Also available at http://amc-journal.eu
ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 12 (2017) 77–87

The distinguishing index of the Cartesian product of finite graphs*

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Received 4 August 2015, accepted 2 March 2016, published online 20 May 2016

Abstract

The *distinguishing index* $D'(G)$ of a graph G is the least natural number d such that G has an edge colouring with d colours that is only preserved by the identity automorphism. In this paper we investigate the distinguishing index of the Cartesian product of connected finite graphs. We prove that for every $k \geq 2$, the k-th Cartesian power of a connected graph G has distinguishing index equal 2, with the only exception $D'(K_2^2) = 3$. We also prove that if G and H are connected graphs that satisfy the relation $2 \leq |G| \leq |H| \leq 2^{|G|}\big(2^{\|G\|} - 1\big) - |G| + 2$, then $D'(G \Box H) \leq 2$ unless $G \Box H = K_2^2$.

Keywords: Edge colouring, symmetry breaking, distinguishing index, Cartesian product of graphs.

Math. Subj. Class.: 05C15, 05E18

1 Introduction

We use standard graph theory notation (cf. [6]). In particular, $\mathrm{Aut}(G)$ denotes the automorphism group of a graph G.

An edge colouring *breaks an automorphism* $\varphi \in Aut(G)$ if φ does not preserve the colouring, i.e., there exists an edge of G that is mapped by φ to an edge of different colour. The *distinguishing index* $D'(G)$ of a graph G is the least natural number d such that G has an edge colouring with d colours that breaks all non-trivial automorphisms of G. Such a d-colouring is called *distinguishing*. This notion was introduced by Kalinowski and Pilśniak [10] as an analogue of the well-known *distinguishing number* $D(G)$ of a graph G defined by Albertson and Collins [1] as the least number of colours in a vertex colouring that breaks all non-trivial automorphisms of G. [1] As the distinguishing index is not defined for K_2, we assume henceforth that K_2 is not a connected component of any graph considered.

*The research was partially supported by the Polish Ministry of Science and Higher Education.

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[1] Fisher and Isaak [5] considered distinguishing edge colourings of complete bipartite graphs, but did not introduce any special notation or terminology.

The distinguishing index of several examples of graphs was exhibited in [10]. For instance, $D'(P_n) = D(P_n) = 2$, for $n \geq 3$; $D'(C_n) = D(C_n) = 2$, for $n \geq 6$, and $D'(C_n) = 3$, for $n = 3, 4, 5$. There exist graphs G for which $D'(G) < D(G)$, for instance $D'(K_n) = D'(K_{p,p}) = 2$, for any $n \geq 6$ and $p \geq 4$, while $D(K_n) = n$ and $D(K_{p,p}) = p+1$. It is also possible that $D'(G) > D(G)$. All trees satisfying this inequality were characterized in [10]. The following general upper bound of the distinguishing index was proved in [10].

Theorem 1.1. [10] *If G is a finite connected graph of order $n \geq 3$, then $D'(G) \leq D(G) + 1$. Moreover, if Δ is the maximum degree of G, then $D'(G) \leq \Delta$ unless G is a C_3, C_4 or C_5.*

The distinguishing index was also investigated for infinite graphs [2] and their Cartesian product [3].

The Cartesian product of graphs G and H is a graph, denoted $G \square H$, whose vertex set is $V(G) \times V(H)$. Two vertices (g, h) and (g', h') are adjacent if either $g = g'$ and $hh' \in E(H)$, or $gg' \in E(G)$ and $h = h'$. Denote $G \square G$ by G^2, and recursively define the *k-th Cartesian power* of G as $G^k = G \square G^{k-1}$.

A non-trivial graph G is called *prime* if $G = G_1 \square G_2$ implies that either G_1 or G_2 is K_1. It is easy to see that every non-trivial finite graph has a prime factorization with respect to the Cartesian product. For connected graphs it is also unique up to isomorphisms and the order of the factors, as has been shown by Sabidussi and Vizing (cf. [6]). Two graphs G and H are called *relatively prime* if K_1 is the only common factor of G and H.

Let v be a vertex of H. A *G^v-layer* (called also a *horizontal layer* of $G \square H$) is the subgraph induced by the vertex set $\{(u, v) : u \in V(G)\}$. An *H^u-layer*, or *vertical* layer, is defined analogously for a vertex u of G. Clearly, each horizontal layer is isomorphic to G and each vertical one is isomorphic to H. Therefore, speaking of a specified layer of $G \square H$, we shall usually use only one coordinate of a vertex. The same refers to edges.

