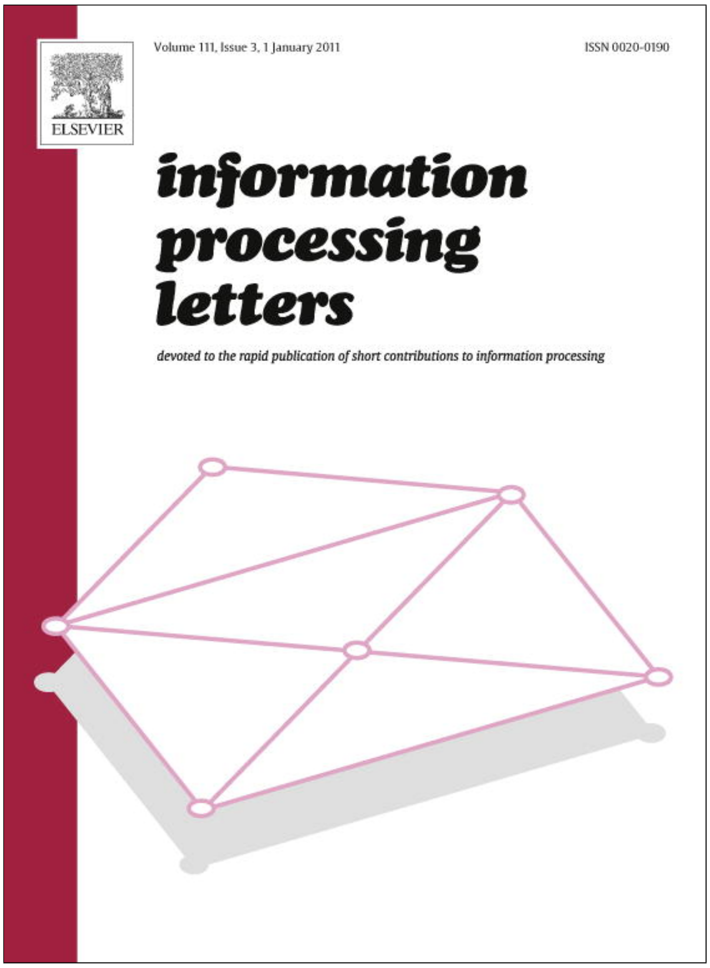
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Coloring chip configurations on graphs and digraphs

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ABSTRACT

Let D be a simple directed graph. Suppose that each edge of D is assigned with some number of chips. For a vertex v of D, let $q^+(v)$ and $q^-(v)$ be the total number of chips lying on the arcs outgoing form v and incoming to v, respectively. Let $q(v) = q^+(v) - q^-(v)$. We prove that there is always a chip arrangement, with one or two chips per edge, such that $q(v)$ is a proper coloring of D. We also show that every undirected graph G can be oriented so that adjacent vertices have different balanced degrees (or even different in-degrees). The arguments are based on peculiar chip shifting operation which provides efficient algorithms for obtaining the desired chip configurations. We also investigate modular versions of these problems. We prove that every k-colorable digraph has a coloring chip configuration modulo k or $k+1$.

1. Introduction

Let G be a simple connected graph with at least two edges. Suppose that each edge of G is assigned with one, two, or three chips. For a vertex v, let $q(v)$ denote the total number of chips lying on the edges incident to v. Is it possible to place the chips so that $q(u)$ is different from $q(v)$ for every pair of adjacent vertices u, v?

This innocently looking question was posed by Karoński, Łuczak, and Thomason [7] as a variant of the *irregularity strength* of a graph (where all numbers $q(v)$, not just for adjacent vertices, are to be different). Despite some efforts using various methods [1,2,7] the question remains open. Currently best result asserts that positive solution exists if we allow up to five chips per edge [6]. The proof gives an efficient algorithm for obtaining the desired arrangement of chips. The main idea appeared first in a slightly modified version of the problem, proposed in [8], in which chips are placed on the edges as well as on the vertices of G (with $q(v)$ denoting the total number of chips lying on the edges incident to v and on the vertex v itself). It is conjectured [8] that appropriate chip configuration is now possible for every graph G with just one or two chips per every edge and every vertex. In [5] Kalkowski proved that this holds if we allow up to three chips per edge (with previous restriction for vertices). A slight refinement of his simple, ingenious argument is used in the proof of Theorem 5.

