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# Improving upper bounds for the distinguishing index

Monika Pilśniak *

*AGH University, Department of Discrete Mathematics,*
*al. Mickiewicza 30, 30-059 Krakow, Poland*

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**Abstract**

The distinguishing index of a graph $G$, denoted by $D'(G)$, is the least number of colours in an edge colouring of $G$ not preserved by any non-trivial automorphism. We characterize all connected graphs $G$ with $D'(G) \geq \Delta(G)$. We show that $D'(G) \leq 2$ if $G$ is a traceable graph of order at least seven, and $D'(G) \leq 3$ if $G$ is either claw-free or 3-connected and planar. We also investigate the Nordhaus-Gaddum type relation: $2 \leq D'(G) + D'(\overline{G}) \leq \max\{\Delta(G), \Delta(\overline{G})\} + 2$ and we confirm it for some classes of graphs.

Keywords: Edge colouring, symmetry breaking in graph, distinguishing index, claw-free graph, planar graph.

Math. Subj. Class.: 05C05, 05C10, 05C15, 05C45

## 1 Introduction

We follow standard terminology and notation of graph theory (cf. [12]). In this paper, we consider general, i.e. not necessarily proper, edge colourings of graphs. Such a colouring $f$ of a graph $G$ *breaks an automorphism* $\varphi \in \mathrm{Aut}(G)$ if $\varphi$ does not preserve colours of $f$. The *distinguishing index* $D'(G)$ of a graph $G$ is the least number $d$ such that $G$ admits an edge colouring with $d$ colours that breaks all non-trivial automorphisms (such a colouring is called a *distinguishing edge $d$-colouring*). Clearly, $D'(K_2)$ is not defined, so in this paper, a graph $G$ is called *admissible* if neither $G$ nor $\overline{G}$ contains $K_2$ as a connected component.

The definition of $D'(G)$ introduced by Kalinowski and Pilśniak in [17] was inspired by the *distinguishing number* $D(G)$ which was defined for general vertex colourings by Albertson and Collins [1]. Another concept is the *distinguishing chromatic number* $\chi_D(G)$

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*E-mail address:* pilsniak@agh.edu.pl (Monika Pilśniak)

introduced by Collins and Trenk [7] for proper vertex colourings. Both numbers, $D(G)$ and $\chi_D(G)$, have been intensively investigated by many authors in recent years [4, 5, 6, 9, 16].

Our investigation was motivated by the renowned result of Nordhaus-Gaddum [18] who proved in 1956 the following lower and upper bounds for the sum of the chromatic numbers of a graph and its complement (actually, the upper bound was first proved by Zykov [22] in 1949).

**Theorem 1.1** ([18]). *If $G$ is a graph of order $n$ with the chromatic number $\chi(G)$, then*

$$2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1.$$

Since then, Nordhaus-Gaddum type bounds were obtained for many graph invariants. An exhaustive survey is given in [2]. Here, we adduce only those closely related to the topic of our paper.

In 1964, Vizing [20] considered proper edge colourings and he proved Nordhaus-Gaddum type bounds for the chromatic index of a graph.

**Theorem 1.2** ([20]). *If $G$ is a graph of order $n$ with the chromatic index $\chi'(G)$, then*

$$n - 1 \leq \chi'(G) + \chi'(\overline{G}) \leq 2(n - 1).$$

In 2013, Collins and Trenk [8] proved Nordhaus-Gaddum type inequalities for the distinguishing chromatic number.

**Theorem 1.3** ([8]). *For every graph of order $n$ and distinguishing number $D(G)$ the following inequalities are satisfied*

$$2\sqrt{n} \leq \chi_D(G) + \chi_D(\overline{G}) \leq n + D(G).$$

Kalinowski and Pilśniak [17] also introduced a *distinguishing chromatic index* $\chi'_D(G)$ of a graph $G$ as the least number of colours in a proper edge colouring that breaks all non-trivial automorphisms of $G$. They proved the following somewhat unexpected result.

**Theorem 1.4** ([17]). *If $G$ is a connected graph of order $n \geq 3$, then*

$$\chi'_D(G) \leq \Delta(G) + 1$$

*unless $G \in \{C_4, K_4, C_6, K_{3,3}\}$ when $\chi'_D(G) \leq \Delta(G) + 2$.*

The following Nordhaus-Gaddum type inequalities for the distinguishing chromatic index are the same as in Theorem 1.2 but we have to be more careful in the proof.

**Theorem 1.5.** *If $G$ is an admissible graph of order $n \geq 3$, then*

$$n - 1 \leq \chi'_D(G) + \chi'_D(\overline{G}) \leq 2(n - 1)$$

*with the only exception $K_{1,4}$.*

*Proof.* Without loss of generality we may assume that $G$ is connected. It can be easily checked that the conclusion holds if $G \in \{K_4, C_6, \overline{C_6}, K_{3,3}\}$. Otherwise, $\chi'_D(G) \leq \Delta(G) + 1$. Suppose first that $\overline{G}$ is also connected. By Theorem 1.4,

$$\Delta(G) + \Delta(\overline{G}) \leq \chi'_D(G) + \chi'_D(\overline{G}) \leq \Delta(G) + \Delta(\overline{G}) + 2.$$

Clearly, $n-1 \leq \Delta(G)+\Delta(\overline{G}) \leq 2(n-2)$ since both $G$ and $\overline{G}$ are connected.

Now, let $\overline{G}$ be disconnected (but admissible). If there are two nonisomorphic components of $\overline{G}$ of orders $k_1$ and $k_2$ such that $3 \leq k_1 \leq k_2$, then $\Delta(\overline{G}) \leq n-k_1-1 \leq n-4$, so $\chi'_D(\overline{G}) \leq n-2$. If $\overline{G}$ has $t \geq 2$ components isomorphic to a graph $H$ of order at least three, then $\chi'_D(H) \leq \frac{n}{t}+1$ as $\Delta(H) \leq \frac{n}{t}-1$. Even if we wastefully add an extra colour for each additional copy of $H$, we get $\chi'_D(tH) \leq \frac{n}{t}+1+t-1 = \frac{n}{t}+t \leq n-2$ unless $G = K_{3,3}$ but this we already checked.

To complete the proof it is enough to settle the case when $\overline{G}$ has only one component $H$ of order at least three and some isolated vertices. Hence, $\Delta(H) \leq n-2$. It is easy to check that $\chi'_D(G)+\chi'_D(\overline{G}) \leq 2(n-1)$ for $H \in \{K_4, C_6, \overline{C_6}, K_{3,3}\}$ except for $H = K_4$ when $G = K_{1,4}$. Otherwise, $\chi'_D(\overline{G}) \leq n-1$ and the conclusion holds unless $|G| = |H|+1$ and $\Delta(H) = n-2$. But then $G$ has a unique vertex $x$ of degree $n-1$ (hence, $x$ is fixed by every automorphism of $G$) with a pendant edge. The graph $G-x$ has a distinguishing colouring with $n-1$ colours by Theorem 1.4 since $\Delta(G-x) \leq n-2$. It suffices to colour the pendant edge with a colour missing at $x$ to see that $\chi'_D(G) \leq n-1$. □

Collins and Trenk observed in [8] that the Nordhaus-Gaddum type relation is trivial for the distinguishing number, as $D(G)+D(\overline{G}) = 2D(G)$ since $\mathrm{Aut}(\overline{G}) = \mathrm{Aut}(G)$ and every colouring of $V(G)$ breaking all non-trivial automorphisms of $G$ also breaks those of $\overline{G}$.

