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Note

A note on breaking small automorphisms in graphs ☆

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ABSTRACT

This paper brings together two concepts in the theory of graph colourings: edge or total colourings distinguishing adjacent vertices and those breaking symmetries of a graph.

We introduce a class of automorphisms such that edge colourings breaking them are connected to edge colouring distinguishing neighbours by multisets or sums. We call an automorphism of a graph G *small* if there exists a vertex of G that is mapped into its neighbour. The small distinguishing index of G, denoted $D'_s(G)$, is the least number of colours in an edge colouring of G such that there does not exist a small automorphism of G preserving this colouring. We prove that $D'_s(G) \leq 3$ for every graph G without K_2 as a component, thus supporting, in a sense, the 1-2-3 Conjecture of Karoński, Łuczak and Thomason.

We also consider an analogous problem for total colourings in connection with the 1-2 Conjecture of Przybyło and Woźniak.

1. Introduction and motivation

We use a standard graph theory terminology and notation [3]. In particular, $G[S]$ stands for the subgraph of G induced by a set $S \subseteq V(G)$, and $\mathrm{Aut}(G)$ denotes the automorphism group of a graph G. By $\chi'(G)$ and $\chi''(G)$ we denote the chromatic index and the total chromatic index of a graph G, respectively.

We say that a (vertex, edge or total) colouring, not necessarily proper, of a graph G *breaks an automorphism* φ of G if φ does not preserve this colouring. If a colouring with d colours breaks all non-trivial automorphisms of G, then we call it a *distinguishing d-colouring*.

The minimum number d such that a graph G admits a distinguishing d-colouring of edges of G is called the *distinguishing index* of G and is denoted by $D'(G)$. This notion was introduced by the first two authors in [4] as an analogue of the well-known *distinguishing number* $D(G)$ of a graph G defined by Albertson and Collins [2] for vertex colourings. Obviously, the distinguishing index is not defined for graphs having K_2 as a component.

In the same paper [4], the first two authors considered also breaking automorphisms by proper edge colourings. They proved that for every connected graph G of order at least 7 there exists a proper edge colouring with $\Delta(G) + 1$ colours that breaks all non-trivial automorphisms. This result is an example of a useful connection between edge colourings breaking automorphisms and those distinguishing all vertices by "extended palettes" defined as colour walks in [5]. Actually, this interesting relationship between two kinds of distinguishing colourings was a motivation for us to investigate connections between breaking some automorphisms and the 1-2-3 Conjecture and the 1-2 Conjecture. We consider not necessarily proper colourings, since proper colourings are less interesting from this point of view, what we explain at the end of this section.

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1.1. Distinguishing adjacent vertices by edge colourings

In this paper we are interested in edge colourings distinguishing adjacent vertices by multisets or by sums. Let $c : E(G) \rightarrow \{1, \ldots, k\}$ be any colouring of edges of a graph G, not necessarily proper. For each vertex v, we denote by $M_c(v)$ the multiset of colours of edges incident to v, and we call it the c-*palette* of v. If $M_c(u) \neq M_c(v)$ for every $uv \in E(G)$, then c is called a *neighbour-distinguishing* edge colouring. The *neighbour distinguishing index* $\chi^e_m(G)$ of a graph G is the least number of colours in a neighbour-distinguishing edge colouring of G. The best upper bound for $\chi^e_m(G)$ was obtained by Addario-Berry, Aldred, Dalal and Reed in [1]. They showed that for every graph G without K_2 as a component, $\chi^e_m(G) \leq 4$. Moreover, they proved that $\chi^e_m(G) \leq 3$ if $\delta(G) \geq 1000$. It is conjectured that $\chi^e_m(G) \leq 3$ for every graph G without isolated edges and with at most one isolated vertex.

Distinguishing adjacent vertices by sums is even better known. By $\sigma_c(v)$ we denote the sum of colours of edges incident to a vertex v. The least number k for which G admits an edge colouring c such that $\sigma_c(u) \neq \sigma_c(v)$, for every edge uv of G, is called the *neighbour distinguishing index by sums* of a graph G, and is denoted by $\chi^e_\Sigma(G)$. Clearly, $\chi^e_m(G) \leq \chi^e_\Sigma(G)$. The following 1-2-3 Conjecture posed by Karoński, Łuczak and Thomason in [9] spawned an avalanche of papers on this and related topics.