In this paper we deal with directed version of the problem. Let D be a simple directed graph. Suppose that each edge of D is assigned with some number of chips. For a vertex v of D, let $q^+(v)$ and $q^-(v)$ be the total number of chips lying on the edges outgoing from v and incoming to v, respectively. Let $q(v) = q^+(v) - q^-(v)$. We prove that there is always a chip arrangement, with one or two chips per edge, such that $q(v)$ is a proper coloring of D (the ends of every edge get distinct values of the function $q(v)$). We also show that every undirected graph G can be oriented so that just one chip per edge suffices (which means that balanced degrees of adjacent vertices are different). Similar argument gives an orientation in which neighbors are distinguished only by in-degrees. We also investigate modular version of the problem.

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2. Results

We start by fixing some terminology and notation. Suppose $G = (V, E)$ is a graph with some number of chips occupying the edges and the vertices. Formally, each element $x \in V \cup E$ is assigned with a nonnegative integer $c(x)$. Let $N(v)$ be the set of all neighbors of the vertex v in G. For a fixed configuration of chips, let $q(v) = c(v) + \sum_{u \in N(v)} c(uv)$ be the total number of chips on the edges around v together with those lying on v itself. The value $q(v)$ will be called the *potential* of the vertex v. A chip assignment $c(x)$ forms a *coloring configuration* on a graph G if the function $q(v)$ constitutes a proper coloring of the vertices of G, that is, if $q(u) \neq q(v)$ for every two adjacent vertices u and v of G. By *shifting* a chip we mean either transferring it from the edge it occupies to one of the two end vertices, or moving it from the vertex to one of the incident edges.

Theorem 1. *Let G be a graph with one chip on every edge and no chips on the vertices. Then it is possible to get a coloring configuration on G by shifting every chip exactly once.*

Proof. Start with finding a vertex v with highest potential $q(v)$ (this will be any vertex with maximum degree) and mark it as v_1. Then shift all chips from the incident edges towards v_1. So we have a new configuration on G in which potentials of all neighbors of v_1 decreased by exactly one. In the second step find a new vertex $v \in V(G) \setminus \{v_1\}$ whose potential $q(v)$ (in the new configuration) is the highest, and mark it as v_2. Then shift all remaining chips from the incident edges towards v_2. In the i-th step do the same: find a vertex v in $V \setminus \{v_1, v_2, \ldots, v_{i-1}\}$ with highest potential $q(v)$ (in the actual configuration), mark it as v_i, and shift all possible chips to v_i. Continue in this way until the very last vertex.

We claim that we obtain a coloring configuration at the end of the algorithm. Indeed, suppose $e = v_i v_j$ is an edge of G with $i < j$. This means that $q(v_i) \geqslant q(v_j)$ in the configuration obtained in the step $i - 1$. Notice that the chip from the edge e was counted in both potentials. However, in the i-th step the chip was moved towards v_i. So, the potential of v_j decreased by exactly one (and of course it can never increase). This completes the proof. □

Shifting chips on a graph G leads in a natural way to orientation of the edges of G: simply direct each edge according to the direction of a chip movement.

Corollary 1. *Every graph G can be oriented so that the in-degrees of every two adjacent vertices are different.*

Suppose now that we have two kinds of chips, red and blue, lying on a graph $G = (V, E)$. Formally, we are given two functions $c_r, c_b : V \cup E \to \mathbb{Z}$. Let $q_r(v) = c_r(v) + \sum_{u \in N(v)} c_r(uv)$ denote the number of red chips lying on v and on the edges incident to v. We will call it the *red potential* of a vertex v. Similarly define the *blue potential* $q_b(v)$ of a vertex v. Let $q(v) = q_b(v) - q_r(v)$ be the *difference potential* of v. By a similar chip shifting argument we get the following result.

