Discrete Applied Mathematics 266 (2019) 56–64

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Asymmetric colorings of products of graphs and digraphs

Izak Broere [a,1], Wilfried Imrich [b,2,*], Rafał Kalinowski [c,3], Monika Pilśniak [c,3]

[a] *University of Pretoria, Pretoria, South Africa*
[b] *Montanuniversität Leoben, A-8700 Leoben, Austria*
[c] *AGH University of Science and Technology, al. Mickiewicza 30, 30-059 Krakow, Poland*

ARTICLE INFO

Article history:
Received 30 January 2018
Received in revised form 5 December 2018
Accepted 10 December 2018
Available online 14 January 2019

Keywords:
Vertex colorings
Asymmetric colorings
Automorphisms
Cartesian, strong, and direct products of graphs and digraphs

ABSTRACT

We extend results about asymmetric colorings of finite Cartesian products of graphs to strong and direct products of graphs and digraphs. On the way we shorten proofs for the existence of prime factorizations of finite digraphs and characterize the structure of the automorphism groups of strong and direct products. The paper ends with results on asymmetric colorings of Cartesian products of finite and infinite digraphs.

1. Introduction

A coloring of the vertex set of a graph is called *asymmetric* if the identity automorphism is the only automorphism that preserves it. Such colorings were independently studied by several authors, notably Babai [1], Cameron [2], and also Polat [3] and Sabidussi [4]. After Albertson and Collins [5], unaware of these developments, introduced the term *distinguishing number* for the smallest positive integer d for which an asymmetric coloring of a graph G exists the subject became more widely known. They introduced the notation $D(G)$ for this number. We prefer to call it the *asymmetric coloring number*, but use the same notation.

This paper extends results of [6], where it was shown that the Cartesian product of two finite graphs has an asymmetric 2-coloring if the sizes of the factors do not differ too much. In Section 4 we extend this result to the strong and the direct product of finite graphs and digraphs. This is possible because the strong and the direct product of finite graphs share many properties with the Cartesian product.

Our methods of proof, first of all, rely on the fact that all finite graphs have prime factorizations with respect to the strong and the direct product, and that these factorizations are unique under suitable, natural conditions. This is treated in Section 3.1. Another property which we will heavily use is the relationship between the automorphism group of a product of prime graphs with the groups of the factors, see Section 3.2.

The unified treatment of the strong and the direct product in Section 3 is possible, because both products are instances of the direct product of digraphs without multiple arcs, which can also be viewed as binary relational structures. We thus begin with the prime factorization of binary relational structures with respect to the direct product.

* Corresponding author.
E-mail addresses: izak.broere@up.ac.za (I. Broere), imrich@unileoben.ac.at (W. Imrich), kalinows@agh.edu.pl (R. Kalinowski), pilsniak@agh.edu.pl (M. Pilśniak).

1 The research of this author is supported in part by the National Research Foundation of South Africa Grant Number 90841.
2 Supported by OEAD grant PL 08/2017.
3 Rafał Kalinowski and Monika Pilśniak were partially supported by the Polish Ministry of Science and Higher Education and OEAD grant PL 08/2017.

https://doi.org/10.1016/j.dam.2018.12.023

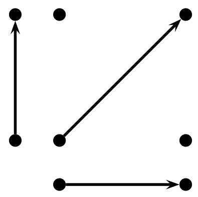

Fig. 1. The direct product of an arc by an arc.

The standard argument is to invoke the so called common refinement property, which in turn implies unique prime factorization for finite structures. The proof of the common refinement property, on the other hand, consists of two parts. The first is a result of Chang, Jónsson and Tarski [7], who proved that it is a consequence of a specific commutativity property of decomposition functions, and the second is due to McKenzie [8], who showed that this property, see Lemma 1, is satisfied under certain natural conditions.

We use the result of McKenzie, and then prove unique factorization directly, using neither the results of [7], nor common refinement. Our main tool is Theorem 5, which we derive directly from a result of McKenzie [8]. It is the main result of Section 3.1 and the basis of Section 3.2, where the structure of the automorphism group of direct products of digraphs is investigated.

The Cartesian product of digraphs is treated in Section 6. Here the restriction to finite graphs is not necessary and we can easily extend results from [6] about asymmetric colorings of finite or infinite Cartesian products to finite or infinite digraphs with or without loops. This section is independent of the results in the previous sections.

For an entirely different approach to prime factorization of digraphs with respect to the direct product see [9–14]. The approach in these papers leads to classes of graphs that have the unique prime factorization property, but are distinct from the class of graphs with the unique prime factorization property of this paper. In other words, that approach secures unique prime factorization of graphs not covered by the methods if this paper, but does not cover all graphs treated here.

These papers also pave the way to prime factorization algorithms, which is not the case with the approach we follow here.

2. Preliminaries

A *directed graph*, or *digraph* for short, is a pair $G = (V, E)$, where V is a set and $E \subseteq V \times V$. The elements of $V = V(G)$ are called the *vertices* and the elements of $E = E(G)$ the *arcs* of G. An arc (u, v) will be also be denoted by uv; if $u = v$ we speak of a *loop*. We will consider undirected graphs as directed graphs with the property that $uv \in E(G)$ if and only if $vu \in E(G)$.