We shall need knowledge of the structure of the automorphism group of the Cartesian product, which was determined by Imrich [7], and independently by Miller [11].

Theorem 1.2. [7, 11] *Suppose ψ is an automorphism of a connected graph G with prime factor decomposition $G = G_1 \square G_2 \square \ldots \square G_r$. Then there is a permutation π of the set $\{1, 2, \ldots, r\}$ and there are isomorphisms $\psi_i \colon G_{\pi(i)} \mapsto G_i$, $i = 1, \ldots, r$, such that*

$$\psi(x_1, x_2, \ldots, x_r) = (\psi_1(x_{\pi(1)}), \psi_2(x_{\pi(2)}), \ldots, \psi_r(x_{\pi(r)})).$$

It follows in particular that every automorphism of the Cartesian product of two relatively prime graphs is a composition of a permutation of vertical layers generated by an automorphism of G and a permutation of horizontal layers generated by an automorphism of H. For additional results about the Cartesian product consult [6].

Our main results are extensions of theorems about the distinguishing number of Cartesian powers and of Cartesian products of connected graphs to the distinguishing index. The results (and some of the proofs) are inspired by a paper [8] by Imrich, Jerebic and Klavžar. In Section 2 we generalize a result of Imrich and Klavžar.

Theorem 1.3. [9] *Let G be a connected graph and $k \geq 2$. Then $D(G^k) = 2$ except for the graphs K_2^2, K_2^3, K_3^2 whose distinguishing number is three.*

The second result that we extend is also due to Imrich and Klavžar:

Theorem 1.4. [9] *Let G and H be connected, relatively prime graphs such that*

$$|G| \leq |H| \leq 2^{|G|} - |G| + 1.$$

Then $D(G \square H) \leq 2$.

In Section 3 we prove an analogous result (Theorem 3.4) for the distinguishing index of the Cartesian product of connected graphs, not necessarily relatively prime (let us note that, using our method of proof, Theorem 3.4 was already strengthened in [4] by omitting the assumption that G and H are relatively prime). We also obtain a slightly stronger result for trees (Theorem 3.1).

In proofs, we usually use colours $1, \ldots, d$. If $d = 2$, then we also use colours 0 and 1, or alternatively red and blue.

2 Distinguishing Cartesian powers

Let us start with the Cartesian powers of K_2. Recall that the k-dimensional hypercube is isomorphic to K_2^k and denoted by Q_k. As mentioned earlier, the distinguished index is not defined for $K_2 = Q_1$. Clearly, $D'(Q_2) = 3$ since $Q_2 = C_4$. The following result was proved in [13].

Theorem 2.1. [13] *If a graph G of order at least 7 contains a Hamiltonian path, then $D'(G) \leq 2$.*

Proposition 2.2. *If $k \geq 3$, then $D'(Q_k) = 2$.*

Proof. For $k \geq 3$ a hypercube Q_k is Hamiltonian and has at least eight vertices. Therefore, $D'(Q_k) = 2$ by Theorem 2.1. □

The distinguishing index of the square of cycles and for arbitrary powers of complete graphs with respect to the Cartesian, direct and strong products has been already considered by Pilśniak [12]. In particular, she proved that $D'(C_m^2) = 2$ for $m \geq 4$, and $D'(K_n^k) = 2$ for any $n \geq 4$ and $k \geq 2$.

Here we consider Cartesian powers of arbitrary connected graphs. We first prove some lemmas.

Lemma 2.3. *Let G and H be connected, relatively prime graphs with $D'(G) = D'(H) = 2$. Then $D'(G \square H) = 2$.*

Proof. We colour one G-layer and one H-layer with distinguishing 2-colourings. The remaining edges can be coloured arbitrarily. Such a colouring breaks all permutations of both horizontal and vertical layers. Since G and H are relatively prime, it follows from Theorem 1.2 that this colouring breaks all automorphisms of $G \square H$. □

Lemma 2.4. *Let G and H be two connected graphs where G is prime, $|G| \leq \|H\| + 1$ and $D'(H) = 2$. Then $D'(G \square H) = 2$.*

Proof. We first colour the H-layers of the graph $G \square H$. There are at least two H-layers, so we colour all edges of one layer blue, all edges of another one with a distinguishing red-blue colouring. If there are more H-layers, then we colour them such that each of them has a different number of blue edges (including the H-layers coloured previously). This is possible since $|G| \leq \|H\| + 1$. Next, we colour all edges in every G-layer red.