1-2-3 Conjecture (Karoński, Łuczak and Thomason [9]) *For every graph G without K_2 as a component,*

$$\chi^e_\Sigma(G) \leq 3.$$

This conjecture has been confirmed for some classes of graphs, but in general it remains open since 2004. Up to now, the best upper bound $\chi^e_\Sigma(G) \leq 5$ was obtained by Kalkowski, Karoński and Pfender [8].

The aim of this paper is to introduce a class of automorphisms such that neighbour-distinguishing edge colourings are also colourings breaking them.

We call an automorphism φ of G *small* if there exists a vertex v such that $\varphi(v)$ is a neighbour of v. We denote the set of all small automorphisms of G by $\mathrm{Aut}_s(G)$. The least number of colours in an edge-colouring of G breaking all small automorphisms is called a *small distinguishing index* of G and denoted by $D'_s(G)$. There are graphs without small automorphisms, e.g., asymmetric graphs, non-balanced bipartite graphs (including trees with a central vertex). Naturally, they have the small distinguishing index equal to 1. Clearly, $D'_s(G) \leq D'(G)$, and the equality holds for all graphs G with $\mathrm{Aut}_s(G) \neq \emptyset$ and $D'(G) = 2$.

In Section 2, assuming that K_2 is not a component of G, we prove that $D'_s(G) \leq 3$ (Theorem 4), while the sharp upper bound for $D'(G)$ of a connected graph of order at least 6 is $\Delta(G)$, as was shown in [4].

If φ is an automorphism of G preserving an edge colouring c, then it obviously preserves the c-palettes of vertices, i.e., $Mc(v) = M_c(\varphi(v))$ for every vertex v. Hence, if c is neighbour-distinguishing, then it breaks all small automorphisms. Consequently, $D'_s(G) \leq \chi^e_m(G) \leq \chi^e_\Sigma(G)$, and Theorem 4 can be viewed as a weaker statement of 1-2-3-Conjecture.

1.2. Distinguishing vertices by total colourings

We also consider an analogous problem for general total colourings $c : V \cup E \rightarrow \{1, 2, 3, \ldots\}$. The *total distinguishing number* $D''(G)$ of a graph G is the least number d such that G has a total colouring with d colours that is preserved only by the identity automorphism of G. This definition was introduced by the authors in [6]. For instance, $D''(K_2) = 2$, and $D''(K_{1,p}) = \lceil \sqrt{p} \rceil$, $p \geq 2$. Observe that $D''(G) \leq \min\{D(G), D'(G)\}$. Clearly, the equality holds for asymmetric graphs, and for graphs with $\min\{D(G), D'(G)\} = 2$. The sharp upper bound was proved in [6]. Namely, if G is a connected graph of order $n \geq 3$, then $D''(G) \leq \lceil \sqrt{\Delta(G)} \rceil$.

The *small total distinguishing number* $D''_s(G)$ of a graph G is the least number d such that G has a total colouring with d colours that breaks all small automorphisms of G.

In Section 3, we prove that $D''_s(G) \leq 2$ for every graph G. It refers to the 1-2 Conjecture formulated by Przybyło and Woźniak. Given a total colouring c, for every vertex v, by $\sigma^t_c(v)$ we denote the sum of a colour of v and colours of all edges incident to v. If $\sigma^t_c(u) \neq \sigma^t_c(v)$ for every edge $uv \in E(G)$, then c is a *neighbour-distinguishing total colouring*. The least number of colours for which G admits such a colouring is called the *total neighbour distinguishing index by sums* of a graph G, and is denoted by $\chi^t_\Sigma(G)$.

1-2 Conjecture (Przybyło and Woźniak [11]) *For every graph G,*

$$\chi^t_\Sigma(G) \leq 2.$$

The 1-2 Conjecture was also confirmed for some classes of graphs, but in general it remains open. Kalkowski [7] proved the best result so far. Since the proof of his result can be found only in Polish in the Ph.D. thesis of Kalkowski we quote it below, adapting it to our terminology and notation.

Theorem 1 (*Kalkowski [7]*). *Every graph admits a total colouring $c : V(G) \cup E(G) \rightarrow \{1, 2, 3\}$ such that the vertices of G do not assign colour 3 and $\sigma^t_c(u) \neq \sigma^t_c(v)$ for every edge $uv \in E(G)$. As a consequence, $\chi^t_\Sigma(G) \leq 3$ for any graph G.*

Proof. The proof stems from a greedy algorithm, which processes the vertices in a given order, and generates a desired total colouring of G. Without loss of generality, we assume G is a connected graph, otherwise we can argue component wise. Let $V(G) = \{v_1, \ldots, v_n\}$. For each v_i, we call its neighbours v_j with $j < i$ the *predecessors* of v_i.

