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How to personalize the vertices of a graph?☆

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ABSTRACT

If f is a proper coloring of edges in a graph $G = (V, E)$, then for each vertex $v \in V$ it defines the palette of colors of v, i.e., the set of colors of edges incident with v. In 1997, Burris and Schelp stated the following problem: how many colors do we have to use if we want to distinguish all vertices by their palettes. In general, we may need much more colors than $\chi'(G)$.

In this paper we show that if we distinguish the vertices by color walks emanating from them, not just by their palettes, then the number of colors we need is very close to the chromatic index. Actually, not greater than $\Delta(G) + 1$.

1. Introduction

In this paper we consider only simple graphs and we use the standard notation of graph theory. Definitions not given here may be found in [3]. Let $G = (V, E)$ be a graph of order n with the vertex set V and the edge set E.

An *edge-coloring* f of a graph G is an assignment of colors to the edges of G. The coloring f is *proper* if no two adjacent edges are assigned the same color. In this paper, we consider only proper colorings. The *palette of a vertex* v is the set $S(v) = \{f(uv) : uv \in E\}$ of colors assigned to the edges incident to v. A proper coloring is said to be *vertex-distinguishing* if $S(u) \neq S(v)$ for any two distinct vertices u, v.

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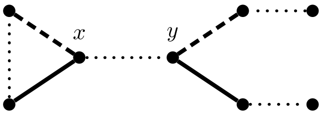

Fig. 1. The vertices x and y are not similar since the sequence (dashed, dotted, continuous line) belongs to $W(x)$ but not to $W(y)$.

Clearly, if G contains more than one isolated vertex or an isolated edge, then such a coloring does not exist. The minimum number of colors required in a vertex-distinguishing coloring of a graph G without isolated edges and with at most one isolated vertex is called the *vertex-distinguishing index* of G and is denoted by $\mathrm{vdi}(G)$.

This invariant was introduced and studied by Burris and Schelp in [4] and, independently, as *observability* of a graph, by Černý, Horňák and Soták in [6].

Among the graphs G of order n, the largest value of $\mathrm{vdi}(G)$ is realized for a complete graph K_n and equals $n + 1$ if n is even. The following result was conjectured by Burris and Schelp in [4] and proved in [1].

Theorem 1 (*[1]*)**.** *If G is a graph of order n, without isolated edges and with at most one isolated vertex, then*

$$\mathrm{vdi}(G) \leq n + 1.$$

For some families of graphs the vertex-distinguishing index is closer to the maximum degree rather than to the order of the graph. Recall that by Vizing's theorem, for any graph G, we need either $\Delta(G)$ or $\Delta(G) + 1$ colors in order to color its edges properly.

The following theorem was proved in [2].

Theorem 2 (*[2]*)**.** *Let G be a graph of order $n \geq 3$ without isolated edges and with at most one isolated vertex. If $\delta(G) > n/3$, then*

$$\mathrm{vdi}(G) \leq \Delta(G) + 5.$$

However, for some families of graphs, the vertex-distinguishing index can be much greater than the maximum degree. For instance, consider a vertex-distinguishing coloring of a cycle of length n with k colors. Since each palette is of size two, and the number of all possible palettes cannot be smaller than n, we have $\binom{k}{2} \geq n$. Hence, $\mathrm{vdi}(C_n) \geq \sqrt{2n}$.

In this paper we propose to distinguish the vertices in another way: we will compare not only the palettes of given vertices, but we will also move using color walks and compare the palettes in attained vertices. We need some formal definitions.

Let $G = (V, E)$ be a graph without isolated edges and with at most one isolated vertex, and let $f : E \to K$ be a proper edge-coloring. For a given vertex $x \in V$, each walk emanating from x defines a sequence of colors (α_i). We then say that the sequence (α_i) is *realizable* at a vertex x. The set of all sequences realizable at x is denoted by $W(x)$.

Definition 3. We say that two vertices x and y of a graph G are similar if $W(x) = W(y)$, and the coloring f personalizes the vertices of G if no two vertices are similar. The minimum number of colors we need to obtain in this property is denoted by $\mu(G)$ and called the vertex distinguishing index by color walks of a graph G. (See Fig. 1.)