All automorphisms of the Cartesian product generated by the automorphisms of H are broken, since one H-layer assumes a distinguishing colouring. Also, no H-layers can be interchanged as every H-layer has different number of blue edges.

If H has a factor H' isomorphic to G, then $G\Box H$ has an automorphism interchanging H' with G. However, since all G-layers have only red edges and there exists an H-layer with only blue edges, such an automorphism does not preserve this colouring. $\Box$

Lemma 2.5. *If H is a graph with $2 \le D'(H) = d$, then*

$$2 \le D'(H\Box K_2) \le d.$$

Proof. We colour the edges of one H-layer with a distinguishing d-colouring, and all the edges of the other H-layer with the same colour, say 1. Next, we colour all edges of K_2-layers with colour 2. Thus all automorphisms of the Cartesian product $H\Box K_2$ generated by the automorphisms of H are broken, since one of the H-layers assumes a distinguishing colouring. Also, the two H-layers cannot be interchanged as they have different numbers of edges coloured with 1.

If H has a factor H' isomorphic to K_2, then $K_2\Box H$ has an automorphism interchanging H' with K_2. However, since all K_2-layers have only colour 2 and there exists an H-layer with all edges coloured with 1, such an automorphism does not preserve the colouring.

The equality for $d = 2$ is obvious since the prism of every graph has a non-trivial automorphism. $\Box$

We now consider the Cartesian powers of arbitrary connected graphs and continue with powers of connected prime graphs on at least three vertices.

Lemma 2.6. *If G is a connected prime graph with $|G| \ge 3$, then $D'(G^k) = 2$ for every $k \ge 2$.*

Proof. The proof goes by induction on k. Let $k = 2$. There are n horizontal and n vertical layers, where $n = |G|$.

Suppose first that G contains a cycle, i.e., $\|G\| \ge n$. We colour horizontal G-layers with two colours such that each of them has a different number of blue edges between 0 and $n-1$. The other edges are coloured such that every vertical G-layer has a different number of blue edges between 1 to n. As every horizontal G-layer has a different number of blue edges they cannot be interchanged. The same is true for vertical G-layers. Therefore automorphisms of the Cartesian product generated by automorphisms of G are broken. Our colouring also breaks interchanging the factors, since there exists a completely red horizontal G-layer but no such vertical G-layer.

Assume now that G is a tree. Every tree has either a central vertex or a central edge fixed by every automorphism. In case of a tree with a central vertex v, we colour the edges of G^2 such that consecutive horizontal layers have $0, \dots, n-1$ blue edges, and consecutive vertical layers have $0, \dots, n-1$ blue edges in such a way that the horizontal G^v-layer and the vertical G^v-layer have all edges coloured red and blue, respectively. The vertex (v, v) is fixed by every automorphism of G^2, hence this colouring is distinguishing. If G has a central edge $e_0 = uv$, we colour the edge $(u, u)(v, u)$ red and the remaining three edges of the subgraph $e_0\Box e_0$ blue. The vertical and horizontal G^v-layers have all edges blue and red, respectively. The remaining edges of G^2 are coloured as in the case of a tree with a central vertex. Such colouring forbids exchange of the horizonal layers with the vertical layers. Thus $D'(G^2) = 2$.

For the induction step, we apply Lemma 2.4 by taking $H = G^{k-1}$ since $|G| \leq \|G^{k-1}\| + 1$. □

Let us now state the main theorem of this section that solves the problem of the distinguishing index of the k-th Cartesian power of a connected graph.

Theorem 2.7. *Let G be a connected graph and $k \geq 2$. Then*

$$D'(G^k) = 2$$

with the only exception: $D'(K_2^2) = 3$.

Proof. Let $G = G_1^{p_1} \square G_2^{p_2} \square \ldots \square G_r^{p_r}$, where $p_i \geq 1, i = 1, \ldots, r$, be the prime factor decomposition of G.

