Theoretical Computer Science 687 (2017) 62–69

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Precise bounds for the distinguishing index of the Cartesian product ☆

Aleksandra Gorzkowska *, Monika Pilśniak

AGH University of Science and Technology, Department of Discrete Mathematics, al. Mickiewicza 30, 30-059 Krakow, Poland

ARTICLE INFO

Article history:
Received 11 August 2016
Received in revised form 25 February 2017
Accepted 7 May 2017
Available online 16 May 2017
Communicated by L.M. Kirousis

Keywords:
Edge colouring
Distinguishing index
Cartesian product

ABSTRACT

The *distinguishing index* $D'(G)$ of a graph G is the least number d such that G has an edge colouring with d colours that is preserved only by the identity automorphism. The distinguishing index of the Cartesian product of graphs was investigated by the authors and Kalinowski. They considered colourings with two colours only and obtained results that do not determine the distinguishing index for all the possible cases.

In this paper we investigate colourings with d colours and determine the exact value of the distinguishing index of the Cartesian product $K_{1,m}\Box K_{1,n}$ for almost all m and n. In particular, we supplement the result of [6] for the case when $2^{2m+1}-\lceil \frac{m}{2}\rceil+1<n\leq 2^{2m+1}$. We also observe the distinguishing index of the Cartesian product of two graphs in general does not have to depend on the size of the graphs and it can be arbitrarily small.

1. Introduction

Let c be an edge colouring $c\colon E(G)\to\{1,\ldots,d\}$ of a graph G. We say that an automorphism $\varphi\in \mathrm{Aut}(G)$ is *broken* by the edge colouring c if there exists an edge $uv\in E(G)$ such that the colour $c(\varphi(u)\varphi(v))$ of its image is different from the colour $c(uv)$ of the edge itself. The *distinguishing index* $D'(G)$ of a graph G is the least number of colours d such that there exists a colouring c breaking every nontrivial automorphism of the graph. We call such a colouring *distinguishing*. Clearly, this parameter does not make sense for K_2. The distinguishing index was introduced by Kalinowski and Pilśniak in [11]. Symmetry breaking (in various ways) has interesting applications to numerous problems of theoretical computer science, for instance to the leader election problem and self-stabilizing algorithms (cf. [4,5,10]).

The Cartesian product of graphs G and H is a graph denoted $G\Box H$ whose vertex set is $V(G)\times V(H)$. Two vertices (g,h) and (g',h') are adjacent if either $g=g'$ and $hh'\in E(H)$, or $gg'\in E(G)$ and $h=h'$. We denote $G\Box G$ by G^2, and we recursively define the *k-th Cartesian power* of G as $G^k=G\Box G^{k-1}$. We say a graph G is *prime* with respect to the Cartesian product if for any decomposition $G=G_1\Box G_2$ the graph G_1 or G_2 is isomorphic to K_1.

For a given vertex $v\in V(H)$ the G^v*-layer* is the subgraph of the Cartesian product $G\Box H$ induced by the vertex set

$$\{(u,v)\in V(G\Box H)\colon u\in V(G)\}.$$

Notice that the G^v-layer is isomorphic with G. We often refer to it as a *horizontal layer*. Analogously, we define a *vertical layer* or an H^u*-layer*.

☆ The research was partially supported by the Polish Ministry of Science and Higher Education.

* Corresponding author.
E-mail addresses: agorzkow@agh.edu.pl (A. Gorzkowska), pilsniak@agh.edu.pl (M. Pilśniak).

http://dx.doi.org/10.1016/j.tcs.2017.05.004
0304-3975/

The automorphism group of the Cartesian product was characterized by Imrich [7] and independently by Miller [13]. Here we present a simplified version of the theorem, for the Cartesian product of only two graphs.

Theorem 1. *[7,13] Suppose ψ is an automorphism of a connected graph G with prime factor decomposition $G = G_1 \Box G_2$. Then there is a permutation π of $\{1, 2\}$ and isomorphisms $\psi_i : G_{\pi(i)} \mapsto G_i$, for $i \in \{1, 2\}$ such that*

$$\psi(x_1, x_2) = (\psi_1(x_{\pi(1)}), \psi_2(x_{\pi(2)})).$$

In this paper we consider the Cartesian product of two stars $K_{1,m} \Box K_{1,n}$. It is easy to see that stars are prime with respect to the Cartesian product. The case when $m = n$ has already been considered in [6]. It has been proved that $D'(K^2_{1,m}) = 2$ for all $m \ge 2$. The authors in [6] also proved that if $2 \le m \le n \le 2^{2m+1} - \lceil \frac{m}{2} \rceil + 1$, then $D'(K_{1,m} \Box K_{1,n}) = 2$.

Further results concerning the case of a larger difference between the sizes of the stars have also been presented in [6].

Proposition 2. [6] *If $m \ge 1$ and $n > d^{2m+1}$, then $D'(K_{1,m} \Box K_{1,n}) > d$.*

The above results appear not to be tight as there is a gap between the presented conditions. In this paper we give a more precise condition, including the remaining cases, i.e., when $d^{2m+1} - \lceil \frac{m}{d} \rceil + 1 < n \le d^{2m+1}$.

The main goal of this paper is the proof of the following theorem.