To establish a connection between graphs and binary relational structures we define a relation $R = R_G$ for every digraph G by letting u be in relation R to v, in symbols uRv, if $uv \in E(G)$.[4] The inverse relation $\breve{R}$ is then defined by $v\breve{R}u$ if and only if uRv, and the pair (V, R) is called a *structure* G_R. We will often write $G_R = (V(G), R_G)$, or simply $G = (V(G), R_G)$.

A structure (V, R) is *connected*,[5] if for any two different elements $x, y \in V$ there exists a sequence $x_0, \ldots, x_k$ such that $x = x_0$, $y = x_k$, and x_iRx_{i+1} for $i = 0, \ldots, k-1$. The *concatenation* $R_1|R_2$ of two relations R_1, R_2 on V is defined by setting

$$x(R_1|R_2)y$$

if there exists a z such that xR_1z and zR_2y. We will mainly consider structures that are both $R|\breve{R}$- and $\breve{R}|R$-connected. This means that for any two vertices u and v there exist vertices $x_0, \ldots, x_{2k}$ such that $u = x_0$, $v = x_{2k}$, $x_{2i}Rx_{2i+1}$ and $x_{2i+1}\breve{R}x_{2i+2}$ for $i = 0, \ldots, k-1$, and vertices $y_0, \ldots, y_{2k}$ such that $u = y_0$, $v = y_{2k}$, $y_{2i}\breve{R}y_{2i+1}$ and $y_{2i+1}Ry_{2i+2}$ for $i = 0, \ldots, k-1$.

By abuse of language we also say a digraph G is $R|\breve{R}$- and $\breve{R}|R$-connected if this is the case for the structure $(V(G), R_G)$.

The *direct product* $G \times H = \{V(G \times H), R_{G\times H}\}$ of two structures G and H is defined on $V(G \times H) = V(G) \times V(H)$ by setting $(g, h)\, R_{G\times H}\, (g', h')$ if both $g\, R_G\, g'$ and $h\, R_H\, h'$ hold. As shown in Fig. 1, an arc uv in a structure G and xy in a structure H thus give rise to an arc $(u, x)(v, y)$ in $G \times H$. The direct product is also known as *cardinal* or *categorical* product. Direct multiplication is commutative, associative, and the one-element set with a loop is its unit (but not the one element set without a loop).

If the relation R of a structure G is symmetric, that is, if $R = \breve{R}$, then there is an arc vu to every arc uv in G and no loss of information occurs if we represent every pair of arcs vu and uv by an undirected edge between u and v. Hence, the direct product of undirected graphs is a special case of the direct product of binary relational structures. Let us recall that one usually defines the direct product $G \times H$ of two undirected graphs G and H as the graph with vertex set $V(G \times H)$ and the edge set

$$E(G \times H) = \big\{(x, u)(y, v) \mid xy \in E(G) \wedge uv \in E(H)\big\}.$$

We always allow loops when speaking of the direct product.

4 Strictly speaking, if one considers R as the set of ordered pairs u, v for which uRv holds, then R and E are identical. Nonetheless it makes sense to distinguish the cases when we consider the pair u, v as being in the relation R from uv being an arc of G.

5 If (V, R) is connected the corresponding directed graph G is usually called *strongly connected*, whereas G is *connected* if $(V, R \cup \breve{R})$ is connected.

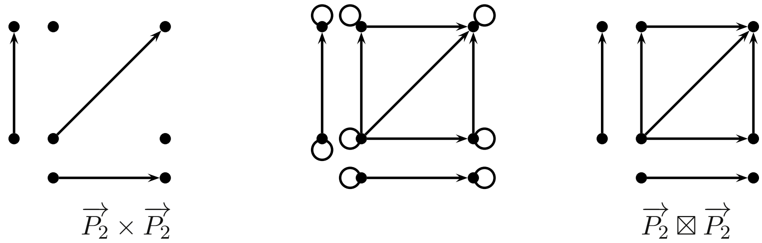

Fig. 2. Illustration to the definition of the strong product.

The *strong product* $G \boxtimes H$ is defined for digraphs G and H without loops. For the definition we add a loop to every vertex of G and to every vertex of H, form the direct product, and then delete the loops from the product; see Fig. 2. Therefore, we can consider also the strong product as a special case of the direct product of relational structures.

For simple graphs, that is, for undirected graphs without loops or multiple edges, our definition of the strong product is equivalent to the definition of the strong product $G \boxtimes H$ of simple graphs, where $V(G \boxtimes H)$ is defined as $V(G) \times V(H)$ and $E(G \boxtimes H)$ as the set of all distinct pairs $(x, u)(y, v)$ of vertices such that

$$(x, u) \neq (y, v) \wedge (xy \in E(G) \vee x = y) \wedge (uv \in E(H) \vee u = v).$$

Notice that the one-vertex graph K_1 is a unit for the strong product.

In this paper thinness plays an important role. For its definition we introduce the concept of neighborhoods. Given a vertex $x \in V(G)$ of a graph, we call the set

$$N^+(x) = \{y \mid xRy\}$$

the *open out-neighborhood*, or simply the *out-neighborhood* of x. The *open in-neighborhood* $N^-(x)$ is defined analogously. Two vertices x, y with the same out-neighborhoods and the same in-neighborhoods are called *equivalent*, in symbols $x \approx_R y$, or simply $x \approx y$. The relation $\approx$ is an equivalence relation, and we call R *thin*, if $\approx$ is trivial.

