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Distinguishing index of maps

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ABSTRACT

The distinguishing number of a group A acting on a finite set Ω, denoted by $D(A, \Omega)$, is the least k such that there is a k-coloring of Ω which is preserved only by elements of A fixing all points in Ω. For a map M, also called a cellular graph embedding or ribbon graph, the action of $\text{Aut}(M)$ on the vertex set V gives the distinguishing number $D(M)$. It is known that $D(M) \leq 2$ whenever $|V| > 10$. The action of $\text{Aut}(M)$ on the edge set E gives the distinguishing index $D'(M)$, which has not been studied before. It is shown that the only maps M with $D'(M) > 2$ are the following: the tetrahedron; the maps in the sphere with underlying graphs C_n, or $K_{1,n}$ for $n = 3, 4, 5$; a map in the projective plane with underlying graph C_4; two one-vertex maps with 4 or 5 edges; one two-vertex map with 4 edges; or any map obtained from these maps using duality or Petrie duality. There are 39 maps in all.

1. Introduction

The *distinguishing number* of a group A acting on a finite set Ω is the least number of colors needed to color the elements of Ω such that the only color-preserving elements of A are those that fix all elements of Ω. If $A = \text{Aut}(G)$ is the automorphism group of a graph G, then the *distinguishing number* of G, denoted by $D(G)$, is the distinguishing number of the action of A on the vertex set of G. Since its introduction by Albertson and Collins [1] more than 20 years ago, there has developed an extensive literature on the distinguishing number of a graph (and

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actually, there is a much earlier, but less known, paper by Babai [3]). Albertson and Collins were motivated by a recreational "necklace" problem equivalent to finding $D(C_n)$ for the cycle C_n. The distinguishing number, however, had an independent separate history in the theory of permutation groups unknown to graph theorists until recently [4]. In particular, the case $D(A, \Omega) = 2$ is equivalent to the action of A on Ω having a regular orbit in the action of A on the set of subsets of Ω. Gluck [8] showed that $D(A, \Omega) = 2$ when $|A| > 1$ is odd, and Cameron et al. [5,17]) showed all but finitely many primitive permutation groups have $D(A, \Omega) = 2$. For examples of the distinguishing number of various other actions of automorphism groups of combinatorial structures see [6]. For graphs G, when we look at the action of Aut(G) on edges instead of vertices, we get the *distinguishing index* $D'(G)$ introduced in [11].

A *map* M is an embedding of a connected graph G in a closed, possibly non-orientable, surface S such that each component of the complement of G in S, called a *face*, is homeomorphic to the interior of the unit disk in $\mathbb{R}^2$. The embedding is determined by a cyclic ordering ("orientation") of the edge-ends incident to each vertex, called a *rotation system*, together with an information on each edge whether it preserves (type 0 or "untwisted") or reverses (type 1 or "twisted") the rotations at its endpoints. This structure was called a *band composition* in [9] but more often now a *ribbon graph* [7].

For a map M, the distinguishing number $D(M)$, respectively the distinguishing index $D'(M)$, come from the action of Aut(M) on vertices, respectively edges. Tucker [18,19] has classified all finite maps with $D(M) > 2$; they all have at most 10 vertices, but their structure is quite complicated, with 31 different underlying graphs and at least 70 different maps.

The classification for $D'(M) > 2$ given in this paper is much easier, but depends on duality and Petrie duality. A formal description of duality and Petrie duality uses the monodromy of the map [10]. We give here a more intuitive, pictorial description. The dual M^* of map M is obtained by placing a vertex at each face center and joining them by edges "perpendicular" to the original edges. The underlying surface remains the same, but the underlying graph can change. For Petrie dual M^P, we imagine the map as a described by oriented small disk neighborhoods around each vertex joined by thin rectangular neighborhoods (or "ribbons") for the edges. If the ribbon has an orientation that agrees with the orientation of the disks at its endpoints, it is *untwisted*; if not, it is *twisted*. The Petrie dual replaces untwisted edges by twisted edges, and twisted edges by untwisted edges. The underlying graph is the same but the underlying surface, including its orientability, may be changed.