Theorem 3. *Let $2 \le m \le n$ and $(d-1)^{2m+1} < n \le d^{2m+1}$. Then*

1. *$D'(K_{1,m} \Box K_{1,n}) = d$, if $n \le d^{2m+1} - \frac{\log_d k}{2} - \frac{1}{2}$,*
2. *$D'(K_{1,m} \Box K_{1,n}) = d + 1$, if $n > d^{2m+1} - \frac{\log_d k}{2}$,*

where $k = \lceil \frac{m}{d} \rceil$.

Let us first remark that such precise result cannot be achieved for the Cartesian product of two arbitrary graphs G and H, even if they are trees. The theorem stated above determines the exact value of the distinguishing index of the Cartesian product of two stars, given the relation between their sizes. In general case, we are unable to introduce a relation between the sizes of two graphs that would allow for the distinguishing index to be determined accurately. Perhaps if another graph invariant was studied, such condition could be found.

The proofs in this paper are based on the one-to-one correspondence between the group of permutations on k elements S_k and the automorphism group of the star $K_{1,k}$. For an arbitrary tree there is no such relation. The colouring of the Cartesian product of trees $T_m \Box T_n$, where m and n are the sizes of the trees, can also be represented as a set of n vectors of length m of pairs from the set $\{1, 2\}$, as was done in [6]. It suffices to define a special numeration of the vertices of both trees. If a tree has a central vertex, numerate it with 1. If a tree has two central vertices, numerate one with 1 and the other one with 2. The remaining vertices are numerated in the Breadth-First Search. Then every term $v_i = (a_i, b_i)$ of any given vector $(v_1, \ldots, v_m)$ represents the colours of edges adjacent to a vertex of a T_m-layer. The first number is the colour of the edge to a father in the BFS tree in the T_m-layer and the other to a father in the BFS tree in the T_n-layer. Considering this enumeration, it is easy to see that for trees T_m and T_n of sizes m and n, respectively, the following inequality holds

$$D'(T_m \Box T_n) \le D'(K_{1,m} \Box K_{1,n}).$$

Observe that the distinguishing index of the Cartesian product of two trees can be equal to 2 for arbitrarily large difference between their sizes. As an example consider $P_m \Box P_n$. As shown in [6] for any $m \ge 2$ and $n > 2$ we have $D'(P_m \Box P_n) = 2$.

The concept of the distinguishing index has been derived from a similar concept for vertex colourings. It has been initiated by Albertson and Collins in [1]. They defined the *distinguishing number* of a graph G, denoted by $D(G)$, as the least number d such that G has a vertex colouring with d colours breaking all nontrivial automorphisms of G. This concept has also been studied for the Cartesian product of graphs, e.g., in [8,9,12] and [3] and recently for the Cartesian product of the countable graph, e.g., in [2]. Similar results to Theorem 3 were obtained by Imrich, Jerebic and Klavžar in [8] for vertex colourings.

Theorem 4. [8] *Let k, n, d be integers so that $d \ge 2$ and $(d-1)^k < n \le d^k$. Then*

$$D(K_k \Box K_n) = \begin{cases} d, & \text{if } n \le d^k - \lceil \log_d k \rceil - 1; \\ d+1, & \text{if } n \ge d^k - \lceil \log_d k \rceil + 1. \end{cases}$$

If $n = d^k - \lceil \log_d k \rceil$, then $D(K_k \Box K_n)$ is either d or $d + 1$ and can be computed recursively in $O(\log^*(n))$.

The proofs in this paper are based on the method presented in [8]. However, substantial modifications are necessary due to the fact that we colour edges instead of vertices.

2. Preliminary results for stars

The following theorem is a generalization of the main result for trees obtained in [6]. Its proof is a straightforward extension of the proof for trees in [6].

Theorem 5. *Let $K_{1,m}$ and $K_{1,n}$ be stars. If*

$$2 \le m \le n \le d^{2m+1} - \left\lceil \frac{m}{d} \right\rceil + 1,$$

then $D'(K_{1,m} \square K_{1,n}) \le d$. □

In this section we present results that lead to the proof of the following theorem, which is Theorem 3 in the case of $d = 2$.

Theorem 6. *Let $m \ge 2$ and $m \le n \le 2^{2m+1}$. Then*

1. $D'(K_{1,m} \square K_{1,n}) = 2$, *if* $n \le 2^{2m+1} - \frac{\log_2 k}{2} - \frac{1}{2}$,
2. $D'(K_{1,m} \square K_{1,n}) = 3$, *if* $n > 2^{2m+1} - \frac{\log_2 k}{2}$,

where $k = \lceil \frac{m}{2} \rceil$.

Throughout this section we assume that $m < n < 2^{2m+1}$. We only consider 2-colourings of the Cartesian product of stars. We denote with $([2]^2)^k$ the set of all vectors of length k whose elements are pairs from the set $\{1, 2\}$, that is

$$([2]^2)^k = \{(v_1, \ldots, v_k) \colon \; v_i = (a_i, b_i) \wedge a_i, b_i \in \{1, 2\}\}.$$

Clearly, $|([2]^2)^k| = 2^{2k}$.

Let $\pi \in S_k$ be a permutation on k elements and let $V = \{v^1, v^2, \ldots, v^r\} \subset ([2]^2)^k$ be a set of r vectors described above. Permutation π acts on an element v^i of the set V by exchanging the terms of the vector v^i, i.e., $\pi v^i = (v^i_{\pi(1)}, \ldots, v^i_{\pi(k)})$. We denote with $\pi V = \{\pi v^1, \pi v^2, \ldots, \pi v^r\}$. We say a set of vectors $V \subset ([2]^2)^k$ is *column invariant* if there exists a permutation $\pi \in S_k \backslash \{\mathrm{id}\}$ such that $\pi V = V$. Otherwise, we say the set V *is not column invariant*.