For a given $(\alpha_i) \in W(x)$, denote by $m(x, (\alpha_i))$ the last vertex on a walk emanating from x and defining the sequence (α_i). The following observation will be used several times in the proof of our main result.

Proposition 4. *Two vertices x and y of G are similar if and only if for each $(\alpha_i) \in W(x)$, we have $(\alpha_i) \in W(y)$ and the vertices $m(x, (\alpha_i))$, $m(y, (\alpha_i))$ have the same palettes.* ■

If the vertices of a graph are distinguished by their palettes, then we may say that they are distinguished by color walks of length one. However, if we are allowed to use walks of arbitrary length, the number of colors we need is surprisingly small. The aim of this paper is to prove the following result.

Theorem 5. *Let G be a connected graph of order $n \geq 3$. Then*

$$\mu(G) \leq \Delta(G) + 1$$

except for four graphs of small orders: C_4, K_4, C_6 and $K_{3,3}$.

The proof of the theorem is divided into two parts. First, in Section 2, we prove that either $\mu(G) \leq \chi'(G) + 1$ or G is one of four exceptional graphs. This implies the validity of the theorem for Class 1 graphs. In Section 3, we prove the theorem for Class 2 graphs.

Obviously, the inequality $\mu(G) \leq \Delta(G) + 1$ is not true for disconnected graphs. For instance, consider a graph $G = rP_3$ being the sum of r pairwise disjoint copies of P_3. Clearly, $\mu(rP_3) = \min\{k: \binom{k}{2} \geq r\} \geq \sqrt{2r} = \sqrt{r\Delta(G)}$.

It has to be noted that Theorem 5 has been already used by the first two authors in [7] to find a sharp upper bound for the *distinguishing chromatic index* of a connected graph, that is the minimum number of colors in a proper edge-coloring such that the identity is the only automorphism preserving this coloring. This is mentioned in the last section of the paper.

In what follows, we will use some additional notation. For a given cycle C we can choose one of two possible orientations. Let v be a vertex of C. We denote by v^+ and v^- the successor and the predecessor, respectively, of v on the cycle C with respect to a given orientation. An analogous notation is used also for paths, where the orientation is often defined by choosing one of its ends as the first vertex of the path.

Let f be a proper edge-coloring of a graph G. If $f(e) = \alpha$ then e is called an *α-edge*. A vertex x is *α-free* if $\alpha \notin S(x)$. An *(α, β)-Kempe subgraph* is a maximal connected subgraph of G formed by the edges colored with α and β. Clearly, it is either a path or an even cycle, and it is called an *(α, β)-Kempe path* or an *(α, β)-Kempe cycle*, respectively. An edge-coloring of G is *minimal* if it uses $\chi'(G)$ colors.

2. Result for Class 1 graphs

In this section, we will prove the following.

Theorem 6. *Let G be a connected graph of order $n \geq 3$. Then*

$$\mu(G) \leq \chi'(G) + 1$$

if and only if $G \notin \{C_4, K_4, C_6, K_{3,3}\}$.

We will first prove the following lemma.

Lemma 7. *Let f be an edge-coloring of a connected graph G of order $n \geq 8$ such that, for each two colors, every Kempe subgraph is a cycle of length four or six. Then there exists another coloring, with the same number of colors, such that at least one Kempe subgraph is a cycle of length at least eight.*

In our proof of this lemma, two observations will be crucial.

Observation 8. *Upon the assumptions of Lemma 7, each color induces a perfect matching in G. In consequence, any two colors induce a 2-factor.*

Proof. Indeed, suppose there exist a vertex x and a color α such that $\alpha \notin S(x)$. Then for any $\beta \in S(x)$, the (α, β)-Kempe subgraph containing x is a path. ■

Observation 9. *Upon the assumptions of Lemma 7, if two consecutive vertices on an (α, β)-Kempe cycle C are connected by two edges of the same color, say γ, to two consecutive vertices on another (α, β)-Kempe cycle, say C', then there exists a coloring of G with the same colors and with an (α, β)-Kempe cycle of length $|C| + |C'|$.*

Proof. Suppose that one of the two above-mentioned γ-edges joins the vertex $x \in C$ with the vertex $y \in C'$. We choose orientations of both cycles C, C' in such a way that the other γ-edge is x^+y^+. Without loss of generality, we may assume that the edge xx^+ is of color α. If the edge yy^+ also has color α, it suffices to exchange the color α on xx^+ and yy^+ with the color γ on xy and x^+y^+ so as to obtain an (α, β)-Kempe cycle of length $|C| + |C'|$. If the edge yy^+ has color β, we exchange first the colors α and β on C'. ■

Proof of Lemma 7. We consider two main cases.