Similarly we call $N^+[x] = \{x\} \cup N^+(x)$ the *closed out-neighborhood* of x, and define $N^-[x]$ analogously. We say a structure is S-*thin* if $x \neq y$ implies $N^+[x] \neq N^+[y]$ or $N^-[x] \neq N^-[y]$. Thinness is relevant for the direct product and S-thinness for the strong product.

We also need the Cartesian product $G \,\square\, H$ of simple graphs. As for all other products considered here its vertex set is $V(G) \times V(H)$. Its edge set is defined as

$$E(G \,\square\, H) = \big\{(x, u)(y, v) \mid (xy \in E(G) \wedge u = v) \vee (x = y \wedge uv \in E(H))\big\}.$$

Cartesian multiplication is commutative, associative and K_1 is a unit. Notice that

$$E(G \boxtimes H) = E(G \,\square\, H) \cup E(G \times H),$$

but that the analogous relation for more than two factors need not be true.

The Cartesian product is best understood and there are numerous strong results about automorphism breaking of Cartesian products. Here we wish to extend some of them to the strong and the direct product of directed and undirected graphs and to the Cartesian product of digraphs.

3. The direct product of digraphs

In this section we study prime factorizations of graphs and digraphs with respect to the direct product and characterize the structure of the automorphism group of the direct product of indecomposable factors.

3.1. Decomposition functions vs. factorizations

We begin with several definitions that apply to all products that are defined on the Cartesian product of the vertex sets of the factors. Let $*$ denote a symbol in $\{\square, \times, \boxtimes\}$, let G be the product $G = G_1 * G_2 * \cdots * G_k$ of k graphs and let i be any index. The map $p_{G_i} : G_1 * G_2 * \cdots * G_k \to G_i$ defined by $p_{G_i}(x_1, x_2, \ldots, x_k) = x_i$ is called the *ith projection map*. Usually we abbreviate it by p_i.

For any fixed vertex $a = (a_1, a_2, \ldots, a_k)$ of a product $G = G_1 * G_2 * \cdots * G_k$ of k graphs, and for any index i, the G_i-*layer (or i-layer) through* a is the subgraph G_i^a of G induced by the set $\{(a_1, a_2, \ldots, x_i, \ldots, a_k) \mid x_i \in V(G_i)\}$ of vertices of G. The projection $p_{G_i^x}(y) : G \to G_i^x$ is then defined by $p_{G_i^x} : y \mapsto (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_k)$.

Note that the i-layers through two vertices a and b are equal, in symbols $G_i^a = G_i^b$, if and only if the jth projection satisfies $p_j(a) = p_j(b)$ for every index $j \neq i$. We call two layers G_i^a, G_j^b *parallel* if $i = j$, and note that any two layers G_i^a, G_j^b are either identical, disjoint, or share exactly one vertex.

A graph G is *prime*, or *irreducible*, with respect to the product $*$ if G has at least two vertices and if $G = G_1 * G_2$ implies that G_1 or G_2 is isomorphic to G (and the other factor is the unit with respect to $*$).

We also need the concept of a decomposition function, as defined in [7].

Definition 1. Let $G = G_1 \times G_2$. Then the function $f : V(G)^2 \to V(G)$ that maps (x, y) into the projection of y into the layer G_2^x is called the *decomposition function* of G with respect to the decomposition $G_1 \times G_2$.

Note that with $x = (x_1, x_2)$ and $y = (y_1, y_2)$, we have that $f(x, y) = f((x_1, x_2), (y_1, y_2)) = (x_1, y_2)$ and, similarly, $f(y, x) = (y_1, x_2)$. The function f can also be used to define two functions of a single variable:

$f_x^d : V(G) \to V(G)$ is defined by $f_x^d(y) = f(x, y)$, and

$f_x : V(G) \to V(G)$ is defined by $f_x(y) = f(y, x)$.

Observe that, for a given decomposition function g, the function g^d, defined by $g^d(x, y) = g(y, x)$, is also a decomposition function.

In Section 6.1 of [15] the concept of a box is defined for the Cartesian product. Generalizing this concept to arbitrary products we define a *box* as a subproduct $U_1 * U_2 * \cdots * U_k$ of a product $G = G_1 * G_2 * \cdots * G_k$, where $U_i \subseteq G_i$. A box is *trivial* if all U_i but one have only one vertex. The vertices $x, f(y, x), y, f(x, y)$ determined by a decomposition function f as described above clearly constitute a box in $G_1 \times G_2$. Furthermore, a subgraph S of $G_1 \times G_2$ is a box in $G_1 \times G_2$ if and only if $x, y \in V(S)$ implies that $f(x, y)$ and $f(y, x)$ are also in $V(S)$.

We are now ready to formulate the following basic result of McKenzie on decomposition functions, namely Lemma 3.1 of [8].

Lemma 1. *Let f, g be decomposition functions of a structure (V, R) that is $R|\breve{R}$- and $\breve{R}|R$-connected. Then $f_x g_x \approx g_x f_x$ for all $x \in V$.*

Recall that a structure is thin if the equivalence classes of $\approx$ are one-element sets. Hence, for thin structures Lemma 1 implies that $f_x g_x = g_x f_x$.

McKenzie then invokes Theorem 5.6 of Chang, Jónsson and Tarski [7] that asserts that the validity of the conclusion of Lemma 1 implies the so-called common refinement property, which in turn yields unique prime factorization for finite $R|\breve{R}$- and $\breve{R}|R$-connected digraphs.

