then First (n_1, x_1) is a path. So it remains to consider $n_1 = 3$. Taking into account that there does not exist an on-line avd tree with four leaves, and the subgraph First $(3, w)$ has to be connected, we have the following possible actions:

(1) First $(3, v_1)$ = First $(3, x_2)$ = Cat $(2, b + 1)$ is on-line avd for b even,
(2) First $(3, v_2)$ = First $(3, y_b)$ = Cat $(3, b)$ is on-line avd for $b \equiv 1, 2 \pmod 3$,
(3) First (y_1) = First (y_{b-2}) = Cat $(5, b - 2)$. By Theorem 8, this tree is on-line avd when $b - 2 = 2, 3, 4, 6, 7, 9, 11, 14, 19$ (note that for $b - 2 < 5$, we get in fact Cat $(b - 2, 5)$). Moreover, if $b = 3$ then First $(3, y_1)$ is a path.

The first two of the above possibilities give together the condition $b \equiv 0, 1, 2, 4, 5 \pmod 6$. The third possibility provides additionally the values: 3, 9 and 21.

Case a = 3. If Sun $(3, b)$ is on-line avd then, by Theorem 4, the number b is even. Note that Sun $(3, b)$ can be obtained from the caterpillar Cat $(4, b + 3)$ by adding an edge. So, our assertion is instantly implied by Proposition 7 and Theorem 8.

Case a = 4. Suppose Sun $(4, b)$ is on-line avd. If $n_1 = 2$ then, up to isomorphism, we have only three possible actions:

(1) First $(2, x_1)$ = Cat $(3, b + 3)$ is on-line avd for $b \equiv 1, 2 \pmod 3$;

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(2) First $(2, v_1) = \text{Cat}\,(5, b+1)$ is on-line avd for $b = 5, 6, 8, 10, 13, 18$;
(3) First $(2, y_1) = \text{Cat}\,(7, b-1)$ is on-line avd for $b = 4, 5, 6, 7$ and for $b = 9, 10, 12, 14, 16$.

If $n_1 = 3$ then four actions are admissible:

(1) First $(3, x_1) = \text{Cat}\,(2, b+3)$ is on-line avd for b even;
(2) First $(3, x_4) = \text{Cat}\,(4, b+1)$ is again on-line avd for b even;
(3) First $(3, y_b) = \text{Cat}\,(5, b)$ is on-line avd for $b = 4, 6, 7, 9, 11, 14, 19$;
(4) First $(3, y_1) = \text{Cat}\,(7, b-2)$ is on-line avd for $b = 4, 5, 6, 7, 8$ and $b = 10, 11, 13, 15, 17$.

Thus, the cases $n_1 = 2$ and $n_1 = 3$ imply that either $b \equiv 2, 4 \pmod 6$ for $b \geq 20$ or $b \in [4, 19] \setminus \{15\}$. We shall show that Sun $(4, b)$ is on-line avd for all these values of b. It is easy to see that, for $n_1 = 4$ and for $n_1 \geq 6$, First (n_1, x_1) is a path. So, it remains to consider $n_1 = 5$. Let us observe that First $(5, x_2) = \text{Cat}\,(2, b+1)$ is on-line avd for even b. Furthermore, First $(5, x_3) = \text{Cat}\,(3, b)$ is on-line avd for all remaining odd b, except for $b = 9$. In the latter case we use First $(5, v_2) = \text{Cat}\,(5, 7)$.

Case a = 5. If Sun $(5, b)$ is on-line avd then, by Theorem 4, b should be even. If $n_1 = 2$ then First $(2, x_1) = \text{Cat}\,(4, b+3)$ is on-line avd for every even b. For $n_1 = 3$, we have

(1) First $(3, x_3) = \text{Cat}\,(3, b+3)$ is on-line avd for $b \equiv 1, 2 \pmod 3$;
(2) First $(3, x_5) = \text{Cat}\,(5, b+1)$ is on-line avd for $b = 6, 8, 10, 18$ (we consider only even b);
(3) First $(3, y_b) = \text{Cat}\,(6, b)$ is never on-line avd for b even;
(4) First $(3, y_1) = \text{Cat}\,(8, b-2)$ is never on-line avd for b even.