In Section 4 we formulate and discuss the following conjecture.

**Conjecture 1.6.** *Let $G$ be an admissible graph of order $n \geq 7$, and let $\Delta = \max\{\Delta(G), \Delta(\overline{G})\}$. Then*

$$2 \leq D'(G)+D'(\overline{G}) \leq \Delta+2.$$

In Section 2 we characterize graphs $G$ which need exactly $\Delta(G)$ colours to break all non-trivial automorphisms. In Section 3 we give upper bounds for the distinguishing index of traceable graphs, claw-free graphs, planar graphs and 2-connected graphs.

## 2 Improved general upper bound

In the sequel, we make use of some facts proved in [17].

**Proposition 2.1** ([17]). *$D'(P_n) = 2$ for every $n \geq 3$.*

**Proposition 2.2** ([17]). *$D'(C_n) = 3$ for $n \leq 5$, and $D'(C_n) = 2$ for $n \geq 6$.*

**Proposition 2.3** ([17]). *$D'(K_n) = 3$ if $3 \leq n \leq 5$, and $D'(K_n) = 2$ if $n \geq 6$.*

**Proposition 2.4** ([17]). *$D'(K_{3,3}) = 3$, and $D'(K_{n,n}) = 2$ if $n \geq 4$.*

By the well-known theorem of Jordan (cf. [12]), every finite tree $T$ has either a central vertex or a central edge, which is fixed by every automorphism of $T$. In the proof of Theorem 2.8, which is the main result of this section, we use Lemma 2.5, a simple generalization of the theorem of Jordan. Recall that the *eccentricity* of a vertex $v$ in a connected graph $G$ is the number

$$\varepsilon_G(v) = \max\{d(v,u) \ : \ u \in V(G)\}.$$

The *center* of a graph $G$ is the set $Z(G)$ of vertices with minimum eccentricity. Clearly, the center of $G$ is setwise fixed by every automorphism $\varphi \in \mathrm{Aut}(G)$, i.e. $\varphi(v) \in Z(G)$ if $v \in Z(G)$. A proper subgraph $H$ of $G$ is called *pendant* if it has only one vertex adjacent to vertices outside $H$.

**Lemma 2.5.** *Let $G$ be a connected graph such that every cycle is contained in a clique. Then the center of $G$ is either a single vertex or a maximal clique.*

*Proof.* The claim is true if $G$ is a clique $K_k$ of order $k \geq 1$. Otherwise, $\kappa(G) = 1$, and each block of $G$ is a clique of order at least two. We then modify the standard proof of the theorem of Jordan for trees. Let $G^-$ be a graph obtained from $G$ by deleting $k-1$ vertices of degree $k-1$ in every pendant clique $K_k$ with $k \geq 2$. Clearly, $\varepsilon_{G^-}(v) = \varepsilon_G(v) - 1$ for each $v \in V(G^-)$. Consequently, $Z(G^-) = Z(G)$. We continue this process until only one clique $K_k$ is left for some $k \geq 1$. This clique is maximal whenever $k \geq 2$. □

A *symmetric tree*, denoted by $T_{h,d}$, is a tree with a central vertex $v_0$, all leaves at the same distance $h$ from $v_0$ and all vertices that are not leaves of equal degree $d$. A *bisymmetric tree*, denoted by $T''_{h,d}$, is a tree with a central edge $e_0$, all leaves at the same distance $h$ from the edge $e_0$ and all vertices which are not leaves of equal degree $d$.

**Theorem 2.6** ([17])**.** *If $T$ is a tree of order $n \geq 3$, then $D'(T) \leq \Delta(T)$. Moreover, equality is achieved if and only if $T$ is either a symmetric or a bisymmetric tree.*

For connected graphs in general there is the following upper bound for $D'(G)$.

**Theorem 2.7** ([17])**.** *If $G$ is a connected graph of order $n \geq 3$, then*

$$D'(G) \leq \Delta(G)$$

*unless $G$ is $C_3$, $C_4$ or $C_5$.*

It follows for connected graphs that $D'(G) > \Delta(G)$ if and only if $D'(G) = \Delta(G) + 1$ and $G$ is a cycle of length at most 5. The equality $D'(G) = \Delta(G)$ holds for cycles of length at least 6, for $K_4$, $K_{3,3}$ and for all symmetric or bisymmetric trees. Now, we show that $D'(G) < \Delta(G)$ for all other connected graphs. A *palette* of a vertex is the multiset of colours of edges incident to it.

**Theorem 2.8.** *Let $G$ be a connected graph that is neither a symmetric nor a bisymmetric tree. If the maximum degree of $G$ is at least 3, then*

$$D'(G) \leq \Delta(G) - 1$$

*unless $G$ is $K_4$ or $K_{3,3}$.*

*Proof.* Denote $\Delta = \Delta(G)$. The conclusion holds for trees due to Theorem 2.6. Then assume that $G$ contains a cycle. The general idea of the proof is the following. If $G$ does not contain a cycle of length greater than three, then we define $G'$ as an empty graph. Otherwise, we consecutively delete pendant trees and pendant triangles until we obtain a subgraph $G'$. Then, we construct an edge colouring $f$ with $\Delta - 1$ colours stabilizing all vertices of $G'$ by every automorphism preserving $f$. Finally, we colour pendant subtrees and pendant triangles to complete a distinguishing colouring with $\Delta - 1$ colours of the whole graph $G$.

If $\Delta(G') = 2$, then $G'$ is a cycle $C_p$ having a distinguishing colouring with $\Delta - 1$ colours unless $p \in \{4, 5\}$ and $\Delta = 3$. In this case, it can be easily checked that the graph $G'_+$ induced by $C_p$ and the independent edges of $G$ incident to $C_p$ can always be coloured with two colours such that the vertices of $C_p$ are fixed by every colour preserving

automorphism. So we can assume that $\Delta(G') \geq 3$. If $G' \in \{K_4, K_{3,3}\}$, then $G' \neq G$ due to the assumption, hence $\Delta \geq 4$, so we can stabilize $K_4$ or $K_{3,3}$ with three colours.

Let $N_i(v)$ denote the *$i$-th sphere in* $v$, i.e. the set of vertices of distance $i$ from the vertex $v$. Let $x$ be a vertex with maximum degree in $G'$. We colour with 1 all edges incident with $x$. In our edge colouring $f$ of the graph $G'$, the vertex $x$ will be the unique vertex of maximum degree with the monochromatic palette $\{1, \ldots, 1\}$. Hence, $x$ will be fixed by every automorphism $\varphi$ preserving $f$. Consequently, $\varphi$ maps each sphere $N_i(x)$ onto itself.

The first sphere $N_1(x)$ can be partitioned into subsets $M_k$, for $k = 0, \ldots, \Delta - 1$, defined as

$$M_k = \{v \in N_1(x) \ : \ |N_1(v) \cap N_2(x)| = k\}.$$

Denote $M_k = \{v_1, \ldots, v_{l_k}\}$. Thus, $l_0 + l_1 + \ldots + l_{\Delta-1} = \Delta$.