Assume first that $G_i \neq K_2$, $i = 1, 2, \ldots, r$. Then for every i we have $D'(G_i^{kp_i}) = 2$ due to Lemma 2.6. By repetitive application of Lemma 2.3 we get $D'(G^k) = 2$ since $G_i^{kp_i}$ and $G_j^{kp_j}$ are relatively prime if $i \neq j$.

Suppose now that G has a factor isomorphic to K_2, say $G_1 = K_2$. If $p_1 \geq 2$, then $D'(K_2^{kp_1}) = 2$ and again $D'(G^k) = 2$ by Lemma 2.3 applied to $K_2^{kp_1}$ and $G_2^{p_2} \square \ldots \square G_r^{p_r}$. The same argument applies in case $p_1 = 1$ and $k \geq 3$. Finally, if $p_1 = 1$ and $k = 2$ we apply Lemma 2.4 twice and we also get $D'(G^k) = 2$ unless $r = 1$, i.e., $G^k = K_2^2$. □

3 Distinguishing Cartesian products

In this section we investigate sufficient conditions on the sizes of factors of the Cartesian product of two graphs to have the distinguishing index equal to two.

3.1 Trees

We begin with a result for trees. Observe first that, by Theorem 1.2, the Cartesian product of two non-isomorphic asymmetric trees is an asymmetric graph, so its distinguishing index is equal to 1.

Theorem 3.1. *Let T_m and T_n be trees of size m and n, respectively. If*

$$2 \leq m \leq n \leq 2^{2m+1} - \left\lceil \frac{m}{2} \right\rceil + 1,$$

then $D'(T_m \square T_n) \leq 2$.

Proof. If T_m is isomorphic to T_n, then the conclusion follows from Lemma2.6. Therefore, assume that T_m and T_n are non-isomorphic. Then they are relatively prime, and it is enough to prove the existence of a 2-colouring of edges of $T_m \square T_n$ that breaks the automorphisms generated by automorphisms of T_m and those generated by automorphisms of T_n.

In the proof we use the following well-known fact. In a rooted tree, if a parent vertex is fixed by every automorphism preserving a given colouring and its children cannot be permuted, then the children are also fixed.

Assume first that $n = 2^{2m+1} - \lceil \frac{m}{2} \rceil + 1$. We choose a root u_0 of T_m as follows. If T_m has a central vertex, then we take it as a root u_0. If T_m has a central edge, then we choose one of its end-vertices as u_0 and the other one as u_1. Then we choose an enumeration $u_0, \dots, u_m$ of vertices of the rooted tree T_m satisfying the following condition: if u_i is the parent of u_j, then $i < j$. We enumerate the edge $u_i u_j = e_j$. Thus $E(T_m) = \{e_1, \dots, e_m\}$. Let v_0 be a root of T_n chosen by the same rule as the root u_0 of T_m. Then we analogously enumerate vertices and edges of T_n to obtain $V(T_n) = \{v_0, \dots, v_n\}$, $E(T_n) = \{\varepsilon_1, \dots, \varepsilon_n\}$.

We begin by colouring the $T_m^{v_0}$-layer by putting colour 0 on the edges e_i, for $i = 1, \dots, \lceil \frac{m}{2} \rceil$, and colour 1 on the remaining edges of this layer. It is easy to see that we can choose such an enumeration of vertices, and hence of edges, that the root u_0 is fixed by every automorphism of T_m preserving this colouring. Indeed, this is obvious if u_0 is a central vertex; if $e_1 = u_0 u_1$ is a central edge of T_m, then we enumerate the vertices such that $u_1, \dots, u_{\lfloor \frac{m}{2} \rfloor}$ belong to the same subtree of $T_m - e_1$, therefore our colouring breaks all automorphisms of T_m reversing the end-vertices of e_1.

Then, we similarly colour the $T_n^{u_0}$-layer with 0 and 1 in such a way that the vertex (u_0, v_0) is fixed by every automorphism of $T_m \Box T_n$ preserving this partial colouring. Hence, the $T_m^{v_0}$-layer, as well as the $T_n^{u_0}$-layer, is mapped onto itself by every $\varphi \in \mathrm{Aut}(T_m \Box T_n)$ preserving this colouring.