We infer that $b \equiv 1, 2 \pmod 3$ or $b \in \{6, 18\}$ is the necessary condition for b. It comes out that this is also sufficient. Indeed, if $n_1 = 4$ then First $(4, x_1) = \text{Cat}\,(2, b+3)$ is on-line avd (for all even b). If $n_1 = 5$ or $n_1 \geq 7$, then First (n_1, x_1) is a path. Finally, First $(6, x_2) = \text{Cat}\,(2, b+1)$ is on-line avd.

Case a = 6. If $n_1 = 2$, there are three possible actions:

(1) First $(2, x_5) = \text{Cat}\,(5, b+3)$ is on-line avd for $b = 6, 8, 11, 16$;
(2) First $(2, v_2) = \text{Cat}\,(7, b+1)$ is on-line avd for $b = 7, 8, 10, 12, 14$;
(3) First $(2, y_1) = \text{Cat}\,(9, b-1)$ is on-line avd for $b = 6, 8$ and for $b = 12$.

Let $n_1 = 3$. We have now four chances:

(1) First $(3, x_4) = \text{Cat}\,(4, b+3)$ is on-line avd for b even;
(2) First $(3, x_6) = \text{Cat}\,(6, b+1)$ is on-line avd for $b \equiv 0, 4 \pmod 6$;
(3) First $(3, v_2) = \text{Cat}\,(7, b)$ is on-line avd for $b = 8, 9, 11, 13, 15$;
(4) First $(3, y_1) = \text{Cat}\,(9, b-2)$ is on-line avd for $b = 6, 7, 9$ and $b = 13$;

Hence, the cases $n_1 = 2$ and $n_1 = 3$ imply that either b is even with $6 \leq b \leq 16$, or $b = 7, 11$. Observe that the suns Sun $(6, 6)$, Sun $(6, 8)$, Sun $(6, 10)$ and Sun $(6, 12)$ can be obtained by adding an edge to the on-line avd trees Cat $(7, 9)$, Cat $(7, 11)$, Cat $(7, 13)$ and Cat $(7, 15)$, respectively. So, it remains to investigate four subcases: $b = 7, 11, 14, 16$. Of course, it suffices to consider $n_1 = 4, 5, 7$ since for $n_1 = 6$ and for $n_1 \geq 8$ the subgraph First (n_1, x_1) is always a path.

For $n_1 = 4$, the subgraph First $(4, x_3)$ is isomorphic to Cat $(3, 10)$, Cat $(3, 14)$, Cat $(3, 17)$ and Cat $(3, 19)$, for $b = 7, 11, 14$ and 16, respectively. All these trees are on-line avd. Next, First $(5, x_5)$ gives Cat $(5, 7)$, Cat $(5, 11)$, or Cat $(5, 14)$ for $b = 7, 11, 14$, respectively, and First $(5, x_4) = \text{Cat}\,(4, 17)$ for $b = 16$. Finally, First $(7, y_1)$ is a path for $b = 7$, while First $(7, x_3)$ is Cat $(3, 11)$, Cat $(3, 14)$ or Cat $(3, 16)$ in the remaining cases, respectively.

Case a = 7. Due to Theorem 4, we consider only even b. If $n_1 = 2$ then

(1) First $(2, x_6) = \text{Cat}\,(6, b+3)$ is on-line avd for $b \equiv 2, 4 \pmod 6$;
(2) First $(2, v_2) = \text{Cat}\,(8, b+1)$ is on-line avd for $b = 10, 18$;
(3) First $(2, y_1) = \text{Cat}\,(10, b-1)$ is on-line avd for $b = 12$.

For $n_1 = 3$, only four actions are admissible:

(1) First $(3, x_5) = \text{Cat}\,(5, b+3)$ is on-line avd for $b = 8, 16$;
(2) First $(3, x_7) = \text{Cat}\,(7, b+1)$ is on-line avd for $b = 8, 10, 12, 14$;
(3) First $(3, v_2) = \text{Cat}\,(8, b)$ is on-line avd for $b = 11, 19$;
(4) First $(3, y_1) = \text{Cat}\,(9, b-1)$ is on-line avd for $b = 8, 12$.