We follow a more direct approach that also enables us to describe the structure of the automorphisms groups of products of prime graphs that are thin and $R|\breve{R}$- and $\breve{R}|R$-connected. We first show that Lemma 1 implies that layers in a product representation of such graphs are boxes in any other representation.

Lemma 2. *Let $A \times B$ and $C \times D$ be two representations of a graph G which is thin, $R|\breve{R}$-connected and $\breve{R}|R$-connected. Then every layer of G with respect to A or B is a box in the representation $C \times D$ of G.*

Proof. Let f be the decomposition function for $A \times B$ and g the one for $C \times D$. Clearly $f_x g_x = g_x f_x$ by Lemma 1, because G is thin.

It suffices to show that every A-layer is a box. That means, for any two distinct vertices x, y in an A-layer A^v through a vertex v we have to show that $g_x(y)$ and $g_x^d(y)$ are also in A^v. To facilitate the proof, note that $z \in A^v$ if and only if $z = f_x(z)$.

Let $a = g_x(y)$ and $b = g_x^d(y)$. Applying Lemma 1 to f and g we infer that

$$a = g_x(y) = g_x(f_x(y)) = g_x f_x(y) = f_x g_x(y) = f_x(g_x(y)) = f_x(a),$$

which implies that $a \in A^v$. Similarly, but now by application of Lemma 1 to f and g^d, we have that

$$b = g_x^d(y) = g_x^d(f_x(y)) = g_x^d f_x(y) = f_x g_x^d(y) = f_x(g_x^d(y)) = f_x(b).$$

Hence $b \in A^v$. □

This immediately yields the following unique prime factorization theorem, first proved by McKenzie [8] by invoking results from [7].

Theorem 1. *Let G be a finite digraph that is thin, $R|\breve{R}$- and $\breve{R}|R$-connected. Then G is representable as a direct product of prime graphs, and this presentation is unique up to isomorphisms and the order of the factors.*

Proof. Because G is finite, there must be a representation of G as a product of factors with at least two vertices and a maximum number of factors. Clearly these factors have to be prime, otherwise the number of factors would not be maximal. Hence there always exists a prime factorization.

To prove uniqueness consider two prime factorizations

$$G \cong P_1 \times \cdots \times P_k \cong Q_1 \times \cdots \times Q_l,$$

and let φ be the isomorphism between them. Choose a vertex $v \in V(G)$ and an index $i \in \{1, \ldots, k\}$. By Lemma 2, $\varphi(P_i^v)$ is a box in $Q_1 \times \cdots \times Q_l$. It must be trivial, because P_i is prime, and thus contained in a Q_j-layer for some j. In symbols, $\varphi(P_i^v) \subseteq Q_j^{\varphi(v)}$.

For the same reason, $\varphi^{-1}(Q_j^{\varphi(v)}) \subseteq P_r^v$ for some r. We wish to show that $r = i$. To see this we first observe that P_i has at least two vertices. Hence, there must be another vertex besides v in P_i^v, say u. Clearly both $\varphi(v)$ and $\varphi(u)$ are in $Q_j^{\varphi(v)}$. But then both u and v are in $\varphi^{-1}(Q_j^{\varphi(v)}) \subseteq P_r^v$. Since they are also in P_i^v we infer that $r = i$, and therefore $\varphi(P_i^v) = Q_j^{\varphi(v)}$.

This means that to any $i \in \{1, \ldots, k\}$ there is a $\pi(i) \in \{1, \ldots, \ell\}$ such that $\varphi(P_i^v) = Q_{\pi(i)}^{\varphi(v)}$. If $i \neq i'$ and $i' \in \{1, \ldots, k\}$, then $\pi(i) \neq \pi(i')$, because $P_i^v \neq P_{i'}^v$, and hence also $Q_{\pi(i)}^{\varphi(v)} \neq Q_{\pi(i')}^{\varphi(v)}$. This implies that π is injective, and so $k \leq \ell$. Reversing the argument we see that $k = \ell$ and that π is a permutation.

Because $P_i \cong P_i^v \cong Q_{\pi(i)}^{\varphi(v)} \cong Q_{\pi(i)}$ the prime factorization is unique up to isomorphisms and the order of the factors. □

For the Cartesian product unique prime factorization holds for connected graphs as has been shown first by Sabidussi [16] and then by Vizing [17]. There are many different ways to prove it, but we wish to remark that the proof of the Sabidussi–Vizing Theorem in [15] is similar to the proof of Theorem 1. The proof in [15] uses the fact that convex subgraphs are boxes in Cartesian products. For Cartesian products this implies that layers are boxes, which is used here.

3.2. Automorphisms of direct products of digraphs

The following theorem describes the structure of the automorphism group of the direct product of prime graphs under the above thinness and connectivity conditions. It is exactly the same as the structure of the groups of Cartesian products of connected, prime graphs, see [15, Theorem 6.10].

The key is to prove that the permutation π whose existence was shown in Theorem 1, and which might depend on the choice of v, is actually independent of v.