Next, we colour the other layers. Consider the set S of all 2^{2m+1} binary sequences of length $2m+1$. Each $T_m^{v_i}$-layer with $i \geq 1$ is assigned a distinct sequence

$$s_i = (a_0, a_1, \dots, a_m, b_1 \dots, b_m)$$

from S, where $a_j, j = 0, \dots, m$, is the colour of the edge ε_i joining the vertex (u_j, v_i) with its parent in the $T_n^{u_j}$-layer (observe that a_0 has been already defined for all $i \geq 1$), and $b_j, j = 1, \dots, m$ is the colour of the edge of the $T_m^{v_i}$-layer corresponding to e_j. To break all permutations of T_n-layers we delete some sequences from the set S. First observe that the sum of each coordinate taken over all sequences in S is the same (and equal to 2^{2m}). Clearly, a $T_n^{u_j}$-layer and a $T_n^{u_{j'}}$-layer cannot be permuted whenever $j \leq \lceil \frac{m}{2} \rceil < j'$ since the edges e_j and $e_{j'}$ in the $T_m^{v_0}$-layer have different colours.

Consider the set $A = \{s^k \in S \ : \ k = 1, \dots, \lceil \frac{m}{2} \rceil - 1\}$, where $s^k = (a_0, a_1, \dots, a_m, b_1, \dots, b_m)$ is a sequence such that

$$a_j = a_{\lceil \frac{m}{2} \rceil + j} = 1, \quad j = 1, \dots, k,$$

and all other elements of s^k are equal to 0. Thus $|S \setminus A| = 2^{2m+1} - \lceil \frac{m}{2} \rceil + 1$. We use the set $S \setminus A$ to colour $T_m^{v_i}$-layers, $i = 1, \dots, 2^{2m+1} - \lceil \frac{m}{2} \rceil + 1$, hence the numbers of edges coloured with 1 is distinct for every pair of T_n-layers that could be permuted. Thus, all edges in $T_m \Box T_n$ are coloured, and we obtain a distinguishing 2-colouring of $T_m \Box T_n$, when $n = 2^{2m+1} - \lceil \frac{m}{2} \rceil + 1$.

Now, assume that the difference $l = 2^{2m+1} - \lceil \frac{m}{2} \rceil + 1 - n$ is positive. We have to choose l sequences from $S \setminus A$ that will not be used in the colouring. To do this we apply the idea of complementary pairs used in [8]. Denote $\overline{0} = 1, \overline{1} = 0$. A pair of sequences

$$(a_0, a_1, \dots, a_m, b_1, \dots, b_m), \quad (a_0, \overline{a_1}, \dots, \overline{a_m}, b_1, \dots, b_m)$$

from $S \setminus A$ is called *complementary*. When l is even, we choose $\frac{l}{2}$ complementary pairs. When l is odd, we choose the sequence $(0, \dots, 0) \in S \setminus A$ and $\frac{l-1}{2}$ complementary pairs. Again all permutations of layers in $T_m \Box T_n$ are broken by this colouring since for every pair of T_n-layers that could be permuted, the numbers of edges coloured with 1 is distinct, because $a_j + \overline{a_j} = 1$, $j = 1, \dots, m$. □

The bound $2^{2m+1} - \lceil \frac{m}{2} \rceil + 1$ for the size of a larger tree is perhaps not sharp. However, it cannot be improved much since Proposition 3.2 below shows that the distinguishing index of the Cartesian product of a star $K_{1,n}$ of size n and a path P_m of order m is greater than 2 whenever $n > 2^{2m+1}$. It also shows that the distinguishing index of the Cartesian product of two graphs with widely different orders and sizes can be arbitrarily large.

Proposition 3.2. *If $m \geq 2$ and $n \geq 2$, then*

$$D'(K_{1,n} \square P_m) = \left\lceil \sqrt[2m-1]{n} \right\rceil$$

unless $m = 2$ and $n = r^3$ for some r. In the latter case $D'(K_{1,n} \square P_2) = r + 1$.

Proof. Let d be a positive integer such that $(d-1)^{2m-1} < n \leq d^{2m-1}$. Denote by v the central vertex of the star $K_{1,n}$, by $v_1, \dots, v_n$ its pendant vertices, and by $u_1, \dots, u_m$ consecutive vertices of P_m.