Given any map M the number of different maps one can obtain by repeatedly taking duals and Petrie duals is 1, 2, 3 or 6 [10,12,22]. The set of such maps we call *the triality class of M*. An important observation is that the action of Aut(M^*) on the edges of M^* and the action of Aut(M^P) on the edges of M^P (which are the same as the edges of M) are equivalent to the action of Aut(M) on its edges. Thus $D'(M) = D'(M^*) = D'(M^P)$, so the distinguishing index is constant on triality classes. For this reason, our analysis of $D'(M)$ is dominated by triality.

An edge of a graph is a *loop* if it has one endpoint, and *proper* otherwise. A proper edge is *parallel* if there is another edge with the same endpoints, and *simple* otherwise. A graph is *simple* if all its edges are simple. The graph with a single vertex and n edges is a *bouquet*, and denoted by B_n. The graph with two vertices, n parallel edges, and no loops is a *dipole*, and denoted by D_n. A major difficulty with the distinguishing index of a map is that, unlike the distinguishing number, one must allow loops and parallel edges since they arise naturally in triality classes. In fact, for three of the maps with $D'(M) > 2$, we use maps with underlying graph a bouquet or dipole, because none of the maps in the triality class is simple. For the two bouquets, we describe the map simply by giving the cyclic order of edge-ends (each edge is a loop) at the single vertex. For example for the bouquet of two loops labeled 1, 2, there are two possible cyclic orders: 1122 and 1212.

Throughout this paper, we let S_g denote the orientable surface of genus g, and we let N_c denote the non-orientable surface with c crosscaps (with Euler characteristic $\chi(N_c) = 2 - c$). We list our maps by first giving the surface and then the underlying graphs, so for example the tetrahedron is S_0K_4. In only a very few cases are there two maps with the same surface and underlying graph.

Table 1
Triality classes of maps from Theorem 1.1.

M	M^*	M^P	M^{*P}	M^{P*}	M^{*P*}
S_0K_4	M	N_1K_4	M^P	$N_1C_3^d$	M^{P*}
$S_0K_{1,n}$, $n = 3, 4, 5$	S_0B_n	M	N_nB_n	M^*	M^{*P}
S_0C_3	S_0D_3	N_1C_3	S_1D_3	$N_1B_3^A$	$S_1B_3^A$
S_0C_5	S_0D_5	N_1C_5	S_2D_5	$N_1B_5^A$	$S_2B_5^A$
S_0C_4	S_0D_4	M	S_1D_4	M^*	M^{*P}
N_1C_4	$N_1B_4^A$	M	$S_2B_4^A$	M^*	M^{*P}
S_1B_4	$S_1B_2^d$	N_4B_4	M^*	M^P	M
N_1D_4	$N_1B_2^d$	N_2D_4	M^*	M^P	M
S_2B_5	S_2D_5	N_5B_5	M^*	M^P	M

Theorem 1.1 (*The Classification of Maps M with $D'(M) > 2$*)**.** *Suppose that $D'(M) > 2$. Then M is in the triality class of one of the following maps. All but the first have 5 or fewer edges, and all have $D'(M) = 3$.*
(1) The tetrahedron S_0K_4;
(2) The map $S_0K_{1,n}$ of the star $K_{1,n}$ in the sphere for $n = 3, 4, 5$;
(3) The map S_0C_n of a cycle C_n in the sphere for $n = 3, 4, 5$ and the map N_1C_4 for the cycle C_4 in the projective plane;
(4) The one-vertex maps S_1B_4 and S_2B_5 with cyclic order of edge-ends, respectively 14213243 *and* 1521324354*;*
(5) A two-vertex map N_1D_4 in the projective plane.

There are 39 *maps in all (see Table 1). All have a bouquet or dipole in their triality class except for the tetrahedron S_0K_4, and even it has parallel edges in its triality class. All except S_0C_3, S_0C_5 have three members in their triality class. The only simple graphs underlying a map with $D'(M) > 2$ are K_4, C_n, $K_{1,n}$ for $n = 3, 4, 5$.*

Table 1 lists all maps with $D'(M) > 2$. We identify the map M as in the Theorem, while the columns M^*, M^P, M^{*P}, M^{P*}, M^{*P*} give the other members of the triality class of M. If there are only three members of a triality class, then the other three entries indicate which earlier member of the class the same map. For a bouquet, the superscript A denotes the antipodally symmetric cyclic order of edge-ends: 123123, 12341234, or 1234512345.