Theorem 2. *Suppose φ is an automorphism of a thin, $R|\breve{R}$- and $\breve{R}|R$-connected finite digraph G with prime factorization $G = G_1 \times G_2 \times \cdots \times G_r$. Then there exist a permutation π of $\{1, 2, \ldots, r\}$ and an isomorphism $\varphi_i\colon G_{\pi(i)} \to G_i$ for every i such that*

$$\varphi(x_1, x_2, \ldots, x_r) = (\varphi_1(x_{\pi(1)}), \varphi_2(x_{\pi(2)}), \ldots, \varphi_r(x_{\pi(r)})), \tag{1}$$

for every vertex $(x_1, x_2, \ldots, x_r) \in V(G)$.

Proof. Let φ be an automorphism of $G = G_1 \times G_2 \times \cdots \times G_r$. It is an isomorphism from $G_1 \times G_2 \times \cdots \times G_r$ to itself and by Theorem 1 there is a permutation π_v for every $v \in V(G)$ such that $\varphi(G_i^v) = G_{\pi_v}^{\varphi(v)}$ for every $i \in \{1, \ldots, r\}$.

We show first that $\pi_v = \pi_{v'}$ if v, v' differ in exactly one coordinate, say in coordinate t. Suppose $l = \pi_v(i) \neq \pi_{v'}(i) = m$. Consider a vertex $x \in G_i^v$ and the vertex $x' \in G_i^{v'}$ with $x_i' = x_i$. Since $x_t = v_t$ and $x_t' = v_t'$ we infer that $x_t \neq x_t'$. All other coordinates are the same, hence $x, x' \in G_t^x$, and $\varphi(x), \varphi(x')$ are in $\varphi(G_t^x)$, hence $\varphi(x), \varphi(x')$ differ only in coordinate, namely $\pi_x(t)$, which contradicts

$$\varphi(x)_l \neq \varphi(v)_l = \varphi(v')_l = \varphi(x')_l, \quad \varphi(x)_m = \varphi(v)_m = \varphi(v')_m \neq \varphi(x')_m,$$

unless $l = \pi_v(i) = \pi_{v'}(i) = m$. Hence $\pi_v = \pi_{v'}$ if v, v' differ in exactly one coordinate. Because to any two vertices u, v there is a sequence of vertices $u = u_0, u_1, \ldots, u_r = v$, where successive elements differ in only one coordinate, we infer that π is independent of v.

We also observe that $\varphi(x)_j = \varphi(x')_j$ if $j \neq \pi_x(t) = \pi(t)$.

Suppose x and x' have the same ith coordinate, $x_i = x_i'$. Then there is a chain $x = u_0, u_1, \ldots, u_r = x'$, where successive elements have the same ith coordinate, but otherwise differ in only one coordinate. By the above, $\varphi(x)_{\pi(i)} = \varphi(x')_{\pi(i)}$. This means, if $x_i = x_i'$, then $\varphi(x)_{\pi(i)} = \varphi(x')_{\pi(i)}$. If we now define

$$\varphi_{\pi^{-1}(i)} : x_{\pi^{-1}(i)} \mapsto \varphi(x')_i$$

then

$$\varphi(x_1, x_2, \ldots, x_r) = (\varphi_1(x_{\pi(1)}), \varphi_2(x_{\pi(2)}), \ldots, \varphi_r(x_{\pi(r)})).$$

To show that φ_i is an isomorphism from $G_{\pi(i)}$ to G_i consider an arc $wz \in E(G_{\pi(i)})$. There are elements xy in $E(G)$ with $x_{\pi(i)} = w$ and $y_{\pi(i)} = z$. φ maps xy into $\varphi(x)\varphi(y) \in E(G)$ and so $\varphi(x)_i\varphi(y)_i \in E(G_i)$. The observation that $\varphi(x)_i = \varphi_i(x_{\pi(i)}) = \varphi_i(w)$ and $\varphi(y)_i = \varphi_i(y_{\pi(i)}) = \varphi_i(z)$ completes the proof. □

We remark that the factors in a representation $G = G_1 \times G_2 \times \cdots \times G_r$ are thin and $R|\breve{R}$- and $\breve{R}|R$-connected if and only if this is the case for G.

We continue with the special cases of the direct product of graphs and the strong product of graphs and digraphs.

3.3. The direct product of graphs

Suppose that G is a graph[6] whose corresponding structure is thin and $R|\breve{R}$- and $\breve{R}|R$-connected. Then G has to be thin and there must be a path of even length between any two vertices of G, which is only possible if G is nonbipartite.

This means that Theorems 1 and 2 also hold for the *direct product of thin graphs that are connected and nonbipartite.*

3.4. The strong product of graphs and digraphs

As already mentioned, the strong product $G \boxtimes H$ of two graphs or digraphs G and H without loops can be obtained by addition of a loop to every vertex of G and H, formation of the direct product of the new graphs or digraphs, and subsequent deletion of the loops from the product.

Let $\mathcal{L}(\mathcal{G})$ and $\mathcal{L}(\mathcal{H})$ be obtained from G and H by the addition of loops. Clearly Theorems 1 and 2 hold when $\mathcal{L}(\mathcal{G})$ and $\mathcal{L}(\mathcal{H})$ are thin and $R|\breve{R}$- and $\breve{R}|R$-connected.

Let us consider thinness first. Clearly the in- or out-neighborhood of a vertex x in $\mathcal{L}(\mathcal{G})$ is the closed in- or out-neighborhood of x in G. Hence, two vertices x and y have the same in- or out-neighborhoods in $\mathcal{L}(\mathcal{G})$ if and only if they have the same closed in- or out-neighborhoods in G. This implies that $\mathcal{L}(\mathcal{G})$ is thin if and only if G is S-thin.