Initially all vertices get colour 1 and all edges are coloured with 2, so every vertex $v_i \in V(G)$ has an initial value $\sigma^t_c(v_i) = 2d(v_i) + 1$. Then, iteratively, for all vertices v_i, $i = 1, \ldots, n$, the algorithm performs the following steps:

1. Compute the numbers s and t of predecessors of v_i that have colour 1 or 2, respectively.
2. Compute the minimal $r \in [-s, t]$ such that for no predecessor v_j of v_i we have $\sigma_c^t(v_j) = 2d(v_i) + 1 + r$. (Observe that such an r always exists since the number of integers in the interval $[-s, t]$ is greater by one than the number of predecessors of v_i.)
3. If $r > 0$ choose r predecessors v_j with colour 2 and set $c(v_j) = 1$ and $c(v_j v_i) = 3$.
4. If $r < 0$ choose $|r|$ predecessors v_j with colour 1 and set $c(v_j) = 2$ and $c(v_j v_i) = 1$.

The algorithm yields a desired total colouring. Indeed, for each v_i the value $\sigma_c^t(v_i)$ changes at most once, only when it is processed, because the algorithm only changes the colour of (edges to) predecessors v_j and step 3 and 4 do it in such a way that ensures $\sigma_c^t(v_j)$ remains the same. Thus it suffices to ensure that $\sigma_c^t(v_i)$ is unique among the current vertex and its predecessors. This is guaranteed by the choice of r in step 2. $\square$

1.3. Proper colourings

An analogous problem of breaking small automorphisms can also be formulated for proper edge or total colourings. Recall that the minimum number of colours in a proper edge colouring of G that breaks all non-trivial automorphisms is bounded from above by $\Delta(G)+1$ for all graphs except for four graphs $C_4, K_4, C_6, K_{3,3}$ (cp. [4]). It follows from Vizing's theorem that the *small distinguishing chromatic index*, which can be naturally defined, of a graph G may attain one of only two possible values: $\chi'(G)$ or $\chi'(G) + 1$. Therefore, this is less interesting than for general colourings.

For proper total colourings, an analogous problem even does not exist, because every proper total colouring breaks all small automorphisms of a graph.

2. Small distinguishing index

We begin this section with a lemma used in the proof of Theorem 4. A graph G is *traceable* if it contains a Hamiltonian path.

Lemma 2. *If G is a traceable graph of order at least 3, then $D'(G) \leq 3$.*

Proof. If H is a spanning subgraph of G, then $D'(G) \leq D'(H) + 1$ (see [10]). As $D'(P_n) = 2$ for $n \geq 3$, we immediately infer the conclusion. $\square$

Let us mention that a stronger result has been proved by Pilśniak in [10].

Theorem 3 (*[10]*)**.** *If G is a traceable graph of order at least 7, then $D'(G) \leq 2$.*

Let r be a non-negative integer. A *sphere* of radius r and centre v_0, denoted $S(v_0, r)$, is a set of vertices of distance r from a vertex v_0. If $W \subset V(G)$, then the rth *layer* with respect to W, denoted $S(W, r)$, is the set of vertices of distance r from W.

Theorem 4. *For every graph G without K_2 as a component*

$$D'_s(G) \leq 3.$$

Proof. Clearly, if G is disconnected and $G_1, \ldots, G_q$ are its components, then $D'_s(G) = \max\{D'_s(G_i) : i = 1, \ldots, q\}$. Hence, it is enough to prove that $D'_s(G) \leq 3$ for every connected graph G distinct from K_2.

Assume first that G contains a vertex v_0 that does not belong to any triangle. Partition the vertex set $V(G)$ into non-empty spheres $S(v_0, 0), S(v_0, 1), \ldots, S(v_0, \rho)$. Thus, $S(v_0, 1)$ is an independent set. For each $r = 2, \ldots, \rho$, let c'_r be a neighbour-distinguishing total colouring of $G[S(v_0, r)]$ that follows from Theorem 1 of Kalkowski, where vertices are coloured with 1,2 and edges with 1, 2, 3.

We now define an edge colouring c of G as follows. We leave the colours c'_r of edges between vertices of the same sphere $S(v_0, r)$, $r = 1, \ldots, \rho$. If $v \in S(v_0, r)$ for some $r \geq 2$, then we colour all edges joining v to vertices in $S(v_0, r-1)$ with $c'_r(v)$. Finally, we colour all edges incident to v_0 with 3.