We want to find a colouring $f$ of the edges of $G'[N_1(x) \cup N_2(x)]$ and, if necessary, of some subsequent spheres, such that each vertex of $N_1(x) \cup N_2(x)$ is fixed by every automorphism preserving this colouring. To do this, we proceed in a number of steps $M_k$, for $k = 0, \ldots, \Delta - 1$. In each step $M_k$, we find a colouring that fixes the vertices of $M_k$ and their neighbours in $N_2(x)$.

**Step $\mathbf{M_0}$.** First we consider the case when the subgraph $G'[M_0]$ induced by the vertices of $M_0$ is connected. Observe that $\Delta(G'[M_0]) \leq \Delta - 1$ and, by Theorem 2.7, we can colour distinguishingly the edges of $G'[M_0]$ with $\Delta - 1$ colours, even if $G'[M_0]$ is a short cycle $C_p$ with $3 \leq p \leq 5$. Indeed, if $G'[M_0] = C_3$ and $\Delta = 3$, then we would have $G = K_4$, but $K_4$ is excluded. Otherwise, $\Delta \geq 4$ and we can use a third colour in a short cycle $C_p$. It may happen that there exists a vertex $v \in M_0$ of degree $\Delta$ in $G'$ (so $|M_0| = \Delta$) with a monochromatic palette $\{1, \ldots, 1\}$ in a colouring of $G'[M_0]$ given by Theorem 2.7. In this case, either $G$ is a complete graph $K_n$ with $n \geq 5$ so $D'(K_n) \leq \Delta - 1$ by Proposition 2.3, or it is not difficult to see that there exists a colour $c$ such that there is no vertex with all incident edges coloured with $c$; whence we can exchange $c$ and 1 in this colouring of $G'[M_0]$.

Now, let $G'[M_0]$ be disconnected. Let $z_1, \ldots, z_s$ be isolated vertices or end-vertices of isolated edges in $G'[M_0]$. Clearly, $s \leq \Delta - 1$ by the definition of $G'$. If $s = \Delta - 1$, then we colour with $i$ every edge $z_i u$, where $u \in N_1(x) \setminus M_0$. Otherwise, we colour $z_i u$ with $i + 1$ for $i = 1, \ldots, s$. Thus, we avoid a monochromatic palette of $\{1, \ldots, 1\}$ at another vertex of maximum degree in $G'$.

We also have to distinguish all isomorphic components of $G'[M_0]$ of order greater than 2. Denote such a component by $H$ and suppose that $G'[M_0]$ contains $t$ components isomorphic to $H$, for some $t \geq 2$. Hence $t \leq \frac{\Delta}{3}$ and $\Delta(H) \leq \frac{\Delta}{t} - 1$. Therefore, we can choose distinct sets of $\frac{\Delta}{t}$ colours for every component since

$$\binom{\Delta - 1}{\frac{\Delta}{t}} \geq \binom{\Delta - 1}{3} \geq \frac{\Delta}{3} \geq t.$$

Thus each vertex of $M_0$ is fixed.

**Step $\mathbf{M_1}$.** For every $i = 1, \ldots, l_1$, we colour the edge $v_i u$, where $u \in N_2(x)$, with a distinct colour from $\{1, \ldots, \Delta - 1\}$. This is impossible only if $l_1 = \Delta$, when we have to have two vertices $a, b \in M_1$ with the same colour of edges $aa'$ and $bb'$, where $a'$ and $b'$ are neighbours of $a$ and $b$ in $N_2(x)$, respectively. If $G'[M_1]$ contains an edge $e$, then we colour it with 1, and all other edges of $G'[M_1]$ with 2. Then we choose exactly one of the vertices $a, b$ incident to $e$. We proceed analogously when $G'[N_2(x)]$ contains an edge. Then all

vertices of $M_1$ are fixed unless $l_1 = \Delta$ and neither $G'[N_1(x)]$ nor $G'[N_2(x)]$ contains an edge.

If $|N_2(x)| = 1$, then $G'$ is isomorphic to $K_{2,\Delta}$. It is easy to see that $D'(K_{2,\Delta}) \leq \Delta - 1$ for $\Delta \geq 3$ (for $\Delta \geq 4$ this immediately follows from Lemma 3.1 and Corollary 3.8). If $2 \leq |N_2(x)| \leq \Delta - 1$, then choosing $a$ and $b$ such that $a'$ has at least two neighbours in $N_1(x)$ and $b' \neq a'$ yields a colouring fixing $N_1(x) \cup N_2(x)$.

Suppose $|N_2(x)| = \Delta$. If there is a vertex $v \in N_2(x)$ with less than $\Delta - 1$ neighbours in $N_3(x)$, then we choose $a$ such that $a' = v$, and it suffices to reserve a unique set of colours for the edges between $a'$ and $N_3(x)$.

Hence, assume that every vertex of $N_2(x)$ has $\Delta - 1$ neighbours in $N_3(x)$. We select two vertices $a, b \in M_1$ and assume that the colours of the edges $aa'$ and $bb'$ are the same. Next, we implement the following Procedure SUBTREES $(a, b)$, which we also use in subsequent steps.

**Procedure** SUBTREES $(a, b)$

We are given two vertices $a, b \in N_1(x)$ such that each their neighbour in $N_2(x)$ is adjacent to $\Delta - 1$ vertices of $N_3(x)$.

Let $T_a$ be a maximal subtree of the graph $G'[\{a\} \cup \bigcup_{i \geq 2} N_i(x)]$, rooted at $a$, such that all leaves of $T_a$ belong to the same sphere $N_{l-1}(x)$ and each vertex of $V(T_a) \cap N_{i-1}(x)$ has $\Delta - 1$ neighbours in $N_i(x)$ for $i = 3, \ldots, l$. Thus $l \geq 3$. Define a graph

$$\widetilde{T_a} = G'[\bigcup_{v \in V(T_a) \setminus \{a\}} N(v)],$$

i.e. $\widetilde{T}_a$ is a graph obtained from $T_a$ by adding all edges incident with the leaves of $T_a$. Analogously, we define a tree $T_b$ and a graph $\widetilde{T}_b$. Observe that the trees $T_a$ and $T_b$ are disjoint and non-empty.

The edges incident to the roots $a$ and $b$ are already coloured. For every other vertex of $T_a$ and $T_b$, we colour its incident edges going to the next sphere with distinct colours from $\{1, \ldots, \Delta - 1\}$. Thus we obtain an edge colouring $f$. The only automorphism of $T_a$ (as well as of $T_b$) preserving $f$ is the identity. The vertex $x$ will be fixed by every colour preserving automorphism $\varphi$. Consequently, $\varphi$ maps $\widetilde{T}_a$ onto $\widetilde{T}_b$ whenever $\varphi(a) = b$. Thus, if $\widetilde{T}_a$ and $\widetilde{T}_b$ are not isomorphic, then $f$ distinguishes all vertices in $V(T_a) \cup V(T_b)$. Hence, assume that the rooted graphs $\widetilde{T}_a$ and $\widetilde{T}_b$ are isomorphic. Observe that there exists exactly one non-trivial isomorphism $\psi_0 \colon V(T_a) \to V(T_b)$ preserving $f$ since each vertex in $T_a$ has a distinct coloured path from the root $a$.

Denote $W_l = (V(\widetilde{T}_a) \cup V(\widetilde{T}_b)) \cap N_l(x)$. By our choice of $G'$, all vertices in $W_l$ are of degree at least two in $G'$. It follows that one of the following three cases has to hold.