The notion of column invariance is better understood with a matrix representation of the set V. Let M_V be a matrix whose rows are the vectors from the set V. The i-th column of M_V consists of i-th terms of vectors from V. Clearly, M_V is a $r \times k$ matrix. The set V is column invariant if and only if there exists a permutation $\pi \in S_k$ of columns of the matrix M_V such that there exists a permutation $\sigma \in S_r$ of rows of M_V such that under the combined action of these two permutations the matrix M_V does not change.

Let us remark that the set $([2]^2)^k$ is column invariant. Moreover, any nontrivial permutation $\pi \in S_k$ has the property that $\pi([2]^2)^k = ([2]^2)^k$. Clearly, for any permutation π and any vector $v \in ([2]^2)^k$, the vector πv is in the set $([2]^2)^k$. Therefore, the condition $\pi([2]^2)^k = ([2]^2)^k$ is trivially fulfilled.

Lemma 7. *For integers k and $r < 2^{2k}$ if a set $V \subset ([2]^2)^k$ of r vectors is column invariant then the set $U = ([2]^2)^k \backslash V$ of $2^{2k} - r$ vectors is column invariant.*

Proof. Let $V \subset ([2]^2)^k$ be a column invariant set of r vectors, i.e., $V = \{v^1, v^2, \ldots, v^r\}$. We know that there exists a permutation $\pi \in S_k \backslash \{\mathrm{id}\}$ such that for every vector $v^i \in V$ the condition $\pi v^i \in V$ is fulfilled. Since $\pi([2]^2)^k = ([2]^2)^k$, then for every vector $u_i \in ([2]^2)^k \backslash V$ the vector πu_i has to also belong in $([2]^2)^k \backslash V$. Therefore, the set $([2]^2)^k \backslash V$ is also column invariant. □

The above reasoning can be applied to an arbitrary set V. This yields that if all sets $V \subset ([2]^2)^k$ of size r are column invariant, then all sets $U \subset ([2]^2)^k$ of size $2^{2k} - r$ are column invariant and the following conclusion can be made.

Lemma 8. (Switching lemma) *Let k and $r < 2^{2k}$ be positive integers, every set $V \subset ([2]^2)^k$ of r vectors is column invariant if and only if every set $U \subset ([2]^2)^k$ of $2^{2k} - r$ vectors is column invariant.*

For an integer m we use the notation $k = \lceil \frac{m}{2} \rceil$ and $k' = \lfloor \frac{m}{2} \rfloor$. Consider sets $V = \{v^1, \ldots, v^r\} \subset ([2]^2)^k$ and $V' = \{v'^1, \ldots, v'^r\} \subset ([2]^2)^{k'}$. Denote with V^* the set of all vectors $v^i v'^i = (v^i_1, v^i_2, \ldots, v^i_k, v'^i_1, \ldots, v'^i_{k'})$. It is obvious that $V^* \subset ([2]^2)^m$. We define the action of a permutation $\pi \in S_k \backslash \{\mathrm{id}\}$ on a vector $vv' \in V^*$ by

$$\pi v v' = (v_{\pi(1)}, \ldots, v_{\pi(k)}, v'_1, \ldots, v'_{k'}).$$

Analogously, a permutation $\pi' \in S_{k'} \backslash \{\mathrm{id}\}$ acts on a vector vv'

$$\pi' vv' = (v_1, \ldots, v_k, v'_{\pi'(1)}, \ldots, v'_{\pi'(k')}).$$

We also write $\pi V^* = \{\pi v^1 v'^1, \ldots, \pi v^r v'^r\}$ and $\pi' V^* = \{\pi' v^1 v'^1, \ldots, \pi' v^r v'^r\}$. A set of vectors $V^* \subset ([2]^2)^m$ is *half-column invariant* if there exists a permutation $\pi \in S_k \backslash \{\mathrm{id}\}$ or $\pi' \in S_{k'} \backslash \{\mathrm{id}\}$ (or both) such that $\pi V^* = V^*$ or $\pi' V^* = V^*$. Otherwise, we say V^* *is not half-column invariant*.

Observation 9. *If V and V' are not column invariant, then V^* is not half-column invariant.* □

A set of vectors $V \subset ([2]^2)^k$ of size r can be related to a colouring of edges of the Cartesian product of stars $K_{1,r} \square K_{1,k}$. Let u and v be central vertices of the stars, respectively. If the set of all the edges of $K_{1,r}^v$-layer and $K_{1,k}^u$-layer is monochromatic and the colouring is distinguishing, then we call the colouring a *strongly distinguishing colouring*. Numerate the leaves of the star $K_{1,k}$ with numbers from the set $\{1, 2, \ldots, k\}$. Consider the $K_{1,k}$-layers that do not contain the vertex (u, v). Exactly one vertex of each such layer has degree equal to k. Every other vertex has degree equal to two. Assign each vector of the set V to exactly one of the $K_{1,k}$-layers in a way that the consecutive vertices of degree two of the i-th layer are assigned to consecutive pairs of the i-th vector. The first coordinate of the pair corresponds to the colour of the incident edge in the $K_{1,r}$-layer and the other one to the colour in the $K_{1,k}$-layer.