We note that Petrie duality does not change the underlying graph, and duality does not change the underlying surface. Finally, the Petrie dual of an orientable map is non-orientable if and only if the underlying graph is not bipartite. In particular, for a bouquet in an orientable surface, the Petrie dual is the same bouquet in a non-orientable surface, and for a dipole in an orientable surface, the Petrie dual is the same dipole in an orientable surface. The only underlying graphs that are not simple, a bouquet or a dipole are the doubled 3-cycle C_3^d, and the graph B_2^d obtained by inserting a valence 2 vertex in each loop of B_2.

Planar graphs were considered by Pilśniak in [14] in a context of a distinguishing index. In particular, it was shown that $D'(G) \leq 2$ for 4-connected planar graphs and $D'(G) \leq 3$ for 3-connected planar graphs. Theorem 1.1 implies the following corollary, confirming a conjecture that K_4 is the only 3-connected planar graph with the distinguishing index 3.

Corollary 1.2. *If G is 3-connected planar graph, then $D'(G) \leq 2$ except for $D'(K_4) = 3$.*

The organization of this paper is as follows. In Section 2, we address triality in more detail, and describe the triality classes for each of the maps given in the classification theorem. In Section 3 we discuss properties of vertex and edge stabilizers. In Section 4, we use these properties to show that if M has a loop and $D'(M) > 2$, then M is a bouquet. We then classify bouquets with $D'(M) > 2$. In Section 5, we show that if $D'(M) > 2$ and M has parallel edges, then with two exceptions, M is a dipole. We then classify dipoles with $D'(M) > 2$. In Section 6, we show the only maps with simple underlying graph and no vertex of valence 1 or 2 are the tetrahedron and its Petrie dual. In Section 7, we discuss some alternative approaches to proving the classification.

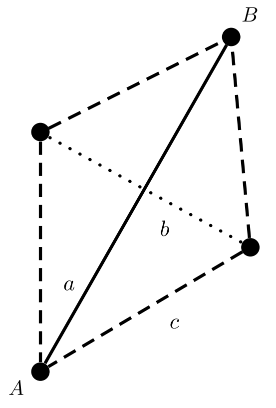

Fig. 1. Flags for the edge AB.

2. Triality and examples

If one views a map as a vertex–edge–face incidence structure represented by the *flags* of a barycentric subdivision, one can give a purely algebraic description of a map as a permutation group generated by three involutions x, y, z with x, y commuting, where the involutions provide the gluing instructions, or *monodromy*, for the flags [10,20,21]. One can view each flag as a right triangle with one leg a between a vertex and an edge midpoint, another leg b between an edge midpoint and a face center on one side of the edge, and a hypotenuse c between a vertex and face center (see Fig. 1). Four flags lie on each edge, even if the edge is a loop or lies on only one face. Then x gives instructions for gluing together pairs of a legs, y gives instructions for gluing together pairs of b legs, and z gives instructions for gluing together pairs of c hypotenuses. An orbit of $\langle x, y\rangle$ gives the four flags incident to an edge: in particular $(xy)^2 = 1$. An orbit of $\langle y, z\rangle$ gives the flags around a face, and an orbit of $\langle x, z\rangle$ gives the flags around a vertex. We note that in general, the involutions may have singleton orbits, leading to maps with boundary and "semi-edges". In this paper, we do not allow this.

The marking of generators x, y, z is as important as the group they generate. For example, generators y, x, z (in that order) give the dual map M^*, and x, xy, z give the Petrie dual M^P. Since $\langle x, y\rangle = \mathbb{Z}_2 \times \mathbb{Z}_2$, there are six possible ordered pairs of non-identity elements we can choose for the marked generators x, y. These correspond to all the possible different maps one can obtain from M by repeated application of duality and Petrie duality. We call this collection the *triality class* of M; in particular it contains 1, 2, 3 or 6 maps.

The automorphism group $\mathrm{Aut}(M)$ for a map with underlying graph G is the centralizer of the monodromy group in the full symmetric group on the flags: for a permutation of the flags to be a map automorphism, it must respect the gluing and take adjacent flags to adjacent flags. The most important observation about $\mathrm{Aut}(M)$ is that only the identity can fix a flag. It follows that the stabilizer of an edge is a subgroup of the Klein four group $V_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$. The most important observation for $D'(M)$ is that no matter what ordered pair of generators from $\langle x, y\rangle$ we choose, the action of $\mathrm{Aut}(M)$ is the same on edges, which correspond to orbits of the subgroup $\langle x, y\rangle$. We summarize:

