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Distinguishing graphs by edge-colourings☆

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ABSTRACT

We introduce the *distinguishing index* $D'(G)$ of a graph G as the least number d such that G has an edge-colouring with d colours that is only preserved by the trivial automorphism. This is an analog to the notion of the distinguishing number $D(G)$ of a graph G, which is defined for colourings of vertices. We obtain a general upper bound $D'(G) \le \Delta(G)$ unless G is a small cycle C_3, C_4 or C_5.

We also introduce the *distinguishing chromatic index* $\chi'_D(G)$ defined for proper edge-colourings of a graph G. A correlation with distinguishing vertices by colour walks introduced in Kalinowski et al. (2004) is shown. We prove that $\chi'_D(G) \le \Delta(G) + 1$ except for four small graphs C_4, K_4, C_6 and $K_{3,3}$. It follows that each connected Class 2 graph G admits a minimal proper edge-colouring, i.e., with $\chi'(G)$ colours, preserved only by the trivial automorphism.

1. Introduction and definitions

We use standard terminology and notation of graph theory. In particular, the minimum and the maximum degree of a graph G will be denoted by $\delta(G)$ and $\Delta(G)$, respectively. We shall use for simplicity δ and Δ when no confusion is possible.

Albertson and Collins [1] introduced the *distinguishing number* $D(G)$ of a graph G as the least number d such that G admits a vertex-colouring with d colours that is only preserved by the trivial automorphism. Ten years later Collins and Trenk [6] defined the *distinguishing chromatic number* $\chi_D(G)$ of a graph G for proper colourings, so $\chi_D(G)$ is the least number d such that G has a proper colouring with d colours that is only preserved by the trivial automorphism. This concept has spawned

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numerous papers on finite graphs (e.g., [2–5,7,10]), as well as on infinite graphs (e.g., [12,15]). For endomorphisms instead of automorphisms this concept was investigated in [11].

Note that $D(G) = 1$ for all asymmetric graphs. This means that almost all finite graphs have distinguishing number one, because almost all graphs are asymmetric (see Erdős and Rényi [9]). Clearly, $D(G) \geq 2$ for all other graphs. Again, it is conjectured that almost all of them have distinguishing number two. This is supported by some observations of Conder and Tucker [8].

On the other hand, for a complete graph K_n, and a complete bipartite graph $K_{n,n}$ we have $D(K_n) = n$, and $D(K_{n,n}) = n+1$. Furthermore, the distinguishing number of cycles C_3, C_4, C_5 is three, while cycles C_n of length $n \geq 6$ have distinguishing number two.

This compares with a more general result of Collins and Trenk [6], and of Klavžar, Wong and Zhu [14].

Theorem 1 (*[6,14]*)**.** *If G is a connected graph with maximum degree Δ, then $D(G) \leq \Delta+1$. Furthermore, equality holds if and only if G is a $K_n, K_{n,n}$ or C_5.*

In the same paper, Collins and Trenk obtained a general result for the distinguishing chromatic number.

Theorem 2 (*[6]*)**.** *If G is a connected graph with maximum degree Δ, then $\chi_D(G) \leq 2\Delta$. Furthermore, equality is achieved if and only if G is a $K_{n,n}$ or C_6.*

The aim of this paper is a presentation of fundamental results for colourings of edges instead of vertices. We obtain general bounds, and an interesting relationship between the distinguishing chromatic index and the vertex distinguishing index by colour walks (introduced in [13]).

Definition 3. The distinguishing index $D'(G)$ of a graph G is the least number d such that G has an edge-colouring with d colours that is preserved only by the identity automorphism of G.

Definition 4. The distinguishing chromatic index $\chi'_D(G)$ of a graph G is the least number d such that G has a proper edge-colouring with d colours that is preserved only by the identity automorphism of G.

One may use the term *labelling* instead of colouring. Obviously, none of these two invariants is defined for graphs having K_2 as a connected component.

Given an edge-colouring c, *the palette at a vertex x* is the set of colours of the edges incident to x. Clearly, if different vertices have different palettes, then the only automorphism preserving c is the identity.

Sometimes $D'(G) = D(G)$. Clearly it holds for all graphs with a trivial automorphism group, and also for paths and cycles.