**Case 1.** There exist vertices in $W_l$ adjacent to more than one vertex of $W_{l-1}$. Then we modify $f$ by colouring again all edges between such vertices and $W_{l-1}$ in order to break any possible permutation of $W_l$. A permutation of a set $L \subseteq W_l$ can be extended to an automorphism of $G'$ that fixes all leaves of $\widetilde{T}_a \cup \widetilde{T}_b$ only if every vertex from $L$ have the same set of neighbours $U = \{u_1, \ldots, u_d\}$ in $W_{l-1}$. Such a set $L$ contains at most $\Delta - 1$ leaves since the number of edges joining $U$ to $W_l$ equals $d(\Delta - 1)$. Every permutation of $L$ will be broken whenever for every vertex $w \in L$ the multiset of colours of the edges $wu_1, \ldots, wu_d$ will be distinct. Clearly, $d \leq \Delta$. There are $\binom{\Delta+d-2}{d}$ such possible multisets of $\Delta - 1$ colours. Clearly, $\binom{\Delta+d-2}{d} - 1 \geq \Delta - 1$ for $\Delta \geq 3$ and $d \geq 2$. We can exclude a

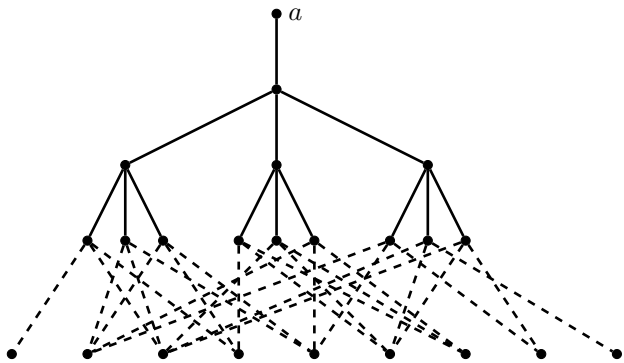


Figure 1: An example of the subgraph $\widetilde{T}_a$ for $\Delta = 4$ and $l = 4$. The edges of $\widetilde{T}_a$ between $W_3$ and $W_4$ that do not belong to the tree $T_a$ are dashed.

rainbow multiset $P = \{1, \ldots, d\}$ (or an almost rainbow multiset $P = \{1, \ldots, \Delta-1, \Delta-1\}$ if $d = \Delta$) and we still have enough multisets to colour the edges incident with vertices of $L$. Moreover, for $d = \Delta$ we can also exclude a monochromatic palette $\{1, \ldots, 1\}$ since $\binom{2\Delta-2}{\Delta} - 2 \geq \Delta - 1$ for $\Delta \geq 3$.

We partition the set $W_l$ into maximal subsets $L$ with the same set of neighbours and assign suitable multisets of colours to each set $L$. We thus obtain a colouring fixing all vertices from $W_l$ unless $\psi_0$ can be extended to an isomorphism $\widetilde{\psi_0}$ of $\widetilde{T}_a$ onto $\widetilde{T}_b$ preserving this colouring. To break every such possible extension $\widetilde{\psi_0}$, it suffices to assign the excluded multiset $P$ to one vertex of one set $L$.

**Case 2.** Every vertex in $W_l$ has only one neighbour in $W_{l-1}$ and the set of edges $F = E(G'[W_l])$ is non-empty. Then we colour one edge of $F$ with 1, and all other edges in $F$ with 2. This colouring fixes all vertices of $\widetilde{T}_a$ and $\widetilde{T}_a$ unless all edges in $F$ are of the form $w\widetilde{\psi_0}(w)$, where $w\widetilde{\psi_0}(w)$ is one of possible extensions of $\psi_0$ to an isomorphism of $\widetilde{T}_a$ onto $\widetilde{T}_b$. In such a case, we choose one edge $ww' \in F$ and exchange colours of the edge $wu$, where $u \in W_{l-1}$, with another edge between $u$ and $W_l$.

**Case 3.** Every vertex in $W_l$ has only one neighbour in $W_{l-1}$ and no neighbours in $W_l$. By the maximality of the trees $T_a$ and $T_b$ and the definition of $G'$, each vertex in $W_l$ has at least one neighbour in $N_{l+1}(x)$ and there exists a vertex $w_0 \in W_l$ with $s < \Delta - 1$ neighbours $y_1, \ldots, y_s \in N_{l+1}(x)$. We colour each edge $w_0 y_j$ with colour $j + 1$ for $j = 1, \ldots, s$. Next, for every vertex $w \in W_l$, we colour the set of edges between $w$ and $N_{l+1}(x)$ with a set of $\Delta - 1$ colours excluding the set $\{2, \ldots, s + 1\}$.

We thus obtained a colouring $f$ of the edges of $G'[V(\widetilde{T}_a) \cup V(\widetilde{T}_b)]$, and the edges incident to $W_l$ in Case 3, fixing all vertices of $\widetilde{T}_a$ and $\widetilde{T}_b$.

**End of Procedure** SUBTREES $(a, b)$

**Step $\mathbf{M_2}$.** For every $i = 1, \ldots, l_2$, we colour the edges $v_i u_i^1$, $v_i u_i^2$ where $\{u_i^1, u_i^2\} \subseteq N_2(x)$, with distinct sets of colours from among $\binom{\Delta-1}{2}$ sets. This is impossible only in the following three cases (in each case, we can assume that neither $G'[N_1(x)]$ nor $G'[N_2(x)]$ contains an edge, otherwise we could construct a distinguishing colouring $f$ of $G'[N_1(x) \cup N_2(x)]$ analogously as in step $M_1$):

a) $l_2 = \Delta = 4$. If there exist two vertices $a$ and $b$ in $M_2$ such that $N(a) \cap N(b) \cap N_2(x) \neq \emptyset$, then we colour with 2 both edges incident with $b$, and for the remaining vertices in $M_2$ we have distinct sets of colours from among $\binom{3}{2}$ sets. If for every two vertices $a, b \in M_2$, the set $N(a) \cap N(b) \cap N_2(x)$ is empty, then two vertices $a$ and $b$ are assign the same pair of distinct colours, and we can distinguish them in next spheres using the procedure SUBTREES $(a, b)$.

b) $l_2 = \Delta - 1$ and $\Delta = 3$. Let $M_2 = \{a, b\}$. If $N(a) \cap N(b) \cap N_2(x) \neq \emptyset$, then we colour edges incident with $a$ with colours 1 and 2, and both edges incident with $b$ with 2. If the set $N(a) \cap N(b) \cap N_2(x)$ is empty, then $a$ and $b$ get the same pair of distinct colours and we can distinguish them in next spheres by the procedure SUBTREES $(a, b)$.

c) $l_2 = \Delta = 3$. Let $M_2 = \{a, b, c\}$. If for two vertices of $M_2$, say $a$ and $b$, the set $N(a) \cap N(b) \cap N_2(x)$ is non-empty, then we can colour with 2 both edges incident with $b$ and we colour edges incident with the remaining vertices of $M_2$ with a couple $\{1, 2\}$. It is not difficult to verify that this way, for every configuration of neighbours of $M_2$, we can obtain colouring fixing the vertices of $N_1(x) \cup N_2(x)$ unless $|N(a) \cap N(b) \cap N(c) \cap N_2(x)| = 2$. But then $G' = G = K_{3,3}$, contrary to the assumption. If every vertex of $N_2(x)$ is adjacent only to one vertex of $M_2$, then the pairs of edges incident to $a$ and $b$ are assign the same pair of colours $\{1, 2\}$, and we distinguish them using the procedure SUBTREES $(a, b)$. Both edges $cu^1, cu^2$ incident with $c$ are coloured with 2, and to distinguish them, we split $c$ into two vertices $c^1$ and $c^2$, each joined by an edge coloured with 2 to $u^1$ and $u^2$, respectively, and apply the procedure SUBTREES $(c^1, c^2)$.