Now suppose that xRy holds in G. Then $xRy\breve{R}y$ and $x\breve{R}yRy$ hold in $\mathcal{L}(\mathcal{G})$, which means that x is $R|\breve{R}$- and $\breve{R}|R$-connected to y in $\mathcal{L}(\mathcal{G})$. Similarly one shows that this is also valid if $x\breve{R}y$ holds.

Therefore the graph or digraph G is S-thin and connected[7] if and only if $\mathcal{L}(\mathcal{G})$ is thin and $R|\breve{R}$- and $\breve{R}|R$-connected.

Hence Theorems 1 and 2 also hold for the *strong product of S-thin, connected graphs and digraphs.*

4. Asymmetric colorings of strong and direct products

In this section we extend two theorems for Cartesian products to the direct and the strong product of finite graphs. The first one is the main result of [6].

Theorem 3 (*[6, Theorem 6]*). *Let G and H be connected graphs such that*

$$|G| \le |H| \le 2^{|G|} - |G| + 1. \tag{2}$$

Then $D(G\Box H) \le 2$ unless $G \Box H \in \{K_2^{\Box,2}, K_3^{\Box,2}\}$.[8]

Actually Theorem 6 in [6] also lists K_2^3 as an exception, but strictly speaking this is not correct, because the product $G \Box H$ does not satisfy Eq. (2) if $G = K_2$ and $H = K_2^{\Box,2}$.

However, $K_2^{\Box,3}$ is a proper exception in the second theorem that we will generalize. It comprises Theorem 1.1 of [18] and the remarks following it.

Theorem 4 (*[18]*). *Let G be a connected graph and $k \ge 2$. Then $\mathrm{D}(G^{\Box,k}) = 2$ except for the graphs $K_2^{\Box,2}$, $K_2^{\Box,3}$ and $K_3^{\Box,2}$, whose asymmetric coloring number is three.*

The key idea in this section is that, given a direct or strong product G of prime graphs or digraphs, there is a Cartesian product H of complete graphs with the same set of vertices such that Aut(G) is a subgroup of Aut(H), both groups being considered as permutation groups. In this case, every asymmetric coloring of the vertices of H also is an asymmetric coloring of G.

To see this, let $G = G_1 \times G_2 \times \cdots \times G_r$ be a prime factorization of a thin, $R|\breve{R}$- and $\breve{R}|R$-connected digraph. Replace every G_i by an undirected complete graph K_{G_i} on the same set of vertices as G_i. Since complete graphs are prime and because the automorphism group of the complete graph on a set V is the full symmetric group on V, Theorem 2 ensures that $\mathrm{Aut}(G) \le \mathrm{Aut}(K_G)$, where $K_G = K_{G_1} \Box K_{G_2} \Box \cdots \Box K_{G_r}$. We formulate this as a lemma.

Lemma 3. *Let $G = G_1 \times G_2 \times \cdots \times G_r$ be a prime factorization of a thin, $R|\breve{R}$- and $\breve{R}|R$-connected digraph and K_{G_i} be the complete graph with vertex set $V(G_i)$. Then*

$$\mathrm{Aut}(G_1 \times G_2 \times \cdots \times G_r) \le \mathrm{Aut}(K_{G_1} \Box K_{G_2} \Box \cdots \Box K_{G_r}).$$

Clearly the lemma also holds when the G_i are thin, connected non-bipartite graphs. It also holds for the strong product $G_1 \boxtimes \cdots \boxtimes G_r$, when the G_i are S-thin connected graphs or digraphs.

We first extend Theorem 3 when $|V(G)| \cdot |V(H)| \ge 10$. Its proof uses the prime factorizations of $G = G_1 \Box \cdots \Box G_r$ and $H = H_1 \Box \cdots \Box H_s$ and also holds when all factors are complete, because complete graphs are prime with respect to the Cartesian product.

6 We allowed loops in the case of directed graphs. We also have to allow them here, otherwise we could not use the results about the direct product of directed graphs.

7 Recall that G is connected if the corresponding structure (V, R) is $R \cup \breve{R}$ connected.

8 Here $G^{\Box,k}$ denotes the kth power of G with respect to the Cartesian product.

This means, if $G = G_1 \times \cdots \times G_r$ and $H = H_1 \times \cdots \times H_s$ are the unique prime factorizations of G and H, where G and H are thin digraphs whose corresponding structures are $R|\breve{R}$- and $\breve{R}|R$-connected then

$$\mathrm{Aut}(G \times H) \leq \mathrm{Aut}(K_{G_1} \,\Box\, \cdots \,\Box\, K_{G_r} \,\Box\, K_{H_1} \,\Box\, \cdots \,\Box\, K_{H_s}).$$

If G, H satisfy (2) and $|V(G \times H)| \geq 10$, then

$$K_{G_1} \,\Box\, \cdots \,\Box\, K_{G_r} \,\Box\, K_{H_1} \,\Box\, \cdots \,\Box\, K_{H_s}$$

is 2-distinguishable, and hence also $G \times H$. We thus infer the following lemma.

Lemma 4. *Let G, H be thin, $R|\breve{R}$- and $\breve{R}|R$-connected digraphs that satisfy (2). If $|V(G \times H)| \geq 10$, then $\mathrm{D}(G \times H) \leq 2$.*

Again, the lemma also holds for thin, connected non-bipartite graphs and with respect to the strong product of S-thin connected graphs or digraphs.