The cases $n_1 = 2$ and $n_1 = 3$ thus imply that possible values for b are $8, 10, 12, 14, 16$. Observe that Sun $(7, 8)$, Sun $(7, 10)$ and Sun $(7, 16)$ can be obtained by adding an edge to the on-line avd trees Cat $(8, 11)$, Cat $(10, 11)$ and Cat $(8, 19)$, respectively. Therefore only two cases $b = 12$ and $b = 14$ are left. Of course, it suffices to consider $n_1 = 4, 5, 6, 8$.

First $(4, x_4) = \text{Cat}\,(4, b+3)$ is on-line avd for both values of b. For $b = 14$, First $(5, x_3) = \text{Cat}\,(3, 17)$ is on-line avd while, for $b = 12$, First $(5, x_7) = \text{Cat}\,(7, 11)$ is on-line avd, too. First $(6, x_2) = \text{Cat}\,(2, b+3)$ and First $(8x_2) = \text{Cat}\,(2, b+1)$ are on-line avd for both values of b.

Case a = 8. If $n_1 = 2$ then

(1) First $(2, x_7) = \text{Cat}\,(7, b+3)$ is on-line avd for $b = 8, 10, 12$;

(2) First $(2, v_2) = \text{Cat}(9, b+1)$ is on-line avd for $b = 10$;
(3) First $(2, y_1) = \text{Cat}(11, b-1)$ is on-line avd for $b = 8, 9, 10, 11, 13$.

Hence, it is enough to consider $b \in [8, 13]$. If $n_1 = 3$, we have

(1) First $(3, x_6) = \text{Cat}(6, b+3)$ is on-line avd for $b = 8, 10$;
(2) First $(3, x_8) = \text{Cat}(8, b+1)$ is on-line avd for $b = 10$;
(3) First $(3, v_2) = \text{Cat}(9, b)$ is on-line avd for $b = 11$;
(4) First $(3, y_1) = \text{Cat}(11, b-2)$ is on-line avd for $b = 8, 9, 10, 11, 12$.

Thus, the cases $n_1 = 2$ and $n_1 = 3$ imply that the possible values for b are 8, 9, 10, 11, 12. Observe that the suns Sun (8, 8), Sun (8, 9) and Sun (8, 11) can be obtained by adding an edge to the on-line avd trees Cat (9, 11), Cat (10, 11) and Cat (11, 12), respectively. Hence, we only have to examine $b = 10$ and $b = 12$. Below, we verify the situations with $4 \le n_1 \le 9, n_1 \ne 8$.
First $(4, x_7)$ gives either Cat (7, 11) or Cat (7, 13).
First $(5, x_6)$ gives either Cat (6, 11) or Cat (6, 13).
First $(6, x_7)$ gives either Cat (7, 9) or Cat (7, 11).
First $(7, x_4)$ gives either Cat (4, 11) or Cat (4, 13).
First $(9, x_2)$ gives either Cat (2, 11) or Cat (2, 13).
All these 10 trees are on-line avd.

Case a = 9. Suppose that Sun $(9, b)$ is on-line avd. By Theorem 4, we consider only even $b \ge 10$. For $n_1 = 2$, there are only three admissible actions:

(1) First $(2, x_8) = \text{Cat}(8, b+3)$ is on-line avd for $b = 16$.
(2) First $(2, v_2) = \text{Cat}(10, b+1)$ is on-line avd for $b = 10$;
(3) First $(2, y_1) = \text{Cat}(12, b-1)$ is on-line avd for $b = 12$.

Let $n_1 = 3$. First $(3, x_7) = \text{Cat}(7, b+3)$ is on-line avd for $b = 10, 12$. It comes out that none of the other possible trees, namely First $(3, x_9) = \text{Cat}(9, b+1)$, First $(3, v_2) = \text{Cat}(10, b)$ and First $(3, y_1) = \text{Cat}(12, b-2)$, is on-line avd for $b = 16$. Therefore $b \in \{10, 12\}$. Observe that Sun (9, 10) can be obtained by adding an edge to the on-line avd tree Cat (11, 12).