Case 1. *There are at least three (α, β)-Kempe cycles C, C' and C'' and there exists a color, say γ, such that one of these cycles, say C', is joined by γ-edges both to C and to C''.*

Suppose that a γ-edge joins the vertex $x \in C$ with the vertex $y \in C'$ and another γ-edge joins the vertex $u \in C'$ with the vertex $z \in C''$. We choose orientations of the cycles C, C', C'' in such a way that the edges x^-x, yy^+ and zz^+ are of color α. We will consider some subcases according to the distance between y and u on the cycle C'.

Subcase 1.1. *$u = y^+$ or $u = y^-$.*

By symmetry, it suffices to consider only the case $u = y^+$. Then the path $x^-xyy^+zz^+$ is an (α, γ)-path of length five. So, the vertex z^+ has to be joined to x^- by an edge colored with γ. Now, on the edges x^-x, yy^+ and zz^+ we can replace the color α by γ, and on the edges xy, y^+z and zx^- we can replace the color γ by α. It is easy to see that in this way we get an (α, β)-Kempe cycle of length $|C| + |C'| + |C''|$.

Subcase 1.2. *$u = y^{++}$ or $u = y^{--}$.*

Again, by symmetry, it suffices to consider only the case $u = y^{++}$. By Observation 8, there is a γ-edge incident to y^+. If the other end vertex of this edge belonged to another (α, β)-Kempe cycle, we would return to Subcase 1.1. Hence, it should be a chord of C'. Suppose first that the second end vertex of this edge is the vertex y^{+++}. Consider now the path $x^-xyy^+y^{+++}y^{++}zz^+$. This is an (α, γ)-path of length seven, a contradiction. This implies that we are done if C' is of length four.

So, consider now the case where C' is of length six. By symmetry, we can omit the case where the γ-edge incident to y^+ ends at y^-. So, assume that this γ-edge ends at y^{--}. Then, the path $x^-xyy^+y^{--}y^-$ is an (α, γ)-path of length five. Therefore, y^- should be joined to x^- by a γ-edge, and, by Observation 9, we are done.

Subcase 1.3. *$u = y^{+++}$ and C' is of length six.*

As above, we may conclude that the γ-edges incident to y^+ and y^{++} are the chords of C'. Therefore, either these two γ-edges are y^+y^- and $y^{++}y^{--}$ or $y^{++}y^-$ and y^+y^{--}. In both cases, an (α, γ)-path of length nine is easy to find.

Case 2. For each three distinct colors α, β, γ, if there is a γ-edge connecting two (α, β)-Kempe cycles C, C', all other γ-edges incident to vertices of $C \cup C'$ end also in $C \cup C'$.

Let C, C' be two (α, β)-cycles connected by a γ-edge. Consider a cubic graph H, induced by three colors α, β, γ on the subgraph $C \cup C'$. Observe that it suffices to show that H is Hamiltonian. For, we may recolor H in the following way: we put the colors α and β on the Hamiltonian cycle and γ on the remaining matching, and obtain an (α, β)-cycle of length $|C| + |C'|$.

Since both cycles have even length, there are at least two γ-edges between C and C'. This implies that H is 2-connected. But all cubic, 2-connected and non-Hamiltonian graphs of small order are known (see, e.g., [5]). There are only two such graphs of order $n \leq 12$: the Petersen graph and the graph obtained from the Petersen graph by replacing a vertex by a triangle. None of them is a candidate for H, for, they do not contain even cycles. This finishes the proof of the lemma. ■

Proof of Theorem 6. Let $f : E \to K$ be a minimal coloring of G. Suppose first that there are two colors α and β such that the (α, β)-Kempe path is of length at least two. Denote it by P. We choose an orientation of P. Without loss of generality we may suppose that the first edge of P, say xx^+, is of color α. We define now a new coloring of G by replacing α on the edge xx^+ by a new color $0 \notin K$. We show that this coloring personalizes the vertices of G. For, suppose that there are two similar vertices u and v. Denote by Q a shortest path from u to the 0-edge $e = xx^+$. Consider now the walk Q' starting at v and inducing the same color sequence as Q. Evidently, the walk Q' should also finish either in x or in x^+. The crucial observation is that since the last edges of Q and Q' are of the same color, they cannot