**Step $\mathbf{M_k}$, for $\mathbf{k \geq 3}$.** For every $i = 1, \ldots, l_k$, we colour the edges between $v_i$ and $N_2(x)$ with distinct sets of $k$ colours from among $\binom{\Delta-1}{k}$ sets. It is always possible whenever $\binom{\Delta-1}{k} \geq l_k$. This inequality does not hold only in two cases:

a) $k = \Delta - 2$ and $l_k = \Delta$. In this case we define a colouring with $\Delta - 1$ colours like in step $M_2$ a). Namely, if either a vertex of $M_k$ or its neighbour in $N_2(x)$ is adjacent to a vertex in the same sphere, then we can define a colouring fixing all these vertices analogously as in step $M_1$ and step $M_2$. Also, if there are two vertices $a, b \in M_{\Delta-2}$ with a common neighbour in $N_2(x)$, we can assign the same palette to $a$ and $b$ as in the previous steps. Otherwise, two vertices $a, b \in M_{\Delta-2}$ are assign the same palette of $\Delta - 2$ colours and we distinguish them using Procedure SUBTREES $(a, b)$.

b) $k = \Delta - 1$ and $l_k \geq 2$. Hence, $\Delta \geq 4$. For every $i = 1, \ldots, l_k$, the set of edges between $v_i \in M_{\Delta-1}$ and $N_2(x)$ will be assign a distinct multiset $P^i$ of colours from the set $\{1, \ldots, \Delta - 1\}$, where only colour $i$ appears twice. Moreover, one vertex can assign a rainbow palette $\{1, \ldots, \Delta - 1\}$. Thus every vertex of $M_{\Delta-1}$ will have a distinct palette, and hence will be stabilized. To stabilize the two vertices of $N_2(x)$ joined to $v_i$ by edges of colour $i$, we examine the vertices $v_1, \ldots, v_{\Delta-1}$ of $M_{\Delta-1}$ in the following order.

First, we consider each vertex $v_i$ that have a neighbour $w_i \in N_2(x)$ with at least one but at most $\Delta - 2$ neighbours in $N_3(x)$. We choose another neighbour $w_i' \in N_2(x)$ of $v_i$ and assign two distinct sets of colours for the edges going to $N_3(x)$ from $w_i$ and $w_i'$, respectively. We colour the edges $v_i w_i$ and $v_i w_i'$ with the same colour $i$. Thus all neighbours of $v_i$ are stabilized.

In the next stage, we consider every vertex $v_i$ with every neighbour in $N_2(x)$ adjacent to $\Delta - 1$ vertices of $N_3(x)$. We colour the set of edges between $v_i$ and $N_2(x)$ with the palette $P^i$, where two edges $v_iu_1, v_iu_2$ are coloured with $i$. Then we delete $v_i$ and introduce two vertices $v_i^1, v_i^2$ and edges $v_i^1u_1$ and $v_i^2u_2$ coloured with $i$. Then we use the procedure SUBTREES $(v_i^1, v_i^2)$ to stabilize $u_1$ and $u_2$.

Further, we consider each vertex $v_i$ with a neighbour $w_i \in N_2(x)$ incident to an edge $w_iu$, where $u \in N_2(x)$. First, we look for such an edge $w_iu$, which is already coloured. If there is no such edge, we take an uncoloured $w_iu$ and colour it with colour 3. In both cases, we put colour $i$ on the edge $v_iw_i$ and another edge $v_iw$ with $w \neq u$. After we examine each such vertex $v_i$, we colour with 2 all remaining edges contained in $N_2(x)$.

Finally, we are left with at most $\Delta$ vertices $v_i$ such that every neighbour of $v_i$ is adjacent only to (at least two) vertices of $N_1(x)$. We take a first such vertex $v_i$ and assign colour $i$ to two its incident edges $v_iw_i$ and $v_iw_i'$. Thus all neighbours of $v_i$ are stabilized unless common neighbours of $w_i$ and $w_i'$ were not considered yet. Then we take such a neighbour $v_j$ and colour its incident edges with the palette $P^j$ such that the edges $v_jw_i$ and $v_jw_i'$ have distinct colours. We repeat this procedure until only one vertex of $M_{\Delta-1}$ is left. We put a rainbow palette $\{1, \ldots, \Delta - 1\}$ on its incident edges.

After we accomplish steps $M_0, \ldots, M_{\Delta-1}$, we colour all uncoloured edges in subgraphs $G'[N_1(x)]$ and $G'[N_2(x)]$ with 2. Each vertex of $N_1(x) \cup N_2(x)$ is now fixed by every automorphism preserving our colouring $f$ of edges of $G'[\{x\} \cup N_1(x) \cup N_2(x)]$, and of some edges between next spheres, if the procedure SUBTREES was used.

Then we recursively colour all yet uncoloured edges incident to consecutive spheres $N_i(x)$ as follows: for $v \in N_i(x)$, $i \geq 2$, we colour all edges $vu$, where $u \in N_{i+1}(x)$, with distinct colours from $\{1, \ldots, \Delta - 1\}$. This is always possible since every vertex of $N_i(x)$ has at most $\Delta - 1$ neighbours in $N_{i+1}(x)$. Finally, we colour all uncoloured edges with end-vertices in the same sphere with 2. Hence, all vertices of $G'$ are fixed by any automorphism preserving our colouring $f$. It is also easily seen that the already coloured edges can save their colours. Moreover, it is not difficult to observe that $x$ is the unique vertex of maximum degree with a monochromatic palette $\{1, \ldots, 1\}$. Thus, the whole subgraph $G'$ (or $G'_+$) is fixed.

To end the proof, we colour pendant trees and triangles deleted from $G$ at the beginning. First assume that $G'$ is not empty. Let $N_i(G')$, for $i \geq 0$, be the set of vertices of distance $i$ from $G'$. Then we recursively colour the edges incident to consecutive spheres $N_i(G')$ in the following way: for $v \in N_i(G')$, $i \geq 0$, we colour all edges $vu$, where $u \in N_{i+1}(G')$, with distinct colours from $\{1, \ldots, \Delta - 1\}$ and the remaining edges incident to $v$, contained in $N_i(x)$, with 2. Hence, all vertices of $G$ will be fixed by any automorphism preserving our colouring $f$.

If $G'$ is empty, then we start with the centre $Z(G)$ that is setwise fixed by every automorphism. It follows from Lemma 2.5 that $Z(G)$ either induces $K_3$, or $K_2$ (not contained in $K_3$), or $K_1$. Let first $Z(G)$ induce a triangle $K_3$. If $\Delta = 3$, then we stabilize $Z(G)$ by colouring with two colours all edges incident with vertices of $Z(G)$. When $\Delta \geq 4$, we can colour the edges of the triangle $Z(G)$ with three colours. Next, we recursively colour edges incident to subsequent spheres $N_i(Z(G))$ with $\Delta - 1$ colours.