Let φ be a small automorphism of G preserving the edge colouring c. The vertex v_0 is fixed by φ since it is the only vertex in G with all incident edges coloured with 3 (except possible pendant neighbours of v_0, but then v_0 is of higher degree). Therefore, for each $r = 0, \ldots, \rho$, the sphere $S(v_0, r)$ is mapped onto itself by φ, i.e., φ reduced to $S(v_0, r)$ is an automorphism of $G[S(v_0, r)]$. Suppose that v and $\varphi(v)$ are neighbours. Then they both belong to the same sphere $S(v_0, r)$ for some $r \geq 2$. Hence all edges joining them to $S(v_0, r-1)$ have the same colour $c'_r(v) = c'_r(\varphi(v))$, and the multisets of colours of edges incident to v and $\varphi(v)$ are identical, contrary to the assumption that c'_r is a neighbour-distinguishing total colouring of $G[S(v_0, r)]$.

Now, assume that every vertex of G belongs to a triangle. Let $P = v_1 \ldots v_t$ be a longest path in G. If P is a Hamiltonian path, then $D'(G) \leq 3$ by Lemma 2. Otherwise, the vertex set $V(G)$ is partitioned into non-empty layers $S(P, 0), \ldots, S(P, \rho)$. For every $r = 1, \ldots, \rho$, let c'_r be a neighbour-distinguishing total colouring of $G[S(P, r)]$, where vertices are coloured with 1,2 and edges with 1, 2, 3.

Again, we define an edge colouring c of G as follows. We leave the colours of edges within the same set $S(P, r)$, $r = 1, \dots, \rho$. For every $r \geq 1$ and every $v \in S(P, r)$, we colour all edges joining v to vertices in $S(P, r-1)$ with $c'_r(v)$. Then we colour all edges of P with 3. Each of the vertices v_1, v_t belongs to a triangle, and all their neighbours are contained in P because P is a longest path. Then we colour one edge incident to v_1 with 1, and all other yet uncoloured edges of $G[V(P)]$ we colour with 2.

Suppose that c is preserved by a small automorphism φ of G. It is well known that any two longest paths in a graph must have at least one vertex in common. Therefore, G does not contain another path coloured with 3 of the same length as P. It follows that each vertex of P is fixed by φ since the end-vertices v_1, v_t of P have distinct c-palettes. Then we argue analogously as in the previous case. Each set $S(P, r)$ is mapped by φ into itself. Hence, if $\varphi(v)$ was a neighbour of $v \in S(P, r)$, then this would contradict that c'_r is a neighbour-distinguishing total colouring of $G[S(P, r)]$. □

The bound in Theorem 4 is sharp since $D'_s(K_n) = 3$, $n = 3, 4, 5$, and $D'_s(C_5) = 3$.

For bipartite graphs, we get even a better upper bound than in 1-2-3 Conjecture. Namely, if G is a bipartite graph, then the neighbour distinguishing index by sums is at most 3 and the equality is achieved since, e.g., $\chi^e_\Sigma(C_{4k+2}) = 3$.

Proposition 5. *If G is a connected bipartite graph with $|G| \geq 3$, then*

$$D'_s(G) \leq 2.$$

Moreover, $D'_s(G) = 2$ if and only if G is balanced and there exists an automorphism of G that switches the sets X and Y of bipartition.

Proof. Observe that every small automorphism of G switches the sets X and Y. Let v_0 be a vertex with $\deg(v_0) = \Delta(G)$. Clearly, $\deg(v_0) \geq 2$. We colour the edges incident to v_0 with 1, and all other ones with 2. Thus, v_0 is the only one vertex of degree greater than one in G with all incident edges coloured with 1. Hence v_0 is fixed by every automorphism preserving the colouring. Therefore $D'_s(G) \leq 2$. □

It follows that $D'_s(T) \leq 2$ for every tree T with equality whenever T has a central edge uv and an automorphism that interchanges the vertices u and v. On the other hand, for every $\Delta \geq 2$, there exist trees with $D'(T) = \Delta$ and $\Delta(T) = \Delta$ (see [4]).

3. Small total distinguishing index

In this section we also start with a lemma concerning traceable graphs.