Theorem 2. *Suppose that each edge of a graph G is assigned with two chips, one red and one blue. Then it is possible to shift each chip exactly once, with chips from the same edge shifted in opposite directions, so that the resulting difference potential is a proper coloring of G.*

Proof. For a fixed red–blue configuration, let $f(v) = q_b(v) - c_r(v)$. Start with finding a vertex v with highest blue potential $q_b(v)$ (this will be any vertex with maximum degree) and put $v_1 = v$. Then shift all blue chips from the incident edges towards v_1 and all red chips in opposite directions. In this way we get new red–blue configuration in which blue potentials of neighbors of v_1 decreased by one. In the second step find a new vertex v in the set $V(G) \setminus \{v_1\}$ with maximum value of the function $f(v)$ (in the new configuration) and mark it as v_2. Then shift all possible blue chips from the incident edges towards v_2 and their red counterparts oppositely. At the i-th step do the same: find a vertex v in $V \setminus \{v_1, v_2, \ldots, v_{i-1}\}$ with highest value of $f(v)$ (in the actual configuration), mark it as v_i, and move all blue chips from the incident edges towards v_i. Continue in this way until the very last vertex.

We claim that we obtain a coloring red–blue configuration at the end of the algorithm. Indeed, suppose $e = v_i v_j$ is an edge of G with $i < j$. This means that $f(v_i) \geqslant f(v_j)$ in the configuration obtained in the step $i - 1$. Notice that the blue chip from the edge e was counted for both vertices. However, in the i-th step this blue chip was moved towards v_i, while the red one floated to v_j. So, the value $f(v_j)$ decreased by exactly two (and it cannot increase later). Finally notice that the final difference potential $q(v_i)$ is equal to $f(v_i)$ computed in the i-th step of the procedure. This completes the proof. □

Let $d^+(v)$ and $d^-(v)$ denote the out-degree and in-degree of a vertex v in a digraph D, respectively. Let $d(v) = d^+(v) - d^-(v)$ denote the *balanced degree* of a vertex v. As before, orienting the edges of a graph G according to the movement of blue chips gives immediately the following result.

Corollary 2. *Every undirected graph G has an orientation in which every two adjacent vertices have different balanced degrees.*

Notice that the two corollaries are logically independent (none of them implies the other). Moreover, there are graphs for which both procedures give orientations satisfying only one of the above properties.

Let $c : V \cup E \to \mathbb{Z}$ be a chip configuration on a digraph $D = (V, E)$. For a vertex $v \in V$, let $E^+(v)$ and $E^-(v)$ denote the set of edges outgoing from v and incoming to v, respectively. Let $q^+(v) = c(v) + \sum_{e \in E^+(v)} c(e)$ be the *outgoing potential* of a vertex v. Similarly define the *incoming potential* $q^-(v)$ by counting chips on v and on the edges incoming to v. Let $q(v) = q^+(v) - q^-(v)$ be the *balanced potential* of the vertex v in the configuration c. By writing $c : E \to \mathbb{Z}$ we mean that there are no chips on the vertices in configuration c.

Theorem 3. *For every directed graph $D = (V, E)$ there is a chip configuration $c : E \to \{1, 2\}$ whose balanced potential is a proper coloring of D.*

Proof. We start with putting one red chip and two blue chips on every edge of D. This configuration will be modified in subsequent steps in accordance with the following two basic rules: (1) blue chips always migrate according to the orientation of the edge they occupy, while red chips in the opposite direction, (2) before shifting chips from an edge e we either add one red chip to e, or subtract one blue chip from e (so, the number of red and blue chips on e right before shifting is the same).

For a fixed red–blue chip configuration on a digraph D, let $q_r^+(v) = c_r(v) + \sum_{e \in E^+(v)} c_r(e)$ denote the *red outgoing potential* of the vertex v. Similarly, let $q_b^-(v) = c_b(v) + \sum_{e \in E^-(v)} c_b(e)$ be the *blue incoming potential* of the vertex v. Finally let $g(v) = q_b^-(v) - q_r^+(v)$.