If $Z(G)$ is an edge $e$, then $G - e$ has two components. We distinguish each of them

by colouring subsequent spheres $N_i(Z(G))$ with $\Delta - 1$ colours. If the components are isomorphic, then by assumption, each of them has a triangle. We colour two edges of these triangles contained in a sphere $N_i(Z(G))$, for some $i \geq 2$, with two distinct colours.

Finally, let $Z(G)$ be a single vertex $z$. Hence, $G - z$ has $q \geq 2$ components, each joined to $z$ by one or two edges. If $q < \Delta$, then we can easily colour distinguishingly the edges incident with subsequent spheres $N_i(z)$, $i \geq 0$, with $\Delta - 1$ colours. If $q = \Delta$, then we choose two components of $G - z$, at least one of them with a triangle, and colour their two edges incident with $z$ with the same colour. Then we distinguish these two components by an edge of the triangle. □

## 3 Some classes of graphs

A graph $G$ is called *asymmetric* if its automorphism group is trivial. Then obviously $D'(G) = 1$.

We say that a graph $G$ is *almost spanned* by a subgraph $H$ (not necessarily connected) if $G - v$ is spanned by $H$ for some $v \in V(G)$. The following observation will play a crucial role in this section.

**Lemma 3.1.** *If a graph $G$ is spanned or almost spanned by a subgraph $H$, then*

$$D'(G) \leq D'(H) + 1.$$

*Proof.* We colour the edges of $H$ with colours $1, \ldots, D'(H)$, and all other edges of $G$ with an additional colour 0. If $\varphi$ is an automorphism of $G$ preserving this colouring, then $\varphi(x) = x$, for each $x \in V(H)$. Moreover, if $H$ is a spanning subgraph of $G - v$, then also $\varphi(v) = v$. Therefore, $\varphi$ is the identity. □

### 3.1 Traceable graphs

Recall that a graph is *traceable* if it contains a Hamiltonian path.

**Theorem 3.2.** *If $G$ is a traceable graph of order $n \geq 7$, then $D'(G) \leq 2$.*

*Proof.* Let $P_n = v_1 v_2 \ldots v_n$ be a Hamiltonian path of $G$. If $G = P_n$, then the conclusion follows from Proposition 2.1. If $G$ is isomorphic to $P_n + v_1 v_3$, then we colour the edge $v_1 v_3$ with 1, and all other edges with 2 breaking all non-trivial automorphisms of $G$. So suppose that $G$ contains an edge $v_i v_j$ distinct from $v_1 v_3$ and $v_{n-2} v_n$ with $i < j - 1$. Without loss of generality we may assume that $i - 1 \leq n - j$ (otherwise we reverse the labeling). It is easy to see that at least one of the graphs $P_n + v_i v_j - v_{j-1} v_j$, $P_n + v_i v_j - v_{j-1}$ or $P_n + v_i v_j - v_n$ is an asymmetric spanning or almost spanning subgraph of $G$ for any $n \geq 7$. The conclusion follows from Lemma 3.1. □

The assumption $n \geq 7$ is substantial in Theorem 3.2 as $D'(K_{3,3}) = 3$.

### 3.2 Claw-free graphs

A $K_{1,3}$-free graph, called also *a claw-free graph*, is a graph containing no copy of $K_{1,3}$ as an induced subgraph. Claw-free graphs have numerous applications, e.g., in operations research and scheduling theory. For a survey of claw-free graphs and their applications consult [10].

A *$k$-tree* of a connected graph is its spanning tree with maximum degree at most $k$. Win [21] investigated spanning trees in 1-tough graphs and proved the following result.

**Theorem 3.3** ([21]). *A 2-connected claw-free graph has a 3-tree.*

We use this result to give an upper bound for the distinguishing number of claw-free graphs.

**Theorem 3.4.** *If $G$ is a connected claw-free graph, then $D'(G) \leq 3$.*

*Proof.* Assume first that $G$ is 2-connected. By Theorem 3.3, $G$ contains a 3-tree $T$. By Theorem 2.6, we have $D'(T) \leq 2$ if $T$ is neither symmetric nor bisymmetric tree. In such a case, $D'(G) \leq 3$ by Lemma 3.1.

Let $T$ be a symmetric tree $T_{h,3}$. Denote a central vertex of $T$ by $x$ and its neighbours by $a$, $b$, $c$. Since $G$ is a claw-free graph, there exists in $G$ at least one edge, say $bc$, in the neighbourhood of $x$ in $T$. Define a subgraph $\widetilde{T} = T + bc$. We colour $bc$, $xa$ and $xb$ with 1, and $xc$ with 2. Thus all vertices $a, b, c, x$ are fixed by every non-trivial automorphism of $\widetilde{T}$. We now colour the remaining edges in $\widetilde{T}$ starting from the edges incident to $a, b, c$ in such a way that two uncoloured adjacent edges obtain two different colours 1 and 2. This 2-colouring breaks all non-trivial automorphisms of $\widetilde{T}$. Hence, $D'(G) \leq 3$ by Lemma 3.1.

Let $T$ be a bisymmetric tree $T''_{h,3}$. Denote a central edge by $xy$ and its neighbours by $a$, $b$ adjacent to $x$, and $c$, $d$ adjacent to $y$. We colour $xy$, $xa$ and $yc$ with 1, and $xb$ and $yd$ with 2. Since $G$ is claw-free, there exists in $G$ either at least one of the edges $by$, $cx$ (or symmetrically $dx$ or $ay$) or both $ab$ and $cd$. We define a subgraph $\widetilde{T}$ obtained from the tree $T$ by adding either one of the edges $by$, $cx$ (or symmetrically, $dx$ or $ay$) or both $ab$ and $cd$. In the first case we colour $by$ or $cx$ (or symmetrically, $dx$ or $ay$) with 1, in the second case we colour $ab$ with 1 and $cd$ with 2. Now all vertices $a, b, c, d, x, y$ are fixed by every non-trivial automorphism of $\widetilde{T}$. We then colour the remaining edges of $\widetilde{T}$ as above, and we obtain the claim.

If a graph $G$ is not 2-connected, then its graph of blocks and cut-vertices is a path, since $G$ is claw-free. We colour every block according to the rules described above. Then to break all non-trivial automorphisms of $G$, it is enough to break a possible automorphism $\psi \in \mathrm{Aut}(G)$ that exchanges two terminal blocks. Let $z$ be a cut-vertex that belongs to a terminal block $B_0$. It follows that $z$ and its neighbours in $B_0$ induce a clique $K$ of order $k \geq 2$. We have three colours in our disposal, so it is easily seen that we can permute the colours to obtain a nonisomorphic colouring of $K$, thus breaking $\psi$. □

The theorem is sharp for graphs of order at most 5. We conjecture that the distinguishing index of claw-free graphs of order big enough is 2.

### 3.3 Planar graphs

First, recall that by the famous Theorem of Tutte [19], every 4-connected planar graph $G$ is Hamiltonian. Hence, its distinguishing index is at most 2, by Theorem 3.2, whenever $|G| \geq 7$. A similar result as for claw-free graphs we obtain for 3-connected planar graphs. In the proof, we use the following result of Barnette about spanning trees of such graphs.