We now consider the case when $|V(G)| \cdot |V(H)| \leq 9$ and G and H satisfy (2). This means that $(|V(G)|, |V(H)|) \in \{(2, 2), (2, 3), (3, 3)\}$.

We begin with a consideration of all thin, $R|\breve{R}$- and $\breve{R}|R$-connected digraphs on 2 and 3 vertices. For two vertices there are two such graphs, one consists of an arc with loops at both ends, say L, and the other of an edge and a loop at one endpoint, say K. Both graphs are asymmetric.

To treat the case with three vertices we introduce the concept of the shadow G^s of a directed graph G.[9] It is a simple graph with the same vertex set as G, where two vertices x and y are adjacent whenever xRy or $x\breve{R}y$ holds in the structure corresponding to G. Clearly $\mathrm{Aut}(G) \leq \mathrm{Aut}(G^s)$, and therefore $\mathrm{D}(G) \leq \mathrm{D}(G^s)$.

The shadow G^s of a connected digraph G on three vertices is a path of length 2 or a K_3. In the first case G^s is 2-distinguishable, and thus also G. In the second case G is 2-distinguishable if there is at least one pair of vertices x, y where xRy or $x\breve{R}y$, but not both. Hence, G must be undirected unless it is 2-distinguishable. If it has two or three loops, then it is not thin, if it has only one loop, then it is 2-distinguishable.

Thus K_3 is the only thin, $R|\breve{R}$- and $\breve{R}|R$-connected digraph that is not 2-distinguishable.

Let us now consider products $G \times H$ of type $(2, 2)$. Both factors are prime and asymmetric. By Eq. (1) a nontrivial automorphism must interchange the factors. If we color the vertices of one G-layer black and the vertices of the other white, this is not possible any more. Hence all such products are 2-distinguishable.

For products $G \times H$ of type $(2, 3)$ it is clear that the factors cannot be interchanged and that the H-layers must be preserved because G is asymmetric. We now color the three G-layers such that one has no black vertex, the second one black vertex, and the third two. This breaks all automorphisms and $G \times H$ is 2-distinguishable.

Now to products $G \times H$ of type $(3, 3)$. Suppose one factor, say G, is 2-distinguishable. Then G has a distinguishing 2-coloring with one black and two white vertices. We use this coloring for one G-layer. In one of the two other G-layers we color the two vertices black whose H-layer does not contain the black vertex of the first layer. In the third all vertices are left white. Hence there is a G-layer with no black vertex, but all H-layers have a black vertex. This ensures that the set of G layers cannot be mapped into the set of H-layers. It is easy to see now that this is an asymmetric 2-coloring.

The only case left is $K_3 \times K_3$. Its automorphism group is the same as that of $K_3 \,\Box\, K_3$, and it is well known (and easy to see) that $\mathrm{D}(K_3 \,\Box\, K_3) = 3$.

We have thus shown that the direct product $G \times H$ of two thin, $R|\breve{R}$- and $\breve{R}|R$-connected digraphs that has at most nine vertices and satisfies (2) is 2-distinguishable unless $G \times H = K_3 \times K_3$.

If we consider graphs instead of digraphs, we have the same exception. Combining this with Lemma 4 and the observation that the lemma also holds for thin, connected non-bipartite graphs, we obtain the following theorem.

Theorem 5. *Let G and H be thin, $R|\breve{R}$- and $\breve{R}|R$-connected digraphs, or thin, connected non-bipartite graphs, such that*

$$|G| \leq |H| \leq 2^{|G|} - |G| + 1.$$

Then $\mathrm{D}(G \times H) \leq 2$, unless $G \times H = K_3 \times K_3$. $\Box$

Strong product of digraphs and graphs. Here we only have to observe that K_3 is not S-thin. Hence we have no exceptions in this case. Together with the observation that Lemma 4 also holds with respect to the strong product for S-thin, connected graphs and digraphs we obtain the following result.

Theorem 6. *Let G and H be S-thin, connected digraphs or graphs such that*

$$|G| \leq |H| \leq 2^{|G|} - |G| + 1.$$

Then $\mathrm{D}(G \boxtimes H) \leq 2$. $\Box$

9 We will use this concept again in the next section.

Powers of direct and strong products. We wish to extend Theorem 4 to the direct and the strong product. The theorem asserts that all powers of connected graphs with respect to the Cartesian product are 2-distinguishable, except for the graphs $K_2^{\square,2}$, $K_2^{\square,3}$, $K_3^{\square,2}$, whose asymmetric coloring number is 3. It turns out that only one exception remains and that the following theorem holds. It is new for digraphs, for graphs it has been shown in [18].

Theorem 7. *Let G be a thin, $R|\breve{R}$- and $\breve{R}|R$-connected digraph, or a thin, connected non-bipartite graph. Then any power of G with respect to the direct product that is different from $K_3 \times K_3$ is 2-distinguishable.*

If G is S-thin and connected, then all powers of G with respect to the strong product are 2-distinguishable.

Proof. By Lemma 4 we only have to consider the exceptional cases, that is, the second power of graphs on two and three vertices, and the third power of graphs on two vertices. The first cases are already covered by Theorems 5, 6 and yield the exception $K_3 \times K_3$.