Now the procedure goes similarly as in the previous proofs. Start with finding a vertex v with maximum value of the function $g(v)$ and mark it as v_1. Then add one red chip to every edge incoming to v_1 and subtract one blue chip from every edge outgoing from v_1. Finally, shift all chips from the edges incident to v_1 according to the rule (1). In the second step find a new vertex v in the set $V(G) \setminus \{v_1\}$ with maximum value of the function $g(v)$ (in the new configuration) and mark it as v_2. Repeat procedure of adding and subtracting chips exactly as for v_1, and next shift them according to rule (1). Similarly in subsequent steps until the very last vertex.

We claim that the function $g(v)$ for the final configuration is a proper coloring of D. Indeed, suppose that $e = (v_i, v_j)$ is a directed edge of D with $i < j$. This means that $g(v_i) \geqslant g(v_j)$ in the configuration obtained in the step $i - 1$. However, in calculating $g(v_i)$ we counted one red chip from e with negative sign, while in calculating $g(v_j)$ we counted two blue chips with positive signs (in view of direction of the edge). But before shifting we subtracted one blue chip from e according to rule (2). Hence, the value $g(v_j)$ decreased by one. Similarly in the other case, where $e = (v_j, v_i)$ with $i < j$; this time we have to add one red chip to the edge e before shifting, which also decreases the value $g(v_j)$ by one. Since $g(v_j)$ cannot grow up in future steps we get the claim.

To obtain a chip configuration satisfying the assertion of the theorem just shift back all red chips and throw away all blue chips (or vice versa). This completes the proof. □

In another variation we look for chip configurations with values in additive group $\mathbb{Z}_m$. Potentials of vertices are defined as before, except that all computations are performed modulo m. In [7] it was proved that for every graph G and for every odd integer $m \geqslant \chi(G)$ there is a coloring configuration $c : E \to \mathbb{Z}_m$. Below we derive a similar statement for directed graphs as a consequence of a more general result.

Theorem 4. *Let $D = (V, E)$ be a directed graph and let $f : V \to \mathbb{Z}_m$ be any function satisfying $\sum_{v \in V} f(v) \equiv 0 \pmod{m}$. Then there is a chip configuration $c : E \to \mathbb{Z}_m$ whose balanced potential satisfies $q(v) = f(v)$ for every vertex $v \in V$.*

Proof. We assume that D is a connected graph, that is, every two vertices are joined by (not necessarily directed) path. Start with putting m chips on every edge of D. If $q(v) \neq f(v)$ we call v a *bad* vertex. We will modify initial configuration until there will be no bad vertices. Suppose that x is a bad vertex. Since for every configuration we have $\sum_{v \in V} q(v) \equiv 0 \pmod{m}$, there must be another bad vertex, say y. Let P be any path joining x to y in D. Let $x = v_0, v_1, \ldots, v_k = y$ be the sequence of consecutive vertices on the path P and let $e_1, e_2, \ldots, e_k$ be the sequence of consecutive edges of P. Start with modifying the number of chips on e_1 so that $q(x) = f(x)$. This may change balanced potential of v_1. But then we may modify the number of chips on the edge e_2 so that the potential of v_1 returns to its previous value. This may influence the potential of v_2, but in that case we modify the number of chips on e_3 so as to bring it back. And so on. In this way we change the configuration so that $q(x)$ has the desired value, and potentials of all other vertices of D, except y, remain unchanged. Clearly this operation reduces the number of bad vertices by at least one. The proof is complete. □

Using the above theorem we easily get the following result.

Corollary 3. *For every digraph $D = (V, E)$ there is a chip configuration $c : E \to \mathbb{Z}_m$, with $m \leqslant \chi(D) + 1$, whose balanced potential forms a proper coloring of D.*

Proof. Let $m = \chi(D)$ be the chromatic number of D and suppose that $f : V \to \{0, 1, \ldots, m - 1\}$ is a proper coloring of D. Let $S = \sum_{v \in V} f(v)$ denote the usual integer sum of all colors. If $S \equiv 0 \pmod{m + 1}$ we are done by the theorem. So, suppose this is not the case and let $S + m \equiv j \pmod{m + 1}$, $0 \leqslant j \leqslant m$. Since S is not zero in $\mathbb{Z}_{m+1}$ we have that $j \in \{0, 1, \ldots, m - 1\}$. Hence there must be a vertex v in D with $f(v) = j$. Switch the color of v into m. Clearly the new sum of colors S' satisfies $S' = S - j + m \equiv 0 \pmod{m + 1}$ which completes the proof. □