**Theorem 3.5** ([3]). *Every 3-connected planar graph has a 3-tree.*

Using a similar method as in the proof of Theorem 3.4, we obtain the following.

**Theorem 3.6.** *If $G$ is 3-connected planar graph, then $D'(G) \leq 3$.*

*Proof.* Let $T$ be a 3-tree of $G$. It follows from Theorem 2.6 that $D'(T) \leq 2$ and hence, $D'(G) \leq 3$ by Lemma 3.1, if $T$ is neither a symmetric nor a bisymmetric tree.

Let then $T$ be a symmetric tree $T_{h,3}$. Denote the central vertex by $x$, and by $T_a$, $T_b$ and $T_c$ the connected components of $T - x$ which are trees rooted at the neighbours $a, b, c$ of a vertex $x$, respectively. Since $G$ is 3-connected, there exist an edge $e$ between $T_a$ and $T_b$ in $G$. Consider a spanning subgraph $\widetilde{T} = T + e$. Then we colour $xa$ and $xc$ with 1, and $xb$ with 2, and extend this colouring as in the proof of Theorem 3.4 to a colouring of $\widetilde{T}$ breaking all non-trivial automorphisms of $\widetilde{T}$ (the colour of $e$ is irrelevant). Consequently, $D'(G) \leq 3$ by Lemma 3.1.

If $T$ is a bisymmetric tree $T''_{h,3}$ with the central edge $xy$, then we can add to $T$ one edge in a subtree of $T - xy$ rooted at $x$, and such a graph can be easily distinguished by two colours. Again, our claim follows from Lemma 3.1. □

### 3.4 2-connected graphs

For a 2-connected planar graph $G$, the distinguishing index may attain $1 + \left\lceil \sqrt{\Delta(G)} \right\rceil$ as it is shown by the complete bipartite graph $K_{2,q}$ with $q = r^2$ for a positive integer $r$. In this case, $D'(K_{2,q}) = r + 1$ as it follows from the result obtained independently by Fisher and Isaak [11] and by Imrich, Jerebic and Klavžar [14]. They proved the following theorem. Actually, they formulated it for the distinguishing number $D(K_p \Box K_q)$ of the Cartesian product of complete graphs, but $D'(K_{p,q}) = D(K_p \Box K_q)$.

**Theorem 3.7** ([11, 14])**.** *Let $p$, $q$, $d$ be integers such that $d \geq 2$ and $(d-1)^p < q \leq d^p$. Then*

$$D'(K_{p,q}) = \begin{cases} d, & \text{if} \quad q \leq d^p - \lceil \log_d p \rceil - 1, \\ d+1, & \text{if} \quad q \geq d^p - \lceil \log_d p \rceil + 1. \end{cases}$$

*If $q = d^p - \lceil \log_d p \rceil$ then the distinguishing index $D'(K_{p,q})$ is either $d$ or $d+1$ and can be computed recursively in $O(\log^*(q))$ time.*

In the next section, we make use of the following immediate corollary.

**Corollary 3.8.** *If $p \leq q$, then $D'(K_{p,q}) \leq \lceil \sqrt[p]{q} \rceil + 1$.*

In the proof of Proposition 3.10 we also make use of an earlier result of Imrich and Klavžar [15] which is a slightly weaker version of Theorem 3.7 for $d = 2$.

**Theorem 3.9** ([15])**.** *If $2 \leq p \leq q \leq 2^p - p + 1$, then $D'(K_{p,q}) = 2$.*

**Proposition 3.10.** *If $p \leq q \leq 2^p - p + 1$ and $p + q \geq 7$, then there exists a distinguishing edge 2-colouring of $K_{p,q}$ such that the edges in one of colours induce a connected spanning or almost spanning, asymmetric subgraph of $K_{p,q}$.*

*Proof.* The assumptions imply that $p \geq 3$, and $D'(K_{p,q}) = 2$ by Theorem 3.9. Let $P$ and $Q$ be the two sets of bipartition of $K_{p,q}$ with $|P| = p$ and $|Q| = q$. If $p = q$, then $p \geq 4$, and there exists a spanning asymmetric tree of $K_{p,p}$ (see [17]). If $p < q \leq 2^p - p + 1$, then for the proof of Theorem 3.9, Imrich and Klavžar in [15] constructed a distinguishing vertex 2-colouring of $K_p \Box K_q$ that corresponds to a distinguishing edge 2-colouring $f$ of $K_{p,q}$, where a colouring of vertices in a $K_q$-layer can be represented by a sequence from $\{1, 2\}^q$ and it corresponds to a colouring of edges incident to a vertex in $P$ (the same is true

for $K_p$-layers and vertices in $Q$). We wish to show that this colouring yields a connected asymmetric subgraph of $K_{p,q}$ which is spanning or almost spanning.

First assume that $q = 2^p - p + 1$. In the coloring $f$, every vertex in $P$ has distinct positive number of edges coloured with 1, and there exists a vertex $v_1$ with all incident edges coloured with 1. Moreover, distinct vertices from $Q$ have distinct sets of neighbours joined by edges coloured with 1, and there exists a vertex, say $v_2$, with all incident edges coloured with 2. Let $S$ be a subgraph induced by edges coloured with 1. Then $S$ is an almost spanning subgraph since $v_2$ is the only vertex outside $S$. The graph $S$ is connected because $v_1$ is adjacent to every vertex in $Q$, and every vertex in $P$ is joined to a vertex in $Q$ by an edge coloured with 1. Moreover, $S$ is also asymmetric since $f$ breaks all non-trivial automorphisms of $K_{p,q}$ and any automorphism interchanging some parts of the sets $P$ and $Q$ does not preserve distances in $S$.

Following [15] for $p < q < 2^p - p + 1$, we exclude a relevant number of such pairs of sequences of colours that the sum of them is a sequence $(3, \ldots, 3)$. Additionally, if both $q$ and $p$ are odd, we exclude the sequence $(0, \ldots, 0)$. Again, we obtain a connected spanning (or almost spanning) asymmetric subgraph $S$ of $K_{p,q}$ induced by the edges coloured with 1. □

Proposition 3.10 and Lemma 3.1 immediately imply the following.

**Corollary 3.11.** *If a graph $G$ of order at least 7 is spanned by $K_{p,q}$ and $p \leq q \leq 2^p - p + 1$, then $D'(G) \leq 2$.*

In general, for 2-connected graphs we conjecture that the complete bipartite graph $K_{2,r^2}$ is the worst case, i.e. attains the highest value of the distinguishing index.

**Conjecture 3.12.** *If $G$ is a 2-connected graph, then*

$$D'(G) \leq 1 + \left\lceil \sqrt{\Delta(G)} \right\rceil .$$

## 4 Nordhaus-Gaddum inequalities for $D'$

In this section, we discuss Conjecture 1.6, formulated at the end of Introduction, stating that

$$2 \leq D'(G) + D'(\overline{G}) \leq \Delta + 2$$

for every admissible graph $G$ of order $n \geq 7$, where $\Delta = \max\{\Delta(G), \Delta(\overline{G})\}$.

The left-hand inequality is obvious. Indeed, if a graph $G$ is asymmetric, then so is $\overline{G}$. Thus we are only interested in the right-hand inequality $D'(G) + D'(\overline{G}) \leq \Delta + 2$. Note also that at least one of the graphs $G$ and $\overline{G}$ is connected.