Lemma 10. *The strongly distinguishing colouring of $K_{1,r} \square K_{1,k}$ with two colours exists if and only if there exists a set of vectors $V \subset ([2]^2)^k$ with $|V| = r$ that is not column invariant.*

Proof. If $K_{1,r} \square K_{1,k}$ has a strongly distinguishing 2-colouring, then every nontrivial automorphism $\varphi \in \mathrm{Aut}(K_{1,r} \square K_{1,k})$ is broken by such a colouring. Therefore, for the set V of r vectors from $([2]^2)^k$ representing this colouring as described above there does not exist a permutation $\pi \in S_k \backslash \{\mathrm{id}\}$ such that $\pi V = V$, so V is not column invariant. Conversely, if there exists a set of r vectors $V \subset ([2]^2)^k$ which is not column invariant, then these vectors generate a strongly distinguishing 2-colouring of the Cartesian product of stars $K_{1,r}$ and $K_{1,k}$. It suffices to assign the vectors from the set V to the $K_{1,k}$-layers. Since there does not exist a permutation $\pi \in S_k \backslash \{\mathrm{id}\}$ such that $\pi V = V$, then there does not exist an automorphism $\varphi \in \mathrm{Aut}(K_{1,r} \square K_{1,k})$ preserving this colouring. □

Colours red and blue correspond to 1 and 2, respectively. For any vector $v = ((a_1, b_1), \ldots, (a_k, b_k)) \in ([2]^2)^k$ consider a vector

$$\bar{v} = ((a_1, \bar{b}_1), \ldots, (a_k, \bar{b}_k)).$$

We call vectors v and $\bar{v}$ *a complementary pair* if $b_i + \bar{b}_i = 3$ for all $i \in \{1, \ldots, k\}$.

We present lemmas and observations leading to the proof of Theorem 6.

Lemma 11. *If $r \le k \le 2^{2r} - r + 1$, then $K_{1,r} \square K_{1,k}$ has a strongly distinguishing colouring.*

Proof. Denote with u and v the central vertices of stars $K_{1,r}$ and $K_{1,k}$, respectively.

We construct a strongly distinguishing colouring of $K_{1,r} \square K_{1,k}$. By definition, all edges of the $K_{1,r}^v$-layer and $K_{1,k}^u$-layer have the same colour, say blue. We colour the edges of the remaining layers based on the vectors in $([2]^2)^r = \{(v_1, \ldots, v_r) \colon v_i = (a_i, b_i), a_i, b_i \in \{1, 2\}\}$.

Assume first that $k = 2^{2r} - r + 1$. From the set of all vectors in $([2]^2)^r$ we remove the vectors of the form

$$v^i = \big((2, b_1^i), (2, b_2^i), \ldots, (2, b_r^i)\big),$$

for $i \in \{1, 2, \ldots, r-1\}$, where $b_j^i = 1$ for $j \le i$ and $b_j^i = 2$ otherwise.

There are exactly $r - 1$ such vectors. We assign the remaining vectors to the $K_{1,r}$-layers. Such colouring is strongly distinguishing. It follows from the fact that in every $K_{1,k}$-layer there is a different number of red edges. Therefore, if any of those layers were interchanged by an automorphism of $K_{1,r} \square K_{1,k}$, then the colouring would be broken. It is obvious that $K_{1,r}$-layers are pairwise distinct, hence they cannot be interchanged either.

In the case when $k < 2^{2r} - r + 1$, let $s = 2^{2r} - r + 1 - k$. If s is even, remove additionally $\frac{s}{2}$ complementary pairs of vectors from $([2]^2)^r$. If s is odd remove a vector of all 1's and $\frac{s-1}{2}$ complementary pairs. The use of the remaining vectors will yield a colouring with the property described above. □

Since, $2^{2r-1} \le 2^{2r} - r + 1$ and by Lemma 11 we have the following result.

Corollary 12. *If $r \leq k \leq 2^{2r-1}$, then $K_{1,r} \square K_{1,k}$ has a strongly distinguishing colouring.* □

We present a result similar to the Switching Lemma for half-column invariant sets of vectors. Let us define a set of permutations

$$\overline{S}_m = \Big\{\pi \in S_m \colon \forall\, i \in \{1, 2, \dots, \left\lceil \frac{m}{2} \right\rceil\}\ \pi(i) \in \{1, 2, \dots, \left\lceil \frac{m}{2} \right\rceil\} \\ \wedge \quad \forall\, i \in \{\left\lceil \frac{m}{2} \right\rceil + 1, \dots, m\}\ \pi(i) \in \{\left\lceil \frac{m}{2} \right\rceil + 1, \dots, m\}\Big\}.$$

Observation 13. *A set of vectors $V^* \subset ([2]^2)^m$ is half-column invariant if and only if there exists a permutation $\pi^* \in \overline{S}_m \backslash \{\mathrm{id}\}$ such that for every vector $v^* \in V^*$ the condition $\pi^* v^* \in V^*$ is fulfilled.*

Proof. For a set $V^* \subset ([2]^2)^m$, let $V \subset ([2]^2)^k$ and $V' \subset ([2]^2)^{k'}$ be the sets like in the definition of V^*, where $k = \lceil \frac{m}{2} \rceil$ and $k' = \lfloor \frac{m}{2} \rfloor$.