Proposition 2.1. *For any map M, we have $D'(M) = D'(M^*) = D'(M^P)$. In particular, the distinguishing index is constant on triality classes.*

We note that one usually assumes the action of A on Ω is faithful, that is, A is a permutation group on Ω. Then one requires that the only color-preserving element of A is the identity. In our definition, we specify only that any color-preserving element of A acts as the identity, but is not necessarily the identity in A. That is, we allow actions that are not faithful. The reason for this is

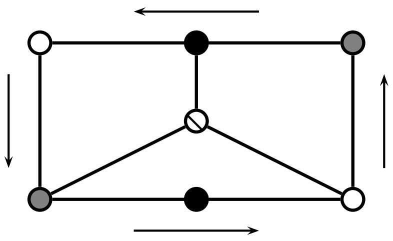

Fig. 2. The Petrie dual of the tetrahedron in the projective plane.

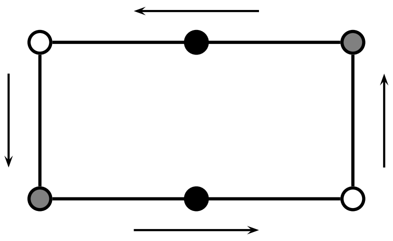

Fig. 3. The Petrie dual of the cycle C_3 in the projective plane.

that although Aut(M) acts semi-regularly (without fixed flags), it might not act faithfully on edges, vertices or faces. As an extreme example, we have

Example 2.2. Let M be the map consisting of a single loop in the projective plane. Then M^P has one vertex, one edge, and one face, and yet $\mathrm{Aut}(M) = \mathbb{Z}_2 \times \mathbb{Z}_2$.

2.1. Explanation of table

Now we explain some of the entries in Table 1.

Example 2.3 (*The Tetrahedron*)**.** The tetrahedron $M = S_0K_4$ is self-dual, so its triality class has at most 3 maps. The Petrie dual N_1K_4 is in the projective plane, as shown in Fig. 2. Its dual $N_1C_3^d$ has the doubled 3-cycle as the underlying graph. The automorphism group $\mathrm{Aut}(S_0K_4)$ is the full symmetric group Σ_4.

Example 2.4 (*Stars*)**.** Let $M = S_0K_{1,n}$ the map of the star $K_{1,n}$ in the sphere. Clearly $M^P = M$ so the triality class has three members. The dual $M^* = S_0B_n$ is the bouquet B_n in the sphere. Its Petrie dual N_nB_n has one vertex, and one face of size $2n$ in the non-orientable surface of Euler characteristic $\chi = 1 - n + 1 = 2 - n$.

Example 2.5 (*Cycles*)**.** Let $M = S_0C_n$ be the map of a cycle of length n as the equator in the sphere. Its dual S_0D_n is the dipole in the sphere with one vertex at the north pole and one at the south pole, joined by n longitudes. When n is odd, the Petrie dual $M^P = N_1C_n$ is a cycle of length n in the projective plane with one face (see Fig. 3); its dual $M^{P*} = N_1B_n^A$, again in the projective plane, has one vertex and n faces of size two.

Also when n is odd, $M^{*P} = S_{1+(n-3)/2}D_n$ is a map with one face and two vertices in the surface of genus $1 + (n-3)/2$. Its dual, $M^{*P*} = S_{1+(n-3)/2}B_n^A$ is a bouquet in the same surface viewed as a $2n$-gon with antipodal sides identified. The automorphism group $\mathrm{Aut}(S_0C_n)$ is $Di_n \times \mathbb{Z}_2$, where Di_n is the dihedral group of order $2n$, and the $\mathbb{Z}_2$ action is the reflection interchanging the northern and southern hemispheres, but fixing all vertices and edges. In particular, the action of $\mathrm{Aut}(S_0C_n)$

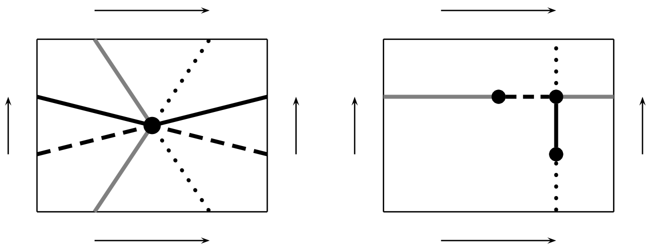