The bound $\Delta + 2$ cannot be improved. To see this, consider a star $K_{1,n-1}$ of any order $n \geq 7$. As $\overline{K_{1,n-1}}$ is a disjoint union of a complete graph $K_{n-1}$ and an isolated vertex, it follows from Proposition 2.3 that $D'(\overline{K_{1,n-1}}) = 2$. Therefore, $D'(K_{1,n-1}) + D'(\overline{K_{1,n-1}}) = n - 1 + 2 = \Delta + 2$.

If $T$ is a tree, then $\Delta(T)$ can be much smaller than $\Delta = \Delta(\overline{T}) = n - 1$. However, the following holds.

**Proposition 4.1.** *If $T$ is a tree of order $n \geq 7$, then*

$$D'(T) + D'(\overline{T}) \leq \Delta(T) + 2.$$

*Proof.* As it was shown above, the conclusion holds for stars. If $T$ is not a star, then $D'(\overline{T}) \leq 2$ by Lemma 3.1. Indeed, as it was proved by Hedetniemi et al. in [13], a complete graph $K_n$ contains edge disjoint copies of any two trees of order $n$ distinct from a star $K_{1,n-1}$. Thus, the complement $\overline{T}$ contains a spanning asymmetric tree. By Theorem 2.6, we have the inequality $D'(T) + D'(\overline{T}) \leq \Delta(T) + 2$. □

This fact emboldened us to formulate the following stronger conjecture.

**Conjecture 4.2.** *Every connected admissible graph $G$ of order $n \geq 7$ satisfies the inequality*

$$D'(G) + D'(\overline{G}) \leq \Delta(G) + 2.$$

Now we show that Conjecture 1.6 holds not only for trees, but also for some other classes of graphs. To do this we use the following fact.

**Theorem 4.3.** *Let $G$ be a connected admissible graph of order $n \geq 7$. If either $G$ or every connected component of $\overline{G}$ has the distinguishing index at most 3, then*

$$D'(G) + D'(\overline{G}) \leq \Delta + 2,$$

*where $\Delta = \max\{\Delta(G), \Delta(\overline{G})\}$.*

*Proof.* Our claim is true for trees by Proposition 4.1. Observe also, that it is true if $G$ is a path or a cycle of order at least 7 since its complement $\overline{G}$ is Hamiltonian, and $D'(G) + D'(\overline{G}) \leq 4$. So, now we can assume that $\Delta(G) \geq 3$ and neither $G$ nor $\overline{G}$ is a tree. We consider two cases.

**Case A.** Every component $H$ of $\overline{G}$ satisfies $D'(H) \leq 3$.
Then $D'(G) \leq \Delta(G) - 1$ by Theorem 2.8, and if $\overline{G}$ is connected, then our claim holds. Assume now that $\overline{G}$ is disconnected. Then $G$ is spanned by $K_{p,q}$ with $p \leq q$ and $\Delta \geq q$, where $p + q = |V(G)|$. Suppose that the graph $\overline{G}$ has $t$ isomorphic components. If we had a distinct set of three colours for every component, then $D'(\overline{G}) \leq \lceil \sqrt[3]{6t} \rceil$. We then consider two cases:

a) If $q \leq 2^p - p + 1$, then $D'(G) = 2$ by Corollary 3.11. Moreover, we then have at most $\frac{n}{3}$ components of $\overline{G}$, so $D'(\overline{G}) \leq \lceil \sqrt[3]{2n} \rceil$. And we can easily see that

$$\lceil \sqrt[3]{2n} \rceil + 2 \leq \frac{n}{2} + 2$$

for every $n \geq 4$.

b) If $q \geq 2^p - p + 1$, then there exists a big component (of order $q$) in $\overline{G}$ and we can assume that $t \leq \frac{p}{3}$ remaining components are isomorphic. In this case, by assumptions we have $p \leq \lceil \log_2(q + p - 1) \rceil$, therefore

$$D'(\overline{G}) \leq \lceil \sqrt[3]{6t} \rceil \leq \sqrt[3]{2\lceil \log_2(q + p - 1) \rceil}.$$

On the other hand, $D'(G) \leq \lceil \sqrt[p]{q} \rceil + 2$ by Corollary 3.8 and Theorem 3.1. Then it is not difficult to check that for $q \geq 2^p - p + 1$

$$\sqrt[3]{2\lceil \log_2(q + p - 1) \rceil} + \lceil \sqrt[p]{q} \rceil + 2 \leq q + 2$$

what finishes the proof in Case A.

**Case B.** $D'(G) \leq 3$.
If graph $\overline{G}$ is connected, then the claim follows immediately from Theorem 2.7 whenever $D'(G) = 2$ or $D'(\overline{G}) = 2$, and it follows from Theorem 2.8 if $D'(G) = 3$. Assume now that $\overline{G}$ has $t \geq 2$ components. Then $\Delta \geq \frac{n}{2}$ and, in the worst case, all components of $\overline{G}$ are isomorphic. Observe that maximal degree of every component is at most $\frac{n}{t} - 1$. If we assign one extra colour to every component, then we need at most $\frac{n}{t} - 1 + (t-1)$ colours to distinguish $\overline{G}$. Hence, if

$$\frac{n}{t} + t \leq \frac{n}{2} - 1,$$

then $D'(\overline{G}) \leq \Delta - 1$, and our claim is true. The above inequality holds unless $t = 2$.

If there exist two isomorphic components in $\overline{G}$, then $D'(G) \leq 2$ due to Corollary 3.11 since $G$ is spanned by $K_{\frac{n}{2},\frac{n}{2}}$. Then $D'(\overline{G}) \leq \frac{n}{2}$, and finally $D'(G) + D'(\overline{G}) \leq \frac{n}{2} + 2$. □

Now we can formulate some consequences of Theorem 4.3 and suitable results proved in Section 3.

**Corollary 4.4.** *Let $G$ be an admissible graph of order $n \geq 7$. If $G$ satisfies at least one of the following conditions:*

*i) $G$ is a traceable graph, or*

*ii) $G$ is a claw-free graph, or*

*iii) $G$ is a triangle-free graph, or*

*iv) $G$ is a 3-connected planar graph,*

*then*

$$D'(G) + D'(\overline{G}) \leq \Delta + 2,$$

*where* $\Delta = \max\{\Delta(G), \Delta(\overline{G})\}$.

*Proof.* It suffices to apply Theorem 4.3 together with Theorem 3.2, Theorem 3.4 and Theorem 3.6, respectively. Observe also that if the girth of a graph $G$ is at least 4, i.e., $G$ is triangle-free, then its complement $\overline{G}$ is claw-free. □

Finally, it has to be noted that there exist graphs of order less than 7 such that the right-hand inequality in Conjecture 1.6 is not satisfied. For example, for the graph $K_{3,3}$ we have $D'(K_{3,3}) = 3$, $D'(\overline{K_{3,3}}) = D'(2K_3) = 4$ and $\Delta = 3$, hence $D'(K_{3,3}) + D'(\overline{K_{3,3}}) = \Delta + 4$. Also, $D'(C_5) + D'(\overline{C_5}) = 3 + 3 = \Delta + 4$, and $D'(K_{1,i}) + D'(\overline{K_{1,i}}) = \Delta + 3$ for $i = 3, 4, 5$.

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