If V^* is half-column invariant, then there exists a permutation $\pi \in S_k \backslash \{\mathrm{id}\}$ or $\pi' \in S_{k'} \backslash \{\mathrm{id}\}$. We consider a permutation $\pi^* \in S_m$ such that $\pi^*|_V = \pi$ and $\pi^*|_{V'} = \pi'$. Clearly, π exchanges only the first k elements and π' only the last k' elements. Therefore, $\pi^* \in \overline{S}_m$ and for every $v^* \in V^*$ it is true that $\pi^* v^* \in V^*$.

Let $\pi^* \in \overline{S}_m$ be a permutation such that for every vector $v^* \in V^*$ the condition $\pi^* v^* \in V^*$ is fulfilled. Consider the restrictions $\pi = \pi^*|_V$ and $\pi' = \pi^*|_{V'}$. Clearly, $\pi' \in S_k$ and $\pi' \in S_{k'}$. Moreover, since for every vector $v^* \in V^*$ we get that $\pi^* v^* \in V^*$, then we obtain that $\pi V = V$ and $\pi' V' = V'$. Which implies that the set V^* is half-column invariant. □

Lemma 14. *If $V^* \subset ([2]^2)^m$ is not a half-column invariant set, then the set $U^* = ([2]^2)^m \backslash V^*$ also is not half-column invariant.*

Proof. For an arbitrary permutation $\pi \in \overline{S}_m \backslash \{\mathrm{id}\}$ its inverse $\pi^{-1} \in \overline{S}_m \backslash \{\mathrm{id}\}$. Since V^* is not half-column invariant, then for the permutation π^{-1} there exists a vector $v^* \in V^*$ such that $u^* = \pi^{-1} v^* \notin V^*$. Then $\pi u^* = (\pi^{-1})^{-1} \pi^{-1} v^* = v^* \in V^*$. Therefore, we showed that for any permutation $\pi \in \overline{S}_m \backslash \{\mathrm{id}\}$ there exists a vector $u^* \in U^*$ such that $\pi u^* \notin U^*$. Hence, U^* is not half-column invariant. □

Theorem 15. *If there exist sets $V \subset ([2]^2)^k$ and $V' \subset ([2]^2)^{k'}$ of size r that are not column invariant, then $D'(K_{1,m} \square K_{1,n}) = 2$, where $k = \lceil \frac{m}{2} \rceil$, $k' = \lfloor \frac{m}{2} \rfloor$ and $n = 2^{2m+1} - r$.*

Proof. Let $V(K_{1,m}) = \{w_0, w_1, \dots, w_m\}$ and $V(K_{1,n}) = \{u_0, u_1, \dots, u_n\}$, where w_0 and u_0 are the central vertices of the stars.

Let $V = \{v^1, v^2, \dots, v^r\} \subset ([2]^2)^k$ and $V' = \{v'^1, v'^2, \dots, v'^r\} \subset ([2]^2)^{k'}$ be sets that are not column invariant. Then the set $V^* \subset ([2]^2)^m$ associated with V and V' is not half-column invariant by Observation 9. It follows from Lemma 14 that the set $U^* = ([2]^2)^m \backslash V^*$ also is not half-column invariant. We construct a colouring that breaks all nontrivial automorphisms of the graph $K_{1,m} \square K_{1,n}$ based on the set U^*.

Colour the edges of the $K_{1,m}^{u_0}$-layer such that $w_0 w_i$ is red if $i \leq k$ and blue otherwise. Colour the edges of the $K_{1,n}^{w_0}$-layer such that $u_0 u_i$ is red if $i \leq 2^{2m} - r$ and blue otherwise. Assign vectors from U^* to the first $2^{2m} - r$ of the $K_{1,m}$-layers and all the vectors from $([2]^2)^m$ to the remaining $K_{1,m}$-layers. Since U^* is not half-column invariant, then there does not exist a permutation $\pi \in \overline{S}_m\{\mathrm{id}\}$ such that for every $u^* \in U^*$ the vector πu^* also belongs in U^*, as stated in Observation 13. Therefore, for every nontrivial automorphism of the Cartesian product of stars $K_{1,m}$ and $K_{1,n}$ there exists at least one $K_{1,m}$-layer that cannot be mapped into any other layer such that the colours are preserved. □

Lemma 16. *If $2^{2r} < k$, then every set $V \subset ([2]^2)^k$ of size r is column invariant.*

Proof. Consider a set $V = \{v^1, \dots, v^r\}$, where $v^i = (v^i_1, \dots, v^i_k)$ for every $i \in \{1, \dots, r\}$. We define a vector $u_j = (v^1_j, \dots, v^r_j) \in ([2]^2)^r$ for every $j \in \{1, \dots, k\}$. Since $k > 2^{2r} = |([2]^2)^r|$, then there are at least two indices m and n, with $m < n$ such that the vectors u_m and u_n are equal, i.e., $v^i_m = v^i_n$ for all i. We consider a transposition $\pi \in S_k$ such that $\pi(m) = n$. Then for every vector $v^i \in V$ we have

$$\pi v^i = \pi(v^i_1, \dots, v^i_m, \dots, v^i_n, \dots, v^i_k) = (v^i_{\pi(1)}, \dots, v^i_{\pi(m)}, \dots, v^i_{\pi(n)}, \dots, v^i_{\pi(k)}) = \\ (v^i_1, \dots, v^i_n, \dots, v^i_m, \dots, v^i_k) = (v^i_1, \dots, v^i_m, \dots, v^i_n, \dots, v^i_k) = v^i.$$

Hence, the set V is column invariant. □

The permutation π from the above proof is a transposition. Moreover, every vector $v \in V$ is its own image under the action of the transposition π.