Proposition 5. *$D'(P_n) = D'(C_p) = 2$, for any $n \geq 3$ and $p \geq 6$, while $D'(C_3) = D'(C_4) = D'(C_5) = 3$.*

Proof. The distinguishing number for paths and cycles equals $D(P_n) = 2$ for $n \geq 2$, $D(C_p) = 2$ for $p \geq 6$ and $D(C_3) = D(C_4) = D(C_5) = 3$. Our observation follows immediately from the fact that the $L(P_n) \cong P_{n-1}$ and $L(C_p) \cong C_p$, where L denotes the line graph. □

It has to be noted that, in general, the distinguishing index of a graph G is not the same as the distinguishing number of its line graph $L(G)$. A simple example is $K_4 - e$, a complete graph of order four with one edge deleted. It can easily be verified that $D'(K_4 - e) = 2$ while $D(L(K_4 - e)) = 3$.

However, quite frequently $D'(G) < D(G)$. Albertson and Collins [1] proved that $D(L(K_n)) = 2$ if $n \geq 6$ by simply showing that there exists an edge-colouring of K_n with two colours that is preserved only by the identity. To do this, they used an argument suggested by Lovász, that K_n contains an asymmetric spanning graph if and only if $n \geq 6$. Thus, without introducing a concept of distinguishing graphs by edge-colourings, they actually proved the following.

Proposition 6 (*[1]*)**.** *$D'(K_n) = 3$ for $n = 3, 4, 5$, and $D'(K_n) = 2$ for any $n \geq 6$.*

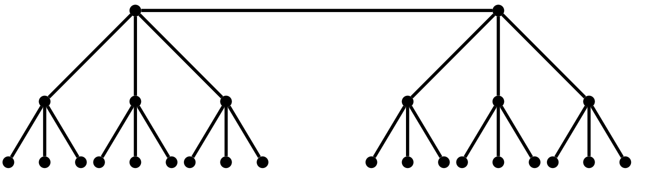

Fig. 1. A bisymmetric tree $T''_{2,4}$.

The argument of Lovász can be also used for balanced complete bipartite graphs $K_{p,p}$ since for each $n \geq 7$ there exists an asymmetric tree of order n. It easily follows that $K_{p,p}$ contains an asymmetric spanning tree if and only if $p \geq 4$.

Proposition 7. *$D'(K_{p,p}) = 3$ for $p = 2, 3$, and $D'(K_{p,p}) = 2$ for $p \geq 4$.* $\square$

In the next section we first present a special class of trees with the distinguishing index greater by one than the distinguishing number, and we show that $D'(T) = D(T)$ for all remaining trees (see Theorem 9). We show finally that $D'(G) \leq \Delta(G)$ except for three small cycles (Theorem 13).

In the last section we investigate proper colourings of edges of G. By Vizing's Theorem every graph has a colouring with $\Delta(G)$ or $\Delta(G)+1$ colours. We show that $\Delta(G)+1$ colours suffice to find a proper colouring preserved only by the trivial automorphism unless G is one of four exceptional small graphs: C_4, K_4, C_6 or $K_{3,3}$.

2. Distinguishing index

2.1. Trees

Recall that every finite tree T has either a central vertex or a central edge, which is fixed by every automorphism of T. A *symmetric tree*, denoted by $T_{h,d}$, is a tree with a central vertex v_0, all leaves at the same distance h from v_0 and all the vertices which are not leaves with degree d. A *bisymmetric tree*, denoted by $T''_{h,d}$, is a tree with a central edge e_0, all leaves at the same distance h from e_0 and all the vertices which are not leaves with degree d. An *almost symmetric* tree, denoted by $T'_{h,d}$, is a tree with a central vertex v_0 of degree $d-1$ such that all other vertices are either leaves or have degree d, and all leaves are at the same distance h from v_0. Thus, the bisymmetric tree $T''_{h,d}$ is a tree obtained by joining central vertices of two almost symmetric trees $T'_{h,d}$ by an edge. Observe that a path is either a symmetric or a bisymmetric tree depending of the parity of its length (see Fig. 1).

Collins and Trenk in [6], obtained a general bound for the distinguishing number of trees. We cite it improving a small mistake in the original paper.