For the remaining case we recall that there are only two thin, $R|\breve{R}$- and $\breve{R}|R$-connected digraphs, namely a single edge with a loop at one endpoint and a single arc with loops at both endpoints. We named them L and K. Let G be a product of three factors $G_1, G_2, G_3 \in \{K, L\}$. Let the vertex set of both K and L be $\{0, 1\}$, where vertex 1 carries the loop in K, and where it is the origin of the arc in L. To obtain an asymmetric 2-coloring of $G_1 \times G_2 \times G_3$ it suffices to color the vertices $(1, 0, 0), (0, 1, 0), (0, 1, 1)$ black and to leave the others white. Hence there is no further exception for the direct product, and thus also not for the strong product as it can be considered as a subcase of the direct product. □

5. Graphs that are not thin

A graph G is not thin if it at least one equivalence class of $\approx$ is nontrivial. If $\tilde{u}$ is such a class, then any two elements $x, y \in \tilde{u}$ have the same neighbors, and the permutation of $V(G)$ that interchanges x, y and fixes all other vertices is an automorphism. In order to break it by a vertex coloring, x and y must be assigned different colors. Hence, the asymmetric coloring number $D(G)$ is at least $\max_{x\in V(G)} |\tilde{x}|$. We denote this number by $b(G)$.

Such a coloring may not break all automorphisms of G, because $\mathrm{Aut}(G)$ may permute equivalence classes of $\approx$. Hence we need extra colors to distinguish the orbits of the action of $\mathrm{Aut}(G)$ on the equivalence classes of $\approx$. This number is clearly bounded by $D(G/\approx)$, that is, by the asymmetric coloring number of the quotient of G by $\approx$. This is the graph whose vertices are the equivalence classes of G by $\approx$, where $\tilde{x}, \tilde{y} \in E(G/\approx)$ if $xy \in E(G)$. Hence

$$b(G) \leq D(G) \leq b(G) + k,$$

where k is the smallest nonnegative integer for which

$$D(G/\approx) \leq \binom{b(G)+k}{b(G)}.$$

If G is a strong or direct product of one of our classes of graphs, then $D(G/\approx)$ also is a direct or strong product. As $D(G/\approx)$ is thin, we can use the above estimates of the asymmetric coloring number of $D(G/\approx)$ for an estimate of $D(G)$.

Let us mention in passing that under our connectivity assumptions unique prime factorization of $D(G/\approx)$ implies unique prime factorization of G.

6. Asymmetric colorings of Cartesian products of digraphs

For the definition of the Cartesian product of digraphs, with or without loops, we can verbatim use the definition of the Cartesian product for undirected graphs given in Section 2. It has K_1 as a unit, and is commutative and associative. Prime factorization of connected graphs is unique, if they have at least one vertex without a loop, see [19].

Here we extend three theorems about the asymmetric coloring number of Cartesian products to Cartesian products of digraphs. The first one is a theorem about the Cartesian product of infinite graphs [6, Theorem 9]: It asserts that the Cartesian product of two countably infinite connected graphs is 2-distinguishable. The other two are Theorems 3 and 4.

The extension is based on two main properties of the Cartesian products of digraphs. The first is that the automorphism group of a directed graph G is a subgroup of the automorphism of its shadow G^s, that is, $\mathrm{Aut}(G) \leq \mathrm{Aut}(G^s)$. Hence $\mathrm{D}(G) \leq \mathrm{D}(G^s)$. The second that $(G \,\square\, H)^s = G^s \,\square\, H^s$.

Combining these remarks we infer that $\mathrm{D}(G \,\square\, H) \leq \mathrm{D}(G^s \,\square\, H^s)$, which immediately yields Theorem 8 as a generalization of [6, Theorem 9].

We wish to remark that the results we invoke use unique prime factorizations for the shadows G^s and H^s, but not of G and H. Hence, we do not have to require that G and H have at least one vertex without a loop.

Theorem 8. *Let G and H be countably infinite, connected digraphs with or without loops. Then* $\mathrm{D}(G \,\square\, H) \leq 2$.

For the extension of the other results it remains to investigate the cases when the shadow is $K_2^{\square,2}$, $K_2^{\square,3}$ or $K_3^{\square,2}$. It is easily seen that a directed graph whose shadow is K_2 or K_3 is 2-distinguishable unless it is K_2, K_3, $\mathcal{L}(K_2)$ or $\mathcal{L}(K_3)$. One also readily shows that products of these 2-distinguishable (and prime) graphs are also 2-distinguishable. This yields the following theorems.

Theorem 9. *Let G and H be two digraphs (possibly with loops) such that*

$$|G| \leq |H| \leq 2^{|G|} - |G| + 1.$$

Then $\mathrm{D}(G \Box H) \leq 2$ *unless* $G, H \in \{K_2, \mathcal{L}(K_2)\}$ *or* $G, H \in \{K_3, \mathcal{L}(K_3)\}$.

Theorem 10. *Let G be a connected digraph and $k \geq 2$. Then* $\mathrm{D}(G^{\Box,k}) = 2$ *except for the second and third powers of the graphs* K_2, $\mathcal{L}(K_2)$ *and the second power of the graphs* K_3, $\mathcal{L}(K_3)$*. In the exceptional cases the asymmetric coloring number is three.*

We conclude with the remark that Theorem 4 was extended to countably infinite graphs and infinite powers in [6], and that these generalizations hold verbatim for digraphs too, thereby extending Theorem 10.

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