Suppose first that $m \geq 3$. Clearly, every automorphism of $K_{1,n} \square P_m$ maps the P_m^v-layer onto itself. We colour the first edge of this layer with 1 and all other edges of it with 2. Thus the P_m^v-layer is fixed by every automorphism, hence the $K_{1,n}$-layers cannot be permuted.

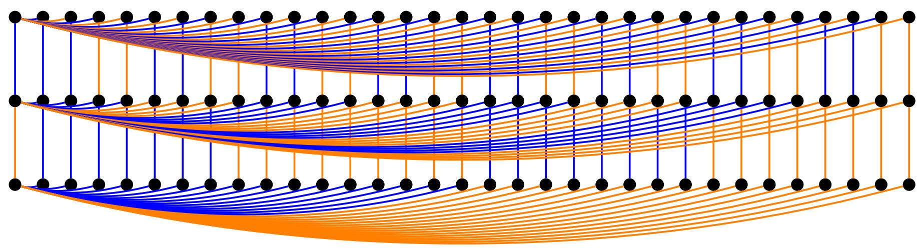

Figure 1: A distinguishing 2-colouring of $K_{1,32} \square P_3$

We want to show that the remaining edges of $K_{1,n} \square P_m$ can be coloured in such a way that P_m-layers also cannot be interchanged. Then all non-trivial automorphisms of $K_{1,n} \square P_m$ will be broken. Note that a colouring of the yet uncoloured edges can be fully described by defining a matrix M with $2m-1$ rows and n columns such that in the j-th column the initial $m-1$ elements are colours of consecutive edges of the $P_m^{v_j}$-layer, and the other m elements are colours of the edges of $K_{1,n}$-layers incident to consecutive vertices of the $P_m^{v_j}$-layer. It is easily seen that there exists a permutation of P_m-layers preserving colours if and only if M contains at least two identical columns. There are exactly d^{2m-1} sequences of length $2m-1$ with elements from the set $\{1, \dots, d\}$, hence there exists a colouring with d colours such that every column of M is distinct. Therefore, $D'(K_{1,n} \square P_m) \leq d = \lceil \sqrt[2m-1]{n} \rceil$. On the other hand, $n > (d-1)^{2m-1}$ so for every edge $(d-1)$-colouring of $K_{1,n} \square P_m$, the corresponding matrix has to contain two identical columns, therefore $D'(K_{1,n} \square P_m) > d-1$. Figure 1 presents the case $n = 32$ and $m = 3$.

For $m = 2$, we colour the edges of $K_{1,n} \square P_2$ in the same way. The only difference is that every P_2-layer has only one edge, hence the two $K_{1,n}$-layers need not be fixed. This is the case when $n = d^3$, because then each element of $\{1, \dots, d\}^3$ is a column in M, and there exists a permutation of columns of M which together with the transposition of rows of M defines a non-trivial automorphism of $K_{1,n} \Box P_2$ preserving the colouring. Thus we need an additional colour for one edge in a $K_{1,n}$-layer. When $n < d^3$, we put the sequence $(1, 1, 2)$ as the first column of M, and we do not use the sequence $(1, 2, 1)$ any more, thus breaking the transposition of the $K_{1,n}$-layers, and all automorphisms of $K_{1,n} \Box P_2$. $\Box$

Let us mention in passing that $D'(K_{1,n} \Box C_m) = \lceil \sqrt[2m]{n} \rceil$, unless $m \leq 5$ and $n = 2^{2m}$. In the latter case $D'(K_{1,n} \Box C_m) = \sqrt[2m]{n} + 1 = 3$. The proof can be led on the lines of the proof of Proposition 3.2.

3.2 Arbitrary factors

We now consider the Cartesian product of arbitrary connected graphs. We first formulate a result for relatively prime factors.

Lemma 3.3. *Let G and H be connected, relatively prime graphs such that*

$$3 \leq |G| \leq |H| \leq 2^{|G|}\left(2^{\|G\|} - 1\right) - |G| + 2.$$

Then $D'(G \Box H) \leq 2$.

Proof. Let $V(G) = \{u_1, \dots, u_{|G|}\}$, $E(G) = \{e_1, \dots, e_{\|G\|}\}$, $V(H) = \{v_1, \dots, v_{|H|}\}$, $E(H) = \{\varepsilon_1, \dots, \varepsilon_{\|H\|}\}$. Assume that v_1 is a root of a spanning tree T_H of the graph H, and the vertices of H are enumerated according to the rooted tree T_H, i.e., each child has an index greater than that of its parent. As G and H are relatively prime, the only automorphisms of $G \Box H$ are permutations of G-layers and H-layers.