Hence, assume that $b = 12$. If $n_1 = 9$ or $n_1 \ge 11$ then First (n_1, x_1) is a path. For the remaining values of n_1, we can also construct suitable on-line avd trees. Namely, First $(4, v_2) = \text{Cat}(10, 11)$, First $(5, x_7) = \text{Cat}(7, 13)$, First $(6, x_4) = \text{Cat}(4, 15)$, First $(7, x_7) = \text{Cat}(7, 11)$, First $(8, x_2) = \text{Cat}(2, 15)$ and First $(10, x_2) = \text{Cat}(2, 13)$.

Case a = 10. If $n_1 = 2$, there are three chances:

(1) First $(2, x_9) = \text{Cat}(9, b+3)$ is on-line avd for $b = 8$;
(2) First $(2, v_2) = \text{Cat}(11, b+1)$ is on-line avd for $b = 11$;
(3) First $(2, y_1) = \text{Cat}(13, b-1)$ is not on-line avd for $b \ge 10$.

If $n_1 = 3$, we have four possible actions:

(1) First $(3, x_8) = \text{Cat}(8, b+3)$ is on-line avd for $b = 16$;
(2) First $(3, x_{10}) = \text{Cat}(10, b+1)$ is on-line avd for $b = 10$;
(3) First $(3, v_2) = \text{Cat}(11, b)$ is on-line avd for $b = 12$;
(4) First $(3, y_1) = \text{Cat}(13, b-2)$ is not on-line avd for $b \ge 10$.

The intersection of the sets of possible values of b for $n_1 = 2$ and $n_1 = 3$ is empty.

Case a = 11. Suppose that Sun $(11, b)$ is on-line avd. This implies that b is even. So, in particular, $b \ge 12$. If $n_1 = 3$ then we have only four choices: First $(3, x_9) = \text{Cat}(9, b+3)$, First $(3, x_{11}) = \text{Cat}(11, b+1)$, First $(3, v_2) = \text{Cat}(12, b)$, and First $(3, y_1) = \text{Cat}(14, b-2)$. None of these trees is on-line avd for admissible values of b.

Case $a \ge 12$. Suppose that the sun Sun (a, b) is on-line avd. Recall that $b \ge a$. Suppose $n_1 = 2$. We have three possible actions giving either Cat $(a-1, b+3)$, or Cat $(a+1, b+1)$ or else Cat $(a+3, b-1)$. The smallest possible values for $a-1$ is 11. But then $b+3$ should equal 12, what is impossible.

Now, let Sun (a, b, c) be an on-line avd **sun with three rays.** Without loss of generality we can assume that $a \le b \le c$, and the sequence

$$v_1, x_1, \dots, x_a, v_2, y_1, \dots, y_b, v_3, z_1, \dots, z_c, v_1$$

defines the orientation of the unique cycle of the sun being considered. We divide the investigation of suns with three rays into cases according to the values of a (see Table 3).

Case a = 0. First observe that the deletion of a vertex u_3 results in a traceable graph Sun$(0, b+c+1)$. Therefore, we may assume that $n_1 \ge 2$.

If $b = 0$ then $c \equiv 1, 2 \pmod 3$, by Theorem 5. To see that every Sun $(0, 0, c)$ with $c \equiv 1, 2 \pmod 3$ is on-line avd, it suffices to observe that for $n_1 = 2$ and for $n_1 \ge 4$ the subgraph First (n_1, v_2) is a path, while First $(3, v_3)$ is an on-line avd caterpillar Cat $(3, c)$.

Now, let $b = 1$. Hence, c has to be even by Theorem 5. If $n_1 \in \{2, 4\}$ then First $(n_1, v_1) \in \{\text{Cat}(4, c+1), \text{Cat}\{2, c+1)\}$, and First (n_1, v_2) is a path for $n_1 = 3$ and $n_1 \ge 5$. All these trees are on-line avd for every even c.

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Table 4
Case Sun $(0, 2, c)$. Note that the subgraph First $(5, z_{c-2})$ is a path for $c = 3$.