Theorem 8 ([6]). *If T is a tree of order $n \geq 3$, then $D(T) \leq \Delta(T)$. Furthermore, equality is achieved if and only if T is a symmetric tree or a path of odd length.*

For $k \geq 0$, a *k-th level* of a tree is the set of all vertices of distance k from either its central vertex or its central edge, respectively. The *height of a tree* is the largest h for which the k-th level is nonempty. For $h \geq 1$ and $d \geq 2$, we define a class $\mathcal{B}(h, d)$ of trees constructed in the following way. Take an almost symmetric tree $T'_{h,d}$, choose l its levels $h_1, \ldots, h_l$ with $0 \leq h_i \leq h-2$, $i = 1, \ldots, l$, and l almost symmetric trees $T'_{k_1,d}, \ldots, T'_{k_l,d}$ such that $1 \leq k_i \leq h - h_i - 1$ for every i, and construct a tree $T'_{h,d}([T'_{k_1,d}]h_1, \ldots, [T'_{k_l,d}]h_l)$ by attaching each tree $T'_{k_i,d}$, $i = 1 \ldots, l$, to every vertex of the h_i-th level of $T'_{h,d}$ (here attaching means identifying the root of $T'_{k_i,d}$ with the vertex of attachment in $T'_{h,d}$). The levels $h_1, \ldots, h_l$ need not be distinct but we assume that $T'_{k_i,d}$ is not isomorphic to $T'_{k_j,d}$ if $h_i \neq h_j$. Next, we can again choose any attached tree, say $T'_{k_i,d}$, then l_1 levels of $T'_{k_i,d}$ and l_1 almost symmetric trees satisfying analogous height constraints, and attach each of them to all vertices of the corresponding levels of each copy of $T'_{k_i,d}$ attached in the first stage. Then we can repeat this operation for any tree

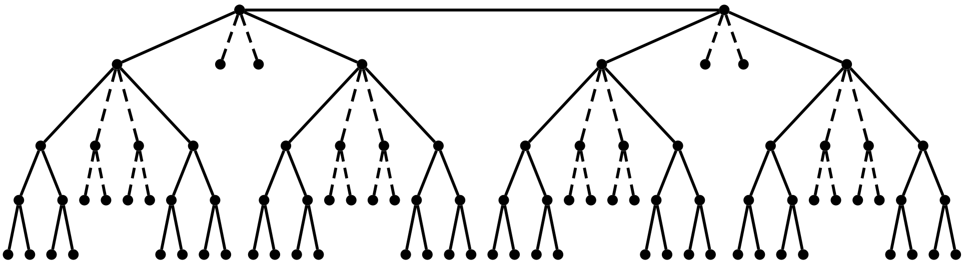

Fig. 2. An example of a tree T from the class $\mathcal{B}(4,3)$ obtained by joining two copies of $T'_{4,3}([T'_{1,3}]_0, [T'_{2,3}]_1)$ by an edge. The trees $T'_{1,3}$ and $T'_{2,3}$ attached to $T'_{4,3}$ in the process of constructing T are drawn with dashed edges.

attached in the previous stages. This way, we obtain a tree T_0. Finally, we take two copies of T_0 and join central vertices of T_0 by an edge e_0 to obtain a tree T that belongs to $\mathcal{B}(h,d)$ (see Fig. 2).

Thus, every tree that belongs to $\mathcal{B}(h,d)$ has a central edge e_0. Clearly, a bisymmetric tree $T''_{h,d}$ belongs to $\mathcal{B}(h,d)$ since we can choose $l = 0$ in the beginning of the above procedure.

Given a vertex x of a tree $T \in \mathcal{B}(h,d)$, let $T^1, \ldots, T^s$ be the connected components of $T - x$ not containing the central edge of T. Let x_i be the vertex of T^i adjacent to x in T, $i = 1, \ldots, s$. The tree $T_{x_i} = T^i + x_i x$ is called a *branch rooted at* x in T. It is not difficult to see that for every tree T that belongs to $\mathcal{B}(h,d)$ the following crucial property holds.

For every vertex $x \in V(T)$, if T_x is a branch of T rooted at x, then there are exactly $d-1$ branches rooted at x (including T_x) isomorphic to T_x. Moreover, every tree of height h with this property belongs to $\mathcal{B}(h,d)$.