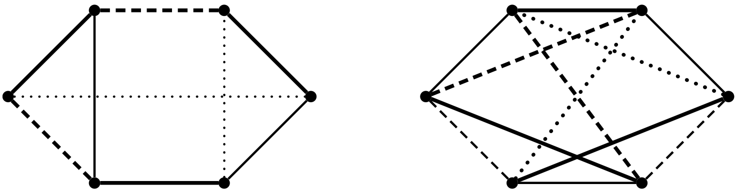

Fig. 2. Personalizing colorings of $K_3 \square K_2$ and $K_6 - M$.

arrive to the edge $e = xx^+$ at the same vertex. But $S(x) \neq S(x^+)$ because $\beta \in S(x^+)$ and $\beta \notin S(x)$, a contradiction.

Consider now the case when every (α, β)-Kempe subgraph is a cycle. By Lemma 7, if $n \geq 8$, the coloring f may be chosen in such a way that at least one such cycle induced by two colors, say α, β, is of length at least eight. Denote this cycle by C. Let x be a vertex of C, and let us choose one of two possible orientations of C. Without loss of generality, we may assume that the edge $e_1 = x^-x$ is colored with α. Then the edge $e_2 = x^{++}x^{+++}$ is colored with β. Let us recolor both these edges with a new color 0. We will show that this new coloring personalizes the vertices of G. For, suppose that there are two similar vertices u and v. As above, denote by Q a shortest path from u to a 0-edge and consider now the walk Q' starting at v and inducing the same color sequence as Q. Evidently, the walk Q' should also terminate either in e_1 or e_2. If the walks Q, Q' terminate at different edges e_1, e_2 we are done, for, the palettes $S(x^-)$ and $S(x)$ do not contain α, a color which is surely present in the palettes $S(x^{++})$ and $S(x^{+++})$. Therefore, both walks have their end vertices on the same 0-edge, but, of course, in different vertices. Without loss of generality we may assume that this is e_1. But then, we continue the walks using alternatively β and α. If we were in x, then after two steps we attain the second 0-edge. If we were in x^-, we need for this at least four steps, a contradiction.

Finally, we are left with the case where for each minimal coloring of G, any two colors induce a cycle of length four or six. By Lemma 7, we have $n \leq 7$. Moreover, Observation 8 (without the assumption $n \geq 8$) implies that G can be decomposed into k perfect matchings, for some k; thus the order n of G is even. For $n = 4$ there is one such graph C_4 with $k = 2$, and one graph K_4 with $k = 3$. Furthermore, for $n = 6$ there are: C_6 with $k = 2$, two graphs $K_3 \square K_2$ and $K_{3,3}$ with $k = 3$, next $K_6 - M$, where M is a perfect matching, with $k = 4$, and K_6 with $k = 5$. Fig. 2 presents personalizing colorings for the Cartesian product $K_3 \square K_2$ and $K_6 - M$ with $\chi' + 1$ colors. Adding the perfect matching M with a new sixth color yields a required coloring of K_6. For the remaining four graphs, C_4, K_4, C_6 and $K_{3,3}$, it is not difficult to see that one new color added to a minimal coloring is not enough to personalize the vertices (in the case of $K_{3,3}$, no matter how we introduce a fourth color on one or two edges, we can always find an edge with similar end vertices). But it suffices to introduce two new colors on two adjacent edges. ■

Thus, we have proved Theorem 5 for Class 1 graphs. Note that some of these graphs may have $\mu(G) = \Delta(G) + 1$. A simple example is a path of even order $n \geq 4$ since the end vertices are similar in any 2-coloring.

3. Result for Class 2 graphs

To complete the proof of Theorem 5 we have to prove the following result.

Theorem 10. *If a connected graph G is of Class 2, then*

$$\mu(G) = \chi'(G).$$

Proof. Let $G = (V, E)$ be a connected Class 2 graph. For a minimal coloring f, let 0 denote the color that is assigned to the least number of edges. Then f is called *biminimal* if the number of 0-edges is the least among all minimal colorings of G.