Lemma 6. *If G is traceable, then $D''(G) \leq 2$.*

Proof. Let P be a Hamiltonian path of G. We colour one end-vertex of P with 1 and all other vertices with 2. The edges of P we colour with 1 and all other edges of G with 2. Clearly, this is a total distinguishing 2-colouring of G. □

Theorem 7. *For every graph G,*

$$D''_s(G) \leq 2.$$

Proof. Without loss of generality, we may assume that G is connected. Let P_0 be a longest path in G. We totally colour the subgraph $G[V(P_0)]$ induced by the vertex set of P_0 with a distinguishing total 2-colouring as in the proof of Lemma 6, i.e., one end-vertex and all edges of P_0 are coloured with 1, and the remaining vertices and edges with 2. If $V(G) = V(P_0)$, then we are done by Lemma 6. Otherwise, the set $V(G) \setminus V(P_0)$ is partitioned into nonempty layers $S(P_0, 1), \dots, S(P_0, \rho)$ with $\rho \geq 1$. We colour all edges between P_0 and $S(P_0, 1)$ with 2. Thus each vertex of P_0 will be fixed by any automorphism preserving this colouring since a connected graph cannot have two disjoint longest paths. Consequently, for every r, the set $S(P_0, r)$ will be mapped onto itself. Then we colour all edges between $S(P_0, r-1)$ and $S(P_0, r)$ with 2.

Now, let $r = 1$. If the edge set of $G[S(P_0, 1)]$ is nonempty, then for every component H of $G[S(P_0, 1)]$ we define a total colouring of H using the following procedure. We take a longest path $P_{1,1}$ of H. We colour one end-vertex and all edges of it with 1 and all other vertices and edges of the subgraph $H[V(P_{1,1})]$ with 2. Then we delete the edges of $H[V(P_{1,1})]$ and we take a longest path $P_{1,2}$ of the subgraph $H_1 = H - E(H[V(P_{1,1})])$. We totally colour the subgraph $H_1[V(P_{1,2})]$ as above. It is not difficult to see that there does not exist a small automorphism of $H[V(P_{1,1}) \cup V(P_{1,2})]$ that preserves this colouring. Indeed, if the paths $P_{1,1}$ and $P_{1,2}$ are of the same length, then they have to share exactly one vertex that is a central vertex of both paths and there cannot exist another path joining a vertex of $P_{1,1}$ with a vertex of $P_{1,2}$ because of the maximality of these paths. Then we delete the edges of $H_1[V(P_{1,2})]$ and continue this procedure until only independent vertices of H remain. We colour them with 1.

Thus we obtain a total colouring of $G[V(P_0) \cup S(P_0, 1)]$ that is not preserved by any small automorphism. We then repeat this procedure for $r = 2, \dots, \rho$, and obtain a total 2-colouring c of G that breaks all small automorphisms of G. □

Note that Theorem 7 implies that for every graph G, we have $D''_s(G) = 2$ if and only if $\mathrm{Aut}_s(G) \neq \emptyset$. Otherwise $D''_s(G) = 1$ if $\mathrm{Aut}_s(G) = \emptyset$.

Clearly, every neighbour-distinguishing total colouring of G breaks all small automorphisms. In this sense, our result supports the 1-2 Conjecture.

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References

[1] L. Addario-Berry, R.E.L. Aldred, K. Dalal, B.A. Reed, Vertex colouring edge partitions, J. Combin. Theory Ser. B 94 (2005) 237–244.
[2] M.O. Albertson, K.L. Collins, Symmetry breaking in graphs, Electron. J. Combin. 3 (1) (1996) R18.
[3] R. Diestel, Graph Theory, fourth ed., Springer-Verlag, Heidelberg, 2010.
[4] R. Kalinowski, M. Pilśniak, Distinguishing graphs by edge-colourings, European J. Combin. 45 (2015) 124–131.
[5] R. Kalinowski, M. Pilśniak, J. Przybyło, M. Woźniak, How to personalize the vertices of a graph? European J. Combin. 40 (2014) 116–123.
[6] R. Kalinowski, M. Pilśniak, M. Woźniak, Distinguishing graphs by total colourings, Ars Math. Contemp. 11 (1) (2016) 79–89.
[7] M. Kalkowski, A note on the 1-2 conjecture, 2009, Preprint.
[8] M. Kalkowski, M. Karoński, F. Pfender, A new upper bound for the irregularity strength of graphs, SIAM J. Discrete Math. 25 (3) (2011) 1319–1321.
[9] M. Karoński, T. Łuczak, A. Thomason, Edge weights and vertex colours, J. Combin. Theory Ser. B 91 (1) (2004) 151–157.
[10] M. Pilśniak, Improving upper bounds for the distinguishing index, Ars Math. Contemp. 13 (2) (2017) 259–274.
[11] J. Przybyło, M. Woźniak, On a 1,2 conjecture, Discrete Math. Theor. Comput. Sci. 12 (1) (2010) 101–108.