It follows that $D(T) = d - 1$ for every tree $T \in \mathcal{B}(h,d)$ with $d \geq 3$. Indeed, we first colour the end vertices of the central edge e_0 with two distinct colours. Then, due to the above property, we need $d-1$ colours to colour children in each of $d-1$ isomorphic branches rooted at an already coloured vertex. If $T \in \mathcal{B}(h,2)$, then $D(T) = D'(T) = 2$ because we can colour one end vertex of a central edge e_0, one edge adjacent to e_0 with 2 and everything else by 1 to obtain colourings of vertices and edges, respectively, that are not preserved by any nontrivial automorphism of T.

Note that the distinguishing index of any tree $T \in \mathcal{B}(h,d)$ equals d. Indeed, for every vertex x of T, we have to colour all the edges joining x to its children in each branch rooted at x with $d-1$ distinct colours. However, the automorphism that switches the two components of $T - e_0$ preserves this colouring. Hence, we need an additional colour for one edge, say the one adjacent to the central edge e_0.

We now show that the distinguishing index of a tree T equals its distinguishing number unless T belongs to the class $\mathcal{B}(h,d)$ with $d \geq 3$.

Theorem 9. *Let T be a tree of order $n \geq 3$. Then $D'(T) = D(T) + 1$ if T belongs to $\mathcal{B}(h,d)$ with $h \geq 1$ and $d \geq 3$, and $D'(T) = D(T)$ for all other trees.*

Proof. Let $\hat{c} : V(T) \to \{1, 2, \ldots, D(T)\}$ be a vertex-colouring preserved only by the identity.

Case 1. A tree T has a central vertex v_0. If xy is an edge of T such that $d(x, v_0) = d(y, v_0) + 1$, then we colour it as $c(xy) := \hat{c}(x)$. Suppose φ is a nontrivial automorphism of T preserving the colouring c. As φ fixes the central vertex v_0, it also fixes the distance from any vertex x to v_0. Hence, $\hat{c}(\varphi(x)) = c(\varphi(x)\varphi(y)) = c(xy) = \hat{c}(x)$, that is, φ preserves the vertex colouring $\hat{c}$, a contradiction.

Case 2. A tree T has a central edge $e_0 = a_1 a_2$. Let T_i be the connected components of $T - e_0$ containing a_i, $i = 1, 2$.

If xy is an edge of $T - e_0$ such that the distance from x to the central edge e_0 is greater by one than the distance from y to e_0, then we colour xy with $c(xy) = \hat{c}(x)$. Finally, we colour e_0 arbitrarily. Suppose that φ is a nontrivial automorphism of T preserving the colouring c. So there exist two edges x_1y_1 and x_2y_2 with the same colour such that $\varphi(x_1)\varphi(y_1) = x_2y_2$. As φ fixes the edge e_0, the distances from e_0 to x_1 and x_2 are equal, and $\hat{c}(x_1) = \hat{c}(x_2)$. If both edges x_1y_1, x_2y_2 belong to the same component of $T - e_0$, then, by the definition of c, the automorphism φ preserves $\hat{c}$, a contradiction.

Otherwise, φ switches the end vertices of e_0 and generates an isomorphism between the subtrees T_1 and T_2. We distinguish three subcases. If $T \in \mathcal{B}(h, 2)$, then we have already shown that $D'(T) = D(T) = 2$. If T belongs to $\mathcal{B}(h, d)$ with $d \geq 3$, then $D'(T) = D(T) + 1$, as we have proved earlier. Otherwise, there exists a vertex x with less than $D(T)$ isomorphic branches rooted at x. Hence, we can re-colour one of the edges incident to x in such a branch with one of $D(T)$ colours.

Thus we have shown that $D'(T) \leq D(T) + 1$ for every tree T, and the equality holds if and only if T belongs to $\mathcal{B}(h, d)$ with $d \geq 3$.