Theorem 17. *For integers m and n, let $k = \lceil \frac{m}{2} \rceil$ and $r = 2^{2m+1} - n$. If $2^{2r} < k$, then $D'(K_{1,m} \square K_{1,n}) > 2$.*

Proof. We show that for any colouring of the Cartesian product $K_{1,m} \square K_{1,n}$ with two colours there exists a nontrivial automorphism that is not broken.

Let v and u be central vertices of $K_{1,m}$ and $K_{1,n}$, respectively. Consider first the $K^u_{1,m}$-layer and $K^v_{1,n}$-layer. Without loss of generality, we assume that the number of blue edges in the $K^u_{1,m}$-layer is greater or equal to the number of red edges in this layer. Denote the number of blue edges in the $K^u_{1,m}$-layer and $K^v_{1,n}$-layer by k' and n', respectively. Further, denote with n'' the number of red edges in the $K^v_{1,n}$-layer. Numerate the vertices of each star such that the blue edges connect the central vertex with first k' vertices of $K_{1,m}$ and first n' vertices of $K_{1,n}$. We call the initial n' of the $K_{1,m}$-layers blue layers and the remaining $K_{1,m}$-layers we call red layers.

Denote with W' and W'' the sets of all vectors of pairs of length m corresponding to the colouring of the blue layers and the red layers, respectively. Denote with V' and V'' the sets of all vectors of pairs of length k' corresponding to the initial k' terms of the vectors from the sets W' and W'', respectively. Let $U' = ([2]^2)^{k'} \setminus V'$ and $U'' = ([2]^2)^{k'} \setminus V''$. Denote with $U = U' \cup U''$ and $r' = |U|$. Since r is the number of all the missing vectors, then $r' \le r$ and $k' \ge \lceil \frac{m}{2} \rceil = k$. Therefore,

$$2^{2r'} \le 2^{2r} < k \le k'.$$

As a result $2^{2r'} < k'$. Then by Lemma 16, the set U is column invariant. There exists a transposition $\pi \in S_{k'} \setminus \{\mathrm{id}\}$ such that $\pi U = U$. Moreover, for every vector $u' \in U'$ we have $\pi u' = u'$. Clearly, the same is true for every vector $u'' \in U''$. Let $\bar{\pi} \in S_m \setminus \{\mathrm{id}\}$ be a permutation such that $\bar{\pi}(i) = \pi(i)$ for all $i \in \{1, \dots, k'\}$ and $\bar{\pi}(j) = j$ for all $j \in \{k'+1, \dots, m\}$. Since $\bar{\pi}([2]^2)^m = ([2]^2)^m$, for every vector $v' \in V'$ ($v'' \in V''$) there exists a vector $w' \in V'$ ($w'' \in V''$) such that $w' = \bar{\pi} v'$ ($w'' = \bar{\pi} v''$). Therefore, there exists an automorphism corresponding to the permutation $\bar{\pi}$ that exchanges two of the initial k' $K_{1,n}$-layers that preserves the colours of all the edges of the Cartesian product $K_{1,m} \square K_{1,n}$. $\square$

Proof of Theorem 6. Let $r = 2^{2m+1} - n$. If $r \ge \frac{\log_2 k}{2} + \frac{1}{2}$, then $k \le 2^{2r-1}$. A graph $K_{1,r} \square K_{1,k}$ has a strongly distinguishing colouring, by Corollary 12. Moreover, by Lemma 10 there exists a set of vectors $V \subset ([2]^2)^k$ of size r that is not column invariant. Finally, by Theorem 15, the distinguishing index of the Cartesian product $K_{1,m} \square K_{1,n}$ equals two.

Otherwise, if $r < \frac{\log_2 k}{2}$, then $2^{2r} < k$. Therefore, by Theorem 17 two colours do not suffice to break all nontrivial automorphisms of the graph $K_{1,m} \square K_{1,n}$. $\square$

3. Proof of the main theorem

The proof of Theorem 3 is a straightforward generalization of the proof of Theorem 6. We consider the set of vectors of length k

$$([d]^2)^k = \{(v_1, \dots, v_k) \colon v_i = (a_i, b_i) \wedge a_i, b_i \in \{1, 2, \dots, d\}\}.$$

For an integer m, denote with $k = \lceil \frac{m}{d} \rceil$ and $k' = \lfloor \frac{m}{d} \rfloor$. For sets $V \subset ([d]^2)^k$ and $V' \subset ([d]^2)^{k'}$ such that $V = \{v^1, \dots, v^r\}$ and $V' = \{v'^1, \dots, v'^r\}$. Let l and l' be nonnegative integers such that $m = lk + l'k'$ and $d = l + l'$. Clearly, this conditions give us a unique solution, unless $d|m$. In this case take $l = d$ and $l' = 0$. Denote with v^{i*} the vector that is a concatenation of l copies of v^i and l' copies of v'^i. Notice that $v^{i*} \in ([d]^2)^m$.