Let us start with a crucial observation. If $e = xy$ is a 0-edge in a biminimal coloring f of G, then

$$S(x) \neq S(y) \quad \text{and} \quad S(x) \cup S(y) = K.$$

Indeed, both properties are implied by the fact that there is no common missing color in the palettes $S(x)$ and $S(y)$. For, if such a color exists, then we could put it on e and decrease the number of 0-edges.

For each 0-edge consider an unordered pair $\{S_1, S_2\}$ of palettes of its end vertices. If there exists a biminimal coloring of G such that a certain pair $\{S_1, S_2\}$ appears only once for all 0-edges, say for an edge $e = xx^+$, we are done. Indeed, suppose that u and v are similar. Denote by Q a shortest path from u to the edge $e = xx^+$. Consider now the walk Q' starting at v and inducing the same color sequence as Q. Evidently, the walk Q' should also terminate either in x or x^+. Since the last edges of Q and Q' are of the same color, they cannot arrive to the edge e at the same vertex. But $S(x) \neq S(x^+)$; hence u and v cannot be similar, a contradiction.

Then suppose that in every biminimal coloring, if $\{S_1, S_2\}$ is a pair of palettes of end vertices of a 0-edge, then there exists another 0-edge whose end vertices also have palettes $\{S_1, S_2\}$. Choose a biminimal coloring f such that a certain pair $\{S_1, S_2\}$ appears the minimum number of times (but at least twice) among all biminimal colorings of G.

Take a 0-edge $e = xy$ with $S(x) = S_1$ and $S(y) = S_2$. Let α be a color missing in S_2, and let β be a color missing in S_1. By the above observation, $\alpha \in S_1$ and $\beta \in S_2$.

Let P_1 be the longest $(\alpha, 0)$-path beginning at x with an α-edge. Note that P_1 together with e creates an $(\alpha, 0)$-Kempe path. Therefore P_1 terminates with an α-edge. For, otherwise, by exchanging the colors α and 0 on $e \cup P_1$ we would get a coloring with a smaller number of 0-edges. Denote by u_1 the last vertex on the path P_1. Of course, $u_1 \neq x$, and, since S_2 do not contain α, also $u_1 \neq y$. Therefore, $f(u_1^- u_1) = \alpha$.

By exchanging the colors α and 0 on $\{xy\} \cup P_1$ we get another coloring, say f', with the same number of 0-edges as in f. Moreover, all vertices between y and u_1 have now the same palettes as in the coloring f. Since, by our choice of f the couple $\{S_1, S_2\}$ cannot disappear, it should be produced somewhere in the coloring f'. The only possible candidate for this is $u_1^- u_1$, the last edge of P_1. More precisely, in the coloring f', we have $S_{f'}(u_1^-) = S_1$ and $S_{f'}(u_1) = S_2$. This means that $S_f(u_1^-) = S_1$ and $S_f(u_1) = S^*$ where $S^* = S_2 - \{0\} \cup \{\alpha\}$.

Hence, S^* contains β. This allows us to consider now the maximal $(\beta, 0)$-path starting at u_1 with a β-edge. Denote it by P_2, and let u_2 be its last vertex. Since, as we have shown above, by exchanging the colors α and 0 on $\{xy\} \cup P_1$ we can get a coloring f' with $u_1^- u_1$ as a 0-edge with $\{S_1, S_2\}$ as the palette pair, with the same argument as above (this time for f'), we conclude that

- $f(u_2^- u_2) = \beta$,
- u_2 does not belong to P_1 (since u_2 is 0-free and $u_2 \neq u_1$),
- $S_f(u_2) = S^{**}$ where $S^{**} = S_1 \setminus \{0\} \cup \{\beta\}$.

Due to the latter condition, if $u_2 \neq y$, we can continue the procedure. Thus, we get a sequence of paths $P_1, P_2, \ldots,$ of length at least one, each starting at u_{i-1} ($u_0 = x$) and ending at u_i. Let us observe that by similar arguments as in the case of u_1 and u_2, we can show that u_i should be outside of the vertex sets of the paths $P_1, \ldots, P_{i-1}$. (Observe, however, that in general, the paths can have common internal vertices). The vertices u_i will be called *change-points*.