Let G be a graph whose vertices are linearly ordered as $v_1, v_2, \ldots, v_n$. A vertex v_i adjacent to v_j, with $i < j$, is a *backward* neighbor of v_j. Let $b(v_j)$ be the number of backward neighbors of v_j. Let $\mathrm{col}(G)$ be the *coloring number* of a graph G, that is, the least integer k such that there is a linear ordering of the vertices of G satisfying $b(v_j) \leqslant k - 1$ for every $j = 1, 2, \ldots, n$. For instance, the coloring number of every planar graph satisfies $\mathrm{col}(G) \leqslant 6$. The proof of the next result is almost entirely based on the idea of Kalkowski from [5].

Theorem 5. *Let $G = (V, E)$ be a graph and let $m = \mathrm{col}(G)$. Then there is a coloring chip configuration $c : V \cup E \to \mathbb{Z}_m$ such that $c(v) \in \{0, 1\}$ and $c(e) \in \{0, 1, 2\}$.*

Proof. First notice that the result is trivial for trees (in fact for all bipartite graphs), so we may assume that $m \geqslant 3$. Start with putting one chip on every edge of G (with no chips on the vertices). Let $v_1, v_2, \ldots, v_n$ be any linear ordering of the vertices of G witnessing that $\mathrm{col}(G) = m$. We will rich a desired configuration by the following chip shifting procedure. In the j-th step we look at the backward neighbors of v_j. Let $v_i v_j \in E$ with $i < j$. At this moment there is one chip on e, as we haven't considered the edge e so far. Let $q(v_j)$ denote the potential of v_j at this moment. If there is a chip on the vertex v_i we may shift it to the edge e or do nothing. Notice that potential of v_i remains the same in both cases, while potential of v_j will be equal to $q(v_j) + 1$ or $q(v_j)$, respectively. If there is no chip on v_i then we may shift the chip from e to v_i or do nothing. This also does not change potential of v_i and gives two possibilities for the new potential of v_j, namely $q(v_j) - 1$ or $q(v_j)$, respectively. Anyway, we have always a choice between two consecutive values for the new potential of v_j, by shifting at only one edge. So, if there are k backward neighbors of v_j we can obtain $k + 1$ different values for the new potential of v_j. Since $k \leqslant m - 1$ there is always a free value for v_j in $\mathbb{Z}_m$ which is different from all potentials of backward neighbors of v_j. This completes the proof. □

3. Problems

We conclude the paper with collecting the most intriguing conjectures in the topic. Maybe some of them could be solved in the near future by using the method of chip configurations.

Conjecture 1. *(See the 123-conjecture, [7].) Every connected graph $G = (V, E)$ (with at least two edges) has a coloring chip configuration $c : E \to \{1, 2, 3\}$.*

Conjecture 2. *(See the 12-conjecture, [8].) Every graph $G = (V, E)$ has a coloring chip configuration $c : V \cup E \to \{1, 2\}$.*

Conjecture 3. *(See antimagic labelings, [4].) For every connected graph $G = (V, E)$ (with at least two edges) there is a bijection $c : E \to \{1, 2, \dots, |E|\}$ such that no two vertices of G have the same potential.*

The last problem we state here concerns a purely vertex version of 123-conjecture. For a chip configuration $c : V \to \{1, 2, \dots, k\}$, let $q(v) = \sum_{u \in N(v)} c(u)$ be the *vertex potential* of a vertex v. How large k must be to guarantee that there is a configuration c whose potential is a proper coloring of G? The case of cliques shows that sometimes $k \geqslant \chi(G)$, but can we always do it with $k = \chi(G)$?

Conjecture 4. *(See Lucky labelings, [3].) For every graph $G = (V, E)$ there is a chip configuration $c : V \to \{1, 2, \dots, \chi(G)\}$ whose vertex potential is a proper coloring of G.*

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