We first colour the edges of the G^{v_1}-layer with a sequence

$$(b_1, \dots, b_{\|G\|}) = (1, \dots, 1).$$

We shall not use this sequence to colour the edges of any other G-layer any more. Thus the G^{v_1}-layer will be mapped onto itself by every automorphism of $G \Box H$ preserving the colouring.

From now on, we proceed similarly as in the proof of Theorem 3.1. For $i = 2, \dots, n$, the G^{v_i}-layer will be assigned a distinct sequence of colours

$$(a_1, \dots, a_{|G|}, b_1, \dots, b_{\|G\|}),$$

where a_j is a colour of the edge joining the vertex (u_j, v_i) to its parent in the rooted tree T_H in the H^{u_j}-layer, and b_j is a colour of e_j in the G^{v_i}-layer. We have $2^{|G|}\left(2^{\|G\|} - 1\right)$ such sequences, as we excluded all sequences of the form $(a_1, \dots, a_{|G|}, 1, \dots, 1)$. Thus all permutations of G-layers are broken. To break permutations of H-layers, we also exclude all sequences $s^k = (a_1, \dots, a_{|G|}, b_1, \dots, b_{\|G\|})$ with $a_1 = \dots = a_k = 1$ and $a_{k+1} = \dots = a_{|G|} = b_1 = \dots = b_{\|G\|} = 0$, for every $k = 1, \dots, |G| - 1$. We have $2^{|G|}\left(2^{\|G\|} - 1\right) - (|G| - 1)$ sequences to colour $|H| - 1$ G-layers. Depending on the size of $|H|$, we also exclude a suitable number of complementary pairs of sequences

$$(a_1, \dots, a_{|G|}, b_1, \dots, b_{\|G\|}), \quad (\overline{a_1}, \dots, \overline{a_{|G|}}, b_1, \dots, b_{\|G\|})$$

and, possibly, a sequence $(0,\dots,0)$. Thus we obtain a colouring of $G\Box H$ with a set of sequences such that the number of 1's is distinct in any of the initial $|G|$ coordinates. Therefore, no permutation of H-layers preserves this colouring. Hence, this is a distinguishing 2-colouring of $G\Box H$. □

Finally, we state the main result of this section.

Theorem 3.4. *Let G and H be connected graphs such that*

$$2 \le |G| \le |H| \le 2^{|G|}\big(2^{\|G\|}-1\big) - |G| + 2.$$

Then $D'(G\Box H) \le 2$ unless $G = H = K_2$.

Proof. If $G = K_2$, then $|H| \le 4$. If $H \ne K_4$, then either $D'(H) = 2$ or H is a cycle or a star, and these cases were already settled in Section 2. To construct a distinguishing 2-colouring of $K_2\Box K_4$, colour one edge in one K_4-layer and two adjacent edges in the other K_4-layer red, and all remaining edges blue.

Let $|G| \ge 3$. The case when G and H are relatively prime was settled by Lemma 3.3. Therefore, we focus here on the situation when G and H have at least one common factor. Then $D'(G\Box H) \ge 2$, since the automorphism group of $G\Box H$ is non-trivial. Let $G = G_1^{k_1}\Box\dots\Box G_t^{k_t}$ and $H = H_1^{l_1}\Box\dots\Box H_s^{l_s}$ be the prime factor decompositions of G and H, respectively. Assume that the initial r factors are common, i.e., $G_i = H_i$ for $i = 1,\dots,r$. Denote

$$G_{II} = G_1^{k_1}\Box\dots\Box G_r^{k_r}, \qquad H_{II} = H_1^{l_1}\Box\dots\Box H_r^{l_r}.$$