Since the procedure has to terminate, the last $(\beta, 0)$-path has to end at y. Let k be the number of these paths, i.e., $u_k = y$. Thus, we finally get a walk (called a *key walk*), from x to y, with the following properties

- for i odd, P_i is an $(\alpha, 0)$-path, with both end edges colored with α,
- for i even, P_i is a $(\beta, 0)$-path, with both end edges colored with β,
- for i odd, $S(u_i) = S^* = S_2 \setminus \{0\} \cup \{\alpha\}$,
- for i even, $S(u_i) = S^{**} = S_1 \setminus \{0\} \cup \{\beta\}$.

Note that one could also start the above-described procedure at the vertex y with a β-edge. We would get the same walk but in the reverse order, with y as the first, x as the last vertex and $(\beta, 0)$-paths with odd indices. Observe however that the change-point u_i with odd (resp. even) index i again has an odd (resp. even) index. That means, in particular, that interpreting the above properties in this situation, by analogy, the change-points u_i have palettes $S(u_i) = S_1 - \{0\} \cup \{\beta\} = S^{**}$. Hence, by the properties listed above, $S^* = S^{**}$.

Therefore, $S_1 = S^* - \{\beta\} \cup \{0\}$ and $S_2 = S^* - \{\alpha\} \cup \{0\}$. On the other hand, we know that $S_1 \cup S_2 = K$. This implies that the palettes differ just by one color; S_1 contains α and does not contain β, S_2 contains β and does not contain α. Denoting by $\hat{S}$ the set $K \setminus \{0, \alpha, \beta\}$, we have

$$S_1 = \hat{S} \cup \{0, \alpha\}, \qquad S_2 = \hat{S} \cup \{0, \beta\}, \qquad S^* = \hat{S} \cup \{\alpha, \beta\}.$$

In particular, $d(x) = d(y) = \Delta(G)$.

We will consider now three cases according to the number of 0-edges contained in the key walk.

Case 1. *There are at least two* 0-*edges contained in the key walk.*

Without loss of generality, we may assume that one of these 0-edges belongs to P_1 (if not, we can exchange the colors 0 and α or β in some initial paths of the key walk). Then consider the subwalk of our key walk joining the vertex u_1^- with v, the first vertex belonging to the second 0-edge on the key walk. Let us observe that $v = u_i^+$, for some *i*. Since the vertex u_1^- is β-free and the vertex $v = u_i^+$ is either α-free or β-free (it depends on the color of the edge $u_i u_i^+$), our subwalk is in fact an (α, β)-Kempe path. By exchanging α and β on this path, we replace at least one α-edge by a β-edge. Then, the $(0, \alpha)$-Kempe path starts at y and being the subpath of $\{xy\} \cup P_1$ ends with 0, a contradiction with the minimality of 0-edges.

Case 2. *There is only one* 0-*edge contained in the key walk.*

Again, without loss of generality, we may assume that this 0-edge belongs to P_1. Then the subwalk of our key walk joining the vertex u_1^- with y is in fact an (α, β)-Kempe path. If we exchange α and β on this path, we obtain a new proper coloring such that the vertices x, y have the same palette S_1. Hence, we can color the edge xy with β decreasing the number of 0-edges, a contradiction.

Case 3. *There is no* 0-*edge contained in the key walk.*

Then, actually, the key walk is an (α, β)-Kempe path of even length.

Subcase 3a. *This path is of length at least four.*

Then all the vertices of the key walk except for x and y have the same palette, namely S^*. So, in particular, they are 0-free. If we replace the color β by 0 on the second edge u_1u_2 of our walk, we obtain a new minimal coloring, denote it by f', such that the edge u_1u_2 is the unique 0-edge in f' having the pair $\{S_1, S_1\}$ as palettes of its ends.

It is not difficult to see that f' personalizes the vertices of G. Indeed, it suffices to observe that if we move from u_1 by an α-edge, we arrive to the vertex x having the palette S_1, while if we move from u_2, the other end vertex of the new 0-edge u_1u_2, by an α-edge, we arrive to u_3 having a different palette S^*.