n_1	w	First (n_1, w)	c	Conclusion
2	v_2	Cat $(3, c+3)$	$\equiv 1, 2 \pmod 3$	$c \equiv 1, 2 \pmod 3$ or $c \in \{3, 6, 18\}$
	v_1	Cat $(5, c+1)$	2, 3, 5, 6, 8, 10, 13, 18	
	y_1	Cat $(3, c+3)$	$\equiv 1, 2 \pmod 3$	
3	z_c	Cat $(5, c)$	3, 4, 6, 7, 11, 14, 19	$c \equiv 2, 4 \pmod 6$ or $c \in \{3, 6, 7, 11, 18, 19\}$
	v_2	Cat $(2, c+3)$	$\equiv 0 \pmod 2$	
4	v_2	Path	Any	As above
5	z_c	Cat $(3, c)$	$\equiv 1, 2 \pmod 3$	As above
	v_1	Cat $(2, c+1)$	$\equiv 0 \pmod 2$	
	z_{c-2}	Cat $(5, c-2)$	3	
≥ 6	v_2	Path	Any	As above

Table 5
Case Sun $(0, 3, c)$. By Theorem 5, c has to be even and non-divisible by three, hence $c \equiv 2, 4 \pmod 6$.

n_1	w	First (n_1, w)	c	Conclusion
2	v_2	Cat $(4, c+3)$	Even	$c \equiv 2, 4 \pmod 6$
3	v_2	Cat $(3, c+3)$	$\equiv 1, 2 \pmod 3$	As above
4	v_2	Cat $(2, c+3)$	Even	As above
5	v_2	Path	Any	As above
6	v_1	Cat $(2, c+1)$	Even	As above
≥ 7	v_2	Path	Any	As above

Table 6
Case Sun $(0, 4, c)$.

n_1	w	First (n_1, w)	c	Conclusion
2	v_1	Cat $(7, c+1)$	4, 5, 7, 8, 10, 12, 14	$c \in \{4, 5, 6, 7, 8, 10, 11, 12, 14, 16\}$
	v_2	Cat $(5, c+3)$	4, 6, 8, 11, 16	
3	v_2	Cat $(4, c+3)$	Even	$c \in \{4, 5, 6, 8, 10, 11, 12, 14, 16\}$
	z_c	Cat $(7, c)$	5, 6, 8, 11, 15	
4	y_1	Cat $(3, c+3)$	4, 5, 8, 10, 11, 14, 16	As above
	z_{c-1}	Cat $(7, c-1)$	6, 12	
5	v_2	Cat $(2, c+3)$	Even	As above
	z_{c-2}	Cat $(7, c-2)$	5, 11	
6	v_2	Path	Any	As above
7	v_1	Cat $(2, c+1)$	Even	As above
	z_c	Cat $(3, c)$	5, 11	
≥ 8	v_2	Path	Any	As above

Investigations of suns of the form Sun $(0, b, c)$, for $b = 2, \ldots, 8$, are exhibited in Tables 4–10. To make things more clear, we shall now describe the case $b = 2$ (cp. Table 4). We start with $n_1 = 2$ and we find the values of c for which subgraphs First $(2, w)$ are on-line avd, for all possible vertices w (i.e. $w \in \{v_1, v_2, y_1\}$). Then we conclude that Sun $(0, 2, c)$ can be on-line avd only if either $c \equiv 1, 2 \pmod 3$ or $c \in \{3, 6, 18\}$. From now on, we are interested exclusively in these values of c. Then we take $n_1 = 3$ and examine First $(3, v_2)$ and First $(3, z_c)$. We conclude that either $c \equiv 2, 4 \pmod 6$ or $c \in \{3, 6, 7, 11, 18, 19\}$. Now, it suffices to show that for each such c and each $n_1 \geq 4$ there exists a vertex w such that First (n_1, w) is on-line avd. This is shown in subsequent rows of Table 4.

Next, let $b \geq 9$. For $n_1 = 2$, we can choose either First $(2, v_1)$ or First $(2, v_2)$. First $(2, v_1) =$ Cat $(b+3, c+1)$ is an on-line avd tree with $9 \leq b \leq c$, only if $b = 9$ and $c = 10$ (in the second case, First $(2, v_2) =$ Cat $(b+1, c+3)$ does not give an on-line avd tree). However, it is not possible to delete a connected subgraph of order three from Sun $(0, 9, 10)$ to obtain an on-line avd graph. Thus, there are no on-line avd suns with three rays for $a = 0$ and $b \geq 9$.
Note that for $a \geq 1$ deleting u_3 does not result in a traceable graph. Thus we add another row for $n_1 = 1$ to Tables 11 and 12.