Thus $G = G_I\Box G_{II}$ and $H = H_I\Box H_{II}$. We use the following notation

$$n_1 = |G_I|, \quad n_2 = |G_{II}|, \quad m_1 = |H_I|, \quad m_2 = |H_{II}|.$$

We first show that the distinguishing index of the Cartesian product

$$G_{II}\Box H_{II} = G_1^{l_i+k_1}\Box\dots\Box G_r^{l_r+k_r}$$

is equal to 2. If $G_{II}\Box H_{II} = K_2^2$, then $|H_I| \ge 2$ and the graphs $G_I\Box K_2^2$ and H_I satisfy the assumptions of Theorem 3.3, hence $D'(G\Box H) = 2$, unless $|G_I\Box K_2^2| > |H_I|$, that is $|H_I| < 4|G_I|$. In the latter case, we can also apply Theorem 3.3 for the graphs G_I and H_I which are relatively prime and satisfy the inequalities $|G_I| \le |H_I| \le 2^{|G_I|}(2^{\|G_I\|}-1) - |G_I| + 2$ unless $|G_I| = 2$ and $\le |H_I| \le 7$, i.e., $G\Box H = K_2^3\Box H_I'$, where H_I' is prime. So we can apply Proposition 2.2 and Lemma 2.4. In any case $D'(G\Box H) = 2$.

If $G_i^{l_i+k_i} \ne K_2^2$ for every $i = 1,\dots,r$, then $D'(G_i^{l_1+k_i}) = 2$ due to Theorem 2.7, and hence $D'(G_{II}\Box H_{II}) = 2$ by repeated application of Lemma 2.3. If $G_1^{l_1+k_1} = K_2^2$, then analogously $D'(G_2^{l_2+k_2}\Box\dots\Box G_r^{l_r+k_r}) = 2$, hence $D'(G_{II}\Box H_{II}) = 2$ by applying Lemma 2.5 twice.

Consider now the graphs $G' = G_I\Box G_{II}\Box H_{II}$ and $H' = H_I$. Clearly, they are relatively prime and

$$|H'| < |H| \le 2^{|G|}\big(2^{\|G\|}-1\big) - |G| + 2 < 2^{|G'|}\big(2^{\|G'\|}-1\big) - |G'| + 2.$$

If also $|G'| = n_1n_2m_2 \leq m_1 = |H'|$, then graphs G' and H' satisfy the conditions of Lemma 3.3, and consequently, $D'(G\square H) = D'(G'\square H') = 2$. Then suppose that $n_1n_2m_2 > m_1$. We consider two cases here.

Assume first that $n_1 \leq n_2m_2$, i.e., $|G_I| \leq |G_{II}\square H_{II}|$. Hence, $|G_I| \leq \|G_{II}\square H_{II}\| + 1$, and by repeated application of Lemma 2.4 we get $D'(G') = 2$. As $|H'| < |G'|$, we infer again from Lemma 2.4 that $D'(G\square H) = D'(G'\square H') = 2$.

In the second case, i.e., when $n_2m_2 < n_1$, suppose first that

$$m_1 = |H_I| \leq 2^{|G_I|}\left(2^{\|G_I\|} - 1\right) - |G_I| + 2.$$

Then $D'(G_I\square H_I) \leq 2$ since the assumptions of Lemma 3.3 are satisfied whenever $|G_I| \leq |H_I|$. Recall that also $D'(G_{II}\square H_{II}) = 2$ and graphs $G_I\square H_I$ and $G_{II}\square H_{II}$ are relatively prime. Hence $D'(G\square H) = 2$ by Lemma 2.3. Otherwise, if $m_1 > 2^{|G_I|}\left(2^{\|G_I\|} - 1\right) - |G_I| + 2$, then we arrive at the sequence of inequalities

$$m_1 < n_1n_2m_2 \leq n_1^2 < 2^{n_1}(2^{n_1} - 1) - n_1 + 2 \leq 2^{|G_I|}\left(2^{\|G_I\|} - 1\right) - |G_I| + 2 < m_1,$$

which is impossible.

Then suppose that $|G_I| = n_1 > m_1 = |H_I|$ (and still $n_2m_2 < n_1$). Let $G'' = G_I$ and $H'' = G_{II}\square H_I\square H_{II}$. Clearly, $|G''| \leq |H''|$ since $|G| \leq |H|$. Moreover, we have

$$|H''| = n_2m_2m_1 < n_1m_1 < n_1^2 < 2^{|G''|}\left(2^{\|G''\|} - 1\right) - |G''| + 2.$$

It follows from Lemma 3.3 that $D'(G\square H) = D'(G''\square H'') = 2$. □

Acknowledgment. The authors are very indebted to Wilfried Imrich for introducing them to the concept of symmetry breaking in graphs, and for many helpful discussions and suggestions.

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