We consider a special subset of the set of permutations S_m. For this purpose, we divide the set $\{1, 2, \dots, m\}$ into the following sets

$$I_s = \{i \colon k(s-1) + 1 \le i \le ks\} \quad \text{for} \quad s \in \{1, \dots, l\}$$
$$I_t = \{i \colon kl + k'(t-1) + 1 \le i \le kl + k't\} \quad \text{for} \quad t \in \{1, \dots, l'\}.$$

Let us define

$$\overline{S}^d_m = \Big\{\pi \in S_m \colon \quad \forall\, s \in \{1, \dots, l\} \quad \forall\, i \in I_s \quad \pi(i) \in I_s$$
$$\wedge \quad \forall\, t \in \{1, \dots, l'\} \quad \forall\, i \in I_t \quad \pi(i) \in I_t\Big\}.$$

We say that a set $V^* \subset ([d]^2)^m$ is *d-column invariant* if there exists a permutation $\pi \in \overline{S}^d_m$ such that for every vector $v^* \in V^*$ the vector $\pi v^* \in V^*$. Otherwise, we say that the set V^* *is not d-column invariant*.

With this notation, lemmas and theorems proved in Section 2 can be extended to the following results.

If there exists a strongly distinguishing colouring of $K_{1,r} \square K_{1,k}$ with d colours, then all of the nontrivial automorphisms of the Cartesian product are broken. This colouring is equivalent to a set of r vectors $V \subset ([d]^2)^k$, as presented in Section 2, such that there does not exist a permutation $\pi \in S_k \backslash \{\mathrm{id}\}$ such that $\pi V = V$. Therefore, V is not column invariant. Reverse argument also holds and the following lemma is justified.

Lemma 18. *A strongly distinguishing colouring of $K_{1,r} \Box K_{1,k}$ with d colours exists if and only if there exists a set of vectors $V \subset ([d]^2)^k$ such that $|V| = r$ and V is not column invariant.* □

A strongly distinguishing colouring with d colours can be constructed based on the set of vectors from $([d]^2)^k$. The vectors are chosen in a similar way as in the proof of Lemma 11. More technical details are necessary for proper description of the complete proof of the following lemma.

Lemma 19. *If $r \le k \le d^{2r} - r + 1$, then $K_{1,r} \Box K_{1,k}$ has a strongly distinguishing colouring.* □

It is easy to notice that the following is also true.

Corollary 20. *If $r \le k \le d^{2r-1}$, then $K_{1,r} \Box K_{1,k}$ has a strongly distinguishing colouring.* □

When V^* is not d-column invariant, then for every permutation $\pi \in \overline{S}_m^d \backslash \{\mathrm{id}\}$ there exists a vector $v^* \in V^*$ such that $\pi v^* \notin V^*$. Therefore, for every permutation $\sigma = \pi^{-1} \in \overline{S}_m^d \backslash \{\mathrm{id}\}$ there exists a vector $\pi v^* \in U^*$ such that the vector $\sigma \pi v^*$ belongs in V^*. This directly implies that U^* is not d-column invariant.

Lemma 21. *If $V^* \subset ([d]^2)^m$ is not a d-column invariant set, then the set $U^* = ([d]^2)^m \backslash V^*$ also is not d-column invariant.* □

Given the sets $V \subset ([d]^2)^k$ and $V' \subset ([d]^2)^{k'}$, where $k = \lceil \frac{m}{d} \rceil$ and $k' = \lfloor \frac{m}{d} \rfloor$, we construct the set V^* that is not d-column invariant. The set $U^* = ([d]^2)^m \backslash V^*$ also is not d-column invariant by Lemma 21. Finally, based on the set U^* we construct the colouring of the Cartesian product $K_{1,m} \Box K_{1,n}$ similarly to the construction in the proof of Theorem 15.

Theorem 22. *If there exist sets $V \subset ([d]^2)^k$ and $V' \subset ([d]^2)^{k'}$ of size r that are not column invariant, then $D'(K_{1,m} \Box K_{1,n}) = d$, where $k = \lceil \frac{m}{d} \rceil$, $k' = \lfloor \frac{m}{d} \rfloor$ and $n = d^{2m+1} - r$.* □

Let $V = \{v^1, \ldots, v^r\} \subset ([d]^2)^k$, where $v^i = (v_1^i, \ldots, v_k^i)$ for every $i \in \{1, \ldots, r\}$. If $d^{2r} < k$, then there exist at least two indices $m < n$ such that $v_m^i = v_n^i$ for all i. Therefore, the transposition $\pi = (mn) \in S_k$ is a permutation such that $\pi v^i = v^i \in V$. This implies that V is column invariant.

Lemma 23. *If $d^{2r} < k$, then every set $V \subset ([d]^2)^k$ of size r is column invariant.* □

We conclude with the condition under which $D'(K_{1,m} \Box K_{1,n}) = d + 1$. As in the proof of Theorem 17, we may show that for any colouring of the Cartesian product $K_{1,m} \Box K_{1,n}$ with d colours here exists a nontrivial automorphism that is not broken. This automorphism corresponds to the transposition $\pi \in S_m$ found in Lemma 23.

Theorem 24. *If $d^{2r} < k$, then $D'(K_{1,m} \Box K_{1,n}) > d$, where $k = \lceil \frac{m}{d} \rceil$ and $n = d^{2m+1} - r$.* □

The proofs do not differ much from the ones presented earlier. However, the notation becomes significantly more complicated and a lot of additional technical details need to be put into consideration. Therefore, we decided to omit these proofs.