To end the proof, it suffices to show that $D(T) \leq D'(T)$. Let c be an edge-colouring of T with $D'(T)$ colours that is preserved only by the identity. By the inverse operation to the one used in the first part of the proof, we get a vertex-colouring $\hat{c}$. That is, if T has a central edge e_0 we put $c(e_0)$ on both its end vertices. In all other cases, we define $\hat{c}(x) = c(xy)$ where xy lies on the path between x and either a central vertex or a central edge. Clearly, $\hat{c}$ is not preserved by any nontrivial automorphism of T, therefore $D(T) \leq D'(T)$. $\square$

Theorems 8 and 9 immediately imply the following result since the only tree in $\mathcal{B}(h, d)$ with maximum degree d is a bisymmetric tree.

Theorem 10. *If T is a tree of order $n \geq 3$, then $D'(T) \leq \Delta(T)$. Moreover, equality is achieved if and only if T is either a symmetric or a bisymmetric tree.* $\square$

2.2. Connected graphs

Theorem 11. *If G is a connected graph of order $n \geq 3$, then $D'(G) \leq D(G) + 1$.*

Proof. If G is a tree then the claim is true by Theorem 9. Suppose that G contains a cycle. If G is just a cycle, then the claim follows from Proposition 5.

Let $\hat{c} : V \to \{1, 2, \dots, D(G)\}$ be a colouring preserved only by the identity. Obviously, if $D(G) = 1$, then G is asymmetric and $D'(G) = 1$. So, let $\hat{c}$ using at least two colours.

We will define an edge-colouring c with $D(G) + 1$ colours. Denote by 0 an additional colour not used by $\hat{c}$. Let C be a shortest cycle of G. We first colour the edges of C with three colours $0, 1, 2$ in such a way that two adjacent edges uv and vw have colours 1 and 2, and all remaining edges of C are coloured with 0. Thus C has no nontrivial automorphism. Then, we will not use more the colour 0, and we colour every edge xy with $c(xy) = \hat{c}(x)$ if the distance from x to the cycle C is one more than the distance from y to C. Finally, we colour every edge xy such that x and y are at the same distance from the cycle C arbitrarily, say with colour 1.

Suppose that φ is a nontrivial automorphism of G preserving the colouring c. First observe that the cycle C is fixed by φ, because 0 appears only on C, and if the vertices u, w have a common neighbour z outside C, then $c(zu) = c(zw)$. Then φ preserves the distances from vertices of G to C. Therefore there exist edges xy such that the distance from x to C is greater by one than that of y, and $c(\varphi(x)\varphi(y)) = c(xy)$. For each such edge, we have $\hat{c}(\varphi(x)) = c(\varphi(x)\varphi(y)) = c(xy) = \hat{c}(x)$. As φ fixes C, it follows that the vertex-colouring $\hat{c}$ is preserved by φ, a contradiction. $\square$

Theorem 11 is interesting in view of the conjecture, mentioned in Section 1, that almost every non-asymmetric graph has the distinguishing number two.

Theorems 1 and 11 immediately imply the following.

Corollary 12. *If G is a connected graph of order $n \geq 3$, then*

$$D'(G) \leq \Delta(G) + 1. \quad \square$$

We can strengthen the above corollary as follows.

Theorem 13. *If G is a connected graph of order $n \geq 3$, then*

$$D'(G) \leq \Delta(G)$$

except for three small cycles C_3, C_4 or C_5.

Proof. Denote $\Delta = \Delta(G)$ and $\delta = \delta(G)$. Due to Proposition 5, we may assume that $\Delta \geq 3$. Denote by $N_r(\xi)$ the set of all vertices of distance r from ξ, where ξ is either a vertex or an edge.

Consider first an irregular graph $G = (V, E)$. Let xy be an edge of G such that $\deg(x) = \delta$. We colour xy with 1, then all other edges incident to x with a colour from $\{2, \ldots, \delta\}$, and all other edges incident with y from $\{\Delta - \deg(y) + 2, \ldots, \Delta\}$. As $\delta \neq \Delta$, the sets of colours of edges incident to x or to y are different. We will not use colour 1 any more, so vertices x and y are fixed by any automorphism of G. Moreover, all vertices from $N_1(xy)$ are fixed by any automorphism. Now, for $r \geq 1$ let u be a vertex from $N_r(xy)$. We colour all edges uv, for $v \in N_{r+1}(xy)$, with colours from $\{2, \ldots, \Delta\}$. Therefore, all vertices of G are fixed by any automorphism of G. Observe that colours of edges between two vertices of the same $N_i(x)$ could be arbitrary.