Subcase 3b. *For each* 0-*edge, the corresponding key walk has only two edges.*

In other words, for each 0-edge, the edges colored with 0, α and β form a triangle with palettes S_1, S^*, S_2. Recall, that there are at least two such 0-edges in G. Let us choose one such triangle with vertices x, u_1, y and a color $\gamma \in S_2 \setminus \{0, \alpha, \beta\}$.

Consider a (β, γ)-Kempe path, say $\tilde{P}$, passing through the edge u_1y and exchange the colors along this path. This operation does not create any new 0-edge. Moreover, the palette at y will not change. Hence, if after this operation the palette at x remains unchanged, we will get a new coloring, with the same number of pairs $\{S_1, S_2\}$ of palettes on 0-edges, but now the edges colored with 0, α and β do not form a triangle. So, we return to one of the previous cases.

Suppose now that the palette at x has changed. This is possible only in the case where x is an end vertex of the path $\tilde{P}$. Then S_1 becomes $S' = S_1 \setminus \{\gamma\} \cup \{\beta\}$.

If the couple $\{S_1, S_2\}$ of palettes on another 0-edge remains unchanged, then we would have a coloring with a fewer number of pairs $\{S_1, S_2\}$ as palettes of 0-edges contrary to our assumptions. But a palette can be changed only at the second end vertex of the path $\tilde{P}$. This implies, in particular, that if the number of 0-edges with the palette couple $\{S_1, S_2\}$ is at least three, we are done.

Thus, the only situation to be considered is as follows: there are exactly two 0-edges with the palette couple $\{S_1, S_2\}$ and with $(0, \alpha, \beta)$-triangles. Denote the vertex sets of these triangles by x, u_1, y and x', u_1', y', respectively. Moreover, for each color γ, other than $0, \alpha, \beta$, there is a (β, γ)-Kempe path $\tilde{P}$ joining x and x' and passing through the β-edges of both triangles. Thus, only two situations may occur, and they are shown in Fig. 3.

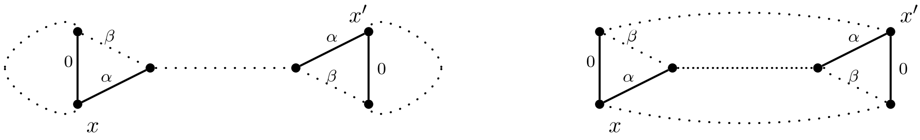

Fig. 3. Subcase 3b of the proof: path $\tilde{P}$ is dotted.

Consider first the left-hand side situation. Observe that by exchanging the colors 0 and β on two edges of the triangle x, u_1, y, the path $\tilde{P}$ is transformed into one (β, γ)-cycle passing through xy and a shorter (β, γ)-path (between u_1 and x'). Now, after changing the colors β and γ only on this cycle we get a coloring with the same number of palette pairs $\{S_1, S_2\}$ of 0-edges as before, but the 0-edge u_1y does not belong to a $(0, \alpha, \beta)$-triangle. Thus, we return to one of the previous cases.

A similar argument can be applied also in the case shown at the right-hand side of Fig. 3. This time, by exchanging the edges colored with α and β in the triangle x', y', u_1', the path $\tilde{P}$ is transformed into one (β, γ)-cycle passing through $u_1'x'$ and a (β, γ)-path (between x and y'). Again, we exchange now the colors β and γ only on this cycle. This time, we destroy the $(0, \alpha, \beta)$-triangle and obtain one of the previous cases.

This finishes the proof of Theorem 10. ■

Theorem 5 follows immediately from Theorems 6 and 10.

4. Final remarks

There are also some other topics concerning the problem of distinguishing all vertices of a graph based on the notion of automorphism preserving given colorings. For instance, the *distinguishing chromatic index*, mentioned in Introduction, is defined as the minimum number of colors in a proper edge-coloring that is preserved only by the trivial automorphism. As a consequence of Theorem 5, the following result was obtained in [7].

Theorem 11 (*[7]*). *Let G be a connected graph of order at least three and let $\chi'_D(G)$ denote its distinguishing index. Then*

$$\chi'_D(G) \le \Delta(G) + 1$$

except for four graphs: C_4, K_4, C_6 and $K_{3,3}$.

It follows that every connected Class 2 graph admits a minimal coloring that is not preserved by any nontrivial automorphism (see [7]).

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