4. Open cases

In this section we show that there exist m and n for which the value of the distinguishing index of the Cartesian product $K_{1,m} \Box K_{1,n}$. Consider two arbitrary stars $K_{1,m}$ and $K_{1,n}$, where $m \le n$. To determine the distinguishing index of their Cartesian product using Theorem 3, we start by determining a natural number d such that $(d-1)^{2m+1} < n \le d^{2m+1}$. We then check which of the two conditions is fullfilled by the sizes of the stars. However, there exists a small gap between the boundary values of n. Namely, if the integers m, n and d satisfy the inequalities $d^{2m+1} - \frac{\log_d k}{2} - \frac{1}{2} < n \le d^{2m+1} - \frac{\log_d k}{2}$, then Theorem 3 does not provide the value of $D'(K_{1,m} \Box K_{1,n})$. Equivalently, this is the case when for given m and d there exists a positive integer l such that $\frac{\log_d k}{2} \le l < \frac{\log_d k}{2} + \frac{1}{2}$ and then $n = d^{2m+1} - l$.

Let us consider the open cases from a different perspective. For a given positive integer d we find that there exists l as above if $m \ge d^3 - d + 1$. Therefore, for $m < d^3 - d + 1$ Theorem 3 determines the value of $D'(K_{1,m} \Box K_{1,n})$. Moreover, if there exists $l \in \mathbb{N}_+$ such that $m = d^{2l} + i$, $i \in \{1, \ldots, d^{2l+1} - d^{2l}\}$, then the distinguishing index of the Cartesian product of $K_{1,m}$

Table 1
Initial values of the sizes of the stars for which $D'(K_{1,m} \square K_{1,n})$ is not known.

$D'(K_{1,m} \square K_{1,n}) =$					
2 or 3		3 or 4		4 or 5	
m	$n = 2^{2m+1} - l$	m	$n = 3^{2m+1} - l$	m	$n = 4^{2m+1} - l$
7	$2^{15} - 1$	25	$3^{51} - 1$	61	$4^{123} - 1$
8	$2^{17} - 1$	26	$3^{53} - 1$	$\vdots$	$(4^m - 1)$
17	$2^{35} - 2$	27	$3^{55} - 1$	64	$4^{129} - 1$
$\vdots$	$(2^{2m+1} - 2)$	82	$3^{165} - 2$	257	$4^{515} - 2$
32	$2^{65} - 2$	$\vdots$	$(3^{2m+1} - 2)$	$\vdots$	$(4^{2m+1} - 2)$
65	$2^{131} - 3$	243	$3^{15} - 2$	1024	$4^{2049} - 2$
...	$(2^{2m+1} - l)$	...	$(3^{2m+1} - l)$	...	$(4^{2m+1} - l)$

and $K_{1,n}$ remains unknown. These are the only such situations. It is worth noting that if Theorem 3 does not give a precise answer, it always leaves only two possible values of the distinguishing index. Below we attach Table 1 with a few initial values of m and respective n with respect to a given d which remain open cases.

References

[1] M.O. Albertson, K.L. Collins, Symmetry breaking in graphs, Electron. J. Combin. 3 (1996) R18.
[2] E. Estaji, W. Imrich, R. Kalinowski, M. Pilśniak, T. Tucker, Distinguishing number for the Cartesian product of countable graphs, Discuss. Math. Graph Theory 37 (2017) 155–164.
[3] M. Fisher, G. Isaak, Distinguishing colouring of the Cartesian products of complete graphs, Discrete Math. 308 (2008) 2240–2246.
[4] M. Ghaffari, B. Haeupler, Near optimal leader election in multi-hop radio networks, in: Proceedings of the Twenty-Fourth Annual ACM-SIAM Symposium on Discrete Algorithms, SIAM, Philadelphia, PA, 2012, pp. 748–766.
[5] W. Goddard, P.K. Srimani, Self-stabilizing master–slave token circulation and efficient size-computation in a unidirectional ring of arbitrary size, Internat. J. Found. Comput. Sci. 23 (2012) 763–777.
[6] A. Gorzkowska, R. Kalinowski, M. Pilśniak, The distinguishing index of the Cartesian product of finite graphs, Ars Math. Contemp. 12 (2017) 77–87.
[7] W. Imrich, Automorphismen und das kartesische Produkt von Graphen, Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II 117 (1969) 203–214.
[8] W. Imrich, J. Jerebic, S. Klavžar, The distinguishing number of Cartesian products of complete graphs, European J. Combin. 29 (2008) 922–929.
[9] W. Imrich, S. Klavžar, Distinguishing Cartesian powers of graphs, J. Graph Theory 53 (2006) 250–260.
[10] G. Itkis, C. Lin, J. Simon, Deterministic, constant space, self-stabilizing leader election on uniform rings, in: Distributed Algorithms, in: Lecture Notes in Comput. Sci., vol. 972, 1993, pp. 288–302.
[11] R. Kalinowski, M. Pilśniak, Distinguishing graphs by edge-colourings, European J. Combin. 45 (2015) 124–131.
[12] S. Klavžar, X. Zhu, Cartesian powers of graphs can be distinguished by two labels, European J. Combin. 28 (2007) 303–310.
[13] D.J. Miller, The automorphism group of a product of graphs, Proc. Amer. Math. Soc. 25 (1970) 24–28.