Now, let G be a regular graph. Due to Proposition 5, we may assume $\Delta \geq 3$. Fix any vertex x of G and colour all edges incident to it with 1. In our edge-colouring c of the graph G, the vertex x will be the unique vertex with the monochromatic palette $\{1\}$, hence it will be fixed by every automorphism φ preserving c. The neighbourhood $N_1(x)$ can be partitioned into subsets M_k, for $k = 0, 1, \ldots, \Delta - 1$, defined as

$$M_k = \{v \in N_1(x) : |N_1(v) \cap N_2(x)| = k\}.$$

Denote $M_k = \{v_1, \ldots, v_{l_k}\}$, $k = 0, 1, \ldots, \Delta - 1$. Thus, $l_0 + l_1 + \cdots + l_{\Delta-1} = \Delta$. If $l_0 = \Delta$, then G is a complete graph $K_{\Delta+1}$, and we done by Proposition 6. Otherwise, if $l_0 \geq 1$, we can colour the edges incident to the vertices of M_0 with two colours 2 and 3 such that the palette of v_i contains exactly $l_0 + 1 - i$ edges coloured with 2. Thus, each vertex of M_0 is fixed.

Let $k \geq 1$. For every $i = 1, \ldots, l_k$, we colour the edges $v_i u$, where $u \in N_2(x)$, with a distinct colour from $\{1, \ldots, k+1\}$ in such a way that the colour i is missing in the palette of v_i. Then we colour all the remaining edges incident to v_i with 2. Clearly, each vertex of $N_1(x) \cup N_2(x)$ is fixed by every automorphism preserving the colouring c.

For $v_j \in N_j(x), j \geq 2$, we colour all edges $v_j u$, $u \in N_{j+1}(x)$ with distinct colours from $\{2, \ldots, \Delta\}$ and the remaining edges incident to v_j arbitrarily.

Then we recursively colour the edges incident to consecutive spheres $N_j(x)$ in such a way that distinct vertices of $N_j(x)$ have distinct palettes. It is easily seen that it is always possible. Hence, all vertices of G are fixed by any automorphism φ preserving our colouring c. □

As we already mentioned, $D'(G) = D(G) = 1$ for every asymmetric graph G. For other graphs there is a trivial lower bound $2 \leq D'(G)$ which is sharp due to Propositions 6 and 7.

3. Distinguishing chromatic index

3.1. General bound

Let c be a proper edge-colouring of a connected graph G of order $n \geq 3$. For every vertex x, each walk starting at x defines a sequence of colours (α_i) called a *colour walk*. Denote by $W_c(x)$ the set of all colour walks starting at x. A new invariant $\mu(G)$, called the *distinguishing index by colour walks* of a graph G, was introduced in [13] as the minimum number of colours required in an edge-colouring of G such that $W_c(x) \neq W_c(y)$ for every two distinct vertices x, y.

Theorem 14 (*[13]*). *Let G be a connected graph of order $n \geq 3$. Then*

$$\mu(G) \leq \Delta(G) + 1$$

except for four graphs of small order $C_4, K_4, C_6, K_{3,3}$.

The next lemma exhibits a relationship between $\mu(G)$ and $\chi'_D(G)$.

Lemma 15. *Every connected graph G of order $n \geq 3$ fulfils the inequality*

$$\chi'_D(G) \leq \mu(G).$$

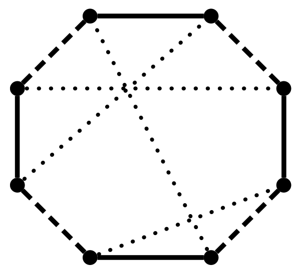

Fig. 3. A graph G such that $\chi'_D(G) < \mu(G)$.

Proof. Let c be an edge-colouring distinguishing vertices by colour walks, i.e., $W_c(x) \neq W_c(y)$ if $x \neq y$. Suppose φ is a nontrivial automorphism of G preserving c. Then there exists a vertex x such that $x \neq \varphi(x)$. An automorphism φ preserves the colouring, so every sequence $(\alpha_i) \in W_c(x)$ belongs also to $W_c(\varphi(x))$. Every sequence (β_i) starting at $\varphi(x)$, starts also at $\varphi^{-1}(\varphi(x)) = x$. Hence, x and $\varphi(x)$ are not distinguished by colour walks in this colouring. □

In consequence, we obtain a sharp upper bound for the distinguishing chromatic index of connected graphs.