Case a = 1. We search for on-line avd suns of the form Sun $(1, b, c)$. By Theorem 5, both b and c have to be even. The case $b = 2$ is considered in Table 11. If $b \geq 4$, then deleting any connected subgraph of order $n_1 = 2$ creates either at least four leaves or an isolated vertex.

Case a = 2. The smallest possible value for b is 2. If $b = 2$ or $b \geq 4$, then deleting any connected subgraph of order $n_1 = 3$ always creates at least four leaves or an isolated vertex. The only remaining case $b = 3$ is described in Table 12.

Case *a* ≥ 3. Take $n_1 = 2$. This is not difficult to see that in this case there does not exist an on-line avd subgraph of Sun (a, b, c) of order $n - 2$.

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Table 7
Case Sun $(0, 5, c)$. By Theorem 5, c has to be even.

n_1	w	First (n_1, w)	c	Conclusion
2	v_2	Cat $(6, c+3)$	$\equiv 2, 4 \pmod 6$	$c \equiv 2, 4 \pmod 6$ or $c \in \{6, 18\}$
	v_1	Cat $(8, c+1)$	6, 10, 18	
3	v_2	Cat $(5, c+3)$	6, 8, 16	$c \in \{6, 8, 16\}$
	z_c	Cat $(8, c)$	None	
4	v_2	Cat $(4, c+3)$	6, 8, 16	As above
5	v_1	Cat $(5, c+1)$	6, 8	As above
	v_2	Cat $(3, c+3)$	8, 16	
6	v_2	Cat $(2, c+3)$	6, 8, 16	As above
7	v_2	Path	Any	As above
8	v_1	Cat $(2, c+1)$	6, 8, 16	As above
≥ 9	v_2	Path	Any	As above

Table 8
Case Sun $(0, 6, c)$. By Theorem 5, c cannot be divisible by three.

n_1	w	First (n_1, w)	c	Conclusion
2	v_1	Cat $(9, c+1)$	10	$c \in \{8, 10\}$
	v_2	Cat $(7, c+3)$	8, 10	
3	v_2	Cat $(6, c+3)$	8, 10	As above
4	v_1	Cat $(7, c+1)$	8, 10	As above
5	v_2	Cat $(4, c+3)$	8, 10	As above
6	v_2	Cat $(3, c+3)$	8, 10	As above
7	v_2	Cat $(2, c+3)$	8, 10	As above
8	v_2	Path	Any	As above
9	v_1	Cat $(2, c+1)$	8, 10	As above
≥ 10	v_2	Path	Any	As above

Table 9
Case Sun $(0, 7, c)$. By Theorem 5, c has to be even.

n_1	w	First (n_1, w)	c	Conclusion
2	v_1	Cat $(10, c+1)$	10	$c \in \{8, 10, 16\}$
	v_2	Cat $(8, c+3)$	8, 16	
3	v_2	Cat $(7, c+3)$	8, 10	$c \in \{8, 10\}$
	z_c	Cat $(10, c)$	None	
4	v_2	Cat $(6, c+3)$	8, 10	As above
5	v_2	Cat $(5, c+3)$	8	As above
	v_1	Cat $(7, c+1)$	10	
6	v_2	Cat $(4, c+3)$	8, 10	As above
7	v_2	Cat $(3, c+3)$	8, 10	As above
8	v_2	Cat $(2, c+3)$	8, 10	As above
9	v_2	Path	Any	As above
10	v_1	Cat $(2, c+1)$	8, 10	As above
≥ 11	v_2	Path	Any	As above

Table 10
Case Sun $(0, 8, c)$. Note that the subgraph First $(11, z_{c-8})$ is a path for $c = 9$.

n_1	w	First (n_1, w)	c	Conclusion
2	v_1	Cat $(11, c+1)$	8, 9, 11	$c \in \{8, 9, 11\}$
	v_2	Cat $(9, c+3)$	8	
3	v_2	Cat $(8, c+3)$	8	$c \in \{8, 9\}$
	z_c	Cat $(11, c)$	8, 9	
4	z_{c-1}	Cat $(11, c-1)$	8, 9	As above
5	z_{c-2}	Cat $(11, c-2)$	8, 9	As above
6	z_{c-3}	Cat $(11, c-3)$	8, 9	As above
7	z_{c-4}	Cat $(11, c-4)$	8, 9	As above
8	z_{c-5}	Cat $(11, c-5)$	8, 9	As above
9	z_{c-6}	Cat $(11, c-6)$	8, 9	As above
10	v_2	Path	Any	As above
11	v_1	Cat $(2, c+1)$	8	As above
	z_{c-8}	Cat $(11, c-8)$	9	
≥ 12	v_2	Path	Any	As above

Now consider **suns with four rays**. Let Sun(a, b, c, d) be on-line avd of order n. Without loss of generality we may assume that $a \leq b \leq d$.

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Table 11
Case Sun $(1, 2, c)$. By Theorem 5, c has to be even.

n_1	w	First (n_1, w)	c	Conclusion
2	y_1	Cat $(4, c+3)$	Even	$c \equiv 0 \pmod 2$
3	x_1	Cat $(3, c+3)$	$\equiv 1, 2 \pmod 3$	$c \equiv 2, 4 \pmod 6$ or $c \in \{6, 18\}$
	v_1	Cat $(5, c+1)$	2, 6, 8, 10, 18	
4	x_1	Cat $(2, c+3)$	Even	As above
5	x_1	Path	Any	As above
6	v_2	Cat $(2, c+1)$	Even	As above
≥ 7	x_1	Path	Any	As above
1	u_2	Sun $(4, c)$	$\equiv 2, 4 \pmod 6$, $[4, 19] \setminus \{15\}$	As above

Table 12
Case Sun $(2, 3, c)$.

n_1	w	First (n_1, w)	c	Conclusion
2	x_1	Cat $(6, c+3)$	$\equiv 2, 4 \pmod 6$	$c \equiv 2, 4 \pmod 6$
3	y_1	Cat $(5, c+3)$	4, 8, 16	$c \in \{4, 8, 16\}$
4	x_1	Cat $(4, c+3)$	Even	As above
5	v_2	Cat $(3, c+3)$	4, 8, 16	As above
6	x_2	Cat $(2, c+3)$	Even	As above
7	x_1	Path	Any	As above
8	v_1	Cat $(2, c+1)$	Even	As above
≥ 9	x_1	Path	Any	As above
1	u_3	Sun $(2, c+4)$	$\not\equiv 5 \pmod 6$, 5, 17	as above

Suppose that $n_1 = 2$. Then G_1 must contain at least one leaf, say u_i, of G (for there is no on-line avd trees with four leaves). This implies that v_i, the neighbour of u_i, should be adjacent to two vertices of degree three. Otherwise, a new leaf would be created. Then it is easy to see that the sun should be of the form Sun $(0, 0, c, d)$, hence $n = c + d + 8$. Now, without loss of generality we can assume that $c \leq d$.

Since the removing of G_1 cannot create neither a new leaf nor an isolated vertex, it is easy to see (taking $n_1 = 3$) that $c \leq 1$. Consider first the case $c = 0$. If $d = 0$ then there is no possible action for $n_1 = 3$. If $d = 1$ then the sequence $(3, 3, 3)$ is not realizable. If $d \geq 2$ then for $n_1 = 3$ each action gives a tree with at least four leaves which is not on-line avd.

So, we may assume that $c = 1$. In this case d should be even, because otherwise the sequence $(2)^{n/2}$ would not be realizable. Let us observe next that the sequence $(3)^{n/3}$, if admissible, also cannot be realized. So it cannot be admissible, therefore n cannot be divisible by three. Since d is even, we have $d \equiv 2, 4 \pmod 6$, and such suns are on-line avd. Indeed, each of the following graphs is on-line avd: First $(1, u_3)$ = Sun $(0, 2, d)$, First $(2, v_2)$ = Cat $(4, d+3)$, First $(3, v_3)$ = Cat $(3, d+3)$, First $(4, v_1)$ = Cat $(4, d+1)$, a path First $(5, v_2)$, and First $(6, v_1)$ = Cat $(2, d+1)$. For $n_1 \geq 7$, we remove the subtree having the vertex between v_3 and v_4 as its last vertex and we get a path.

Consider finally the **suns with at least five rays**. Let $n_1 = 2$. For any action we get a tree with at least four leaves. Such a tree cannot be on-line avd. This ends the proof of Theorem 9. ■

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