Theorem 16. *If G is a connected graph of order $n \geq 3$, then*

$$\chi'_D(G) \leq \Delta(G) + 1$$

except for four graphs of small order $C_4, K_4, C_6, K_{3,3}$. □

This theorem immediately implies the following interesting result. An edge-colouring of G with $\chi'(G)$ colours is called *minimal*.

Theorem 17. *Every connected Class 2 graph admits a minimal edge-colouring that is not preserved by any nontrivial automorphism.* □

3.2. Some class 1 graphs

As it follows from the previous subsection, $\chi'_D(G) = \mu(G) = \chi'(G)$ for every connected Class 2 graph. For Class 1 graphs, one of these two equalities may not hold.

We shall first show that there are graphs for which $\chi'_D < \mu$. Every regular graph G of Class 1 satisfies $\mu(G) = \Delta(G) + 1$. Indeed, for every minimal edge-colouring of G, the palette of each vertex is the same. Hence, for any vertex x, the set $W_c(x)$ is the same, as it comprises all sequences of colours of c. By Theorem 14, one additional colour is enough to distinguish all vertices by colour walks.

Consider the cubic graph G drawn in Fig. 3. The edges of a cycle C_8 are properly coloured with two colours, and the remaining edges, creating a perfect matching, have a third colour. Let φ be an automorphism preserving this colouring. The unique triangle of G has to be mapped onto itself. Regarding the colours of its edges, it has to be fixed by φ. Hence, the cycle C_8 also is fixed. Thus $\chi'_D(G) = 3$ while $\mu(G) = 4$.

This example can easily be generalized to higher orders and degrees by taking a longer even cycle with a perfect matching creating exactly one triangle, and then introducing more arbitrary perfect matchings.

For Class 1 graphs, we sometimes need one colour more than χ' for χ'_D, and in four cases, two additional colours. Also for paths of odd length we have $\chi'_D(P_{2k}) = \chi'(P_{2k}) + 1$ colours. If we have a proper colouring of P_{2k}, then it is enough to recolour a hanging edge with a new additional colour. For paths of even length, any proper colouring is preserved only by the identity. This observation can be extended to trees in general.

Proposition 18. *If T is a tree of order $n \geq 3$, then*

$$\chi'_D(T) = \Delta(T) + 1$$

if and only if T is a bisymmetric tree.

Proof. Consider any proper edge-colouring of T with $\Delta(T)$ colours.

Case 1. T has a central vertex v_0 fixed by every automorphism. A colouring is proper, so every edge incident to v_0 has a distinct colour. Hence, all vertices adjacent to v_0 are fixed by every automorphism of T. By induction on the distance from v_0, all vertices of T are fixed.

Case 2. T has a central edge e_0 fixed by every automorphism. Let T_1 and T_2 be subtrees created by deleting the edge e_0. If these subtrees are not isomorphic, then the end vertices of e_0 are fixed by every automorphism, and we are done by similar arguments as in Case 1. Let then T_1 and T_2 be isomorphic. Suppose there exist vertices $x_1 \in V(T_1)$ and $x_2 \in V(T_2)$, that are not leaves, with the degree in T smaller than $\Delta(T)$. If the sets of colours of edges incident to x_1 and x_2 are different, the unique automorphism preserving this colouring of T is the identity. If not, let 0 be a colour which is not in a set of colours of edges incident to x_2. We re-colour one edge incident to x_2 with 0, and, if necessary, we re-colour a Kempe path in T_2 induced by 0 and the colour of the edge incident with x_2 before re-colouring.

If T_1 and T_2 are isomorphic and all vertices in T that are not leaves have degree $\Delta(T)$, then T is a bisymmetric tree. We need an additional colour for one edge adjacent to the central edge to distinguish its end vertices, which are then fixed by every automorphism, and we can use the same arguments as in Case 1 to finish the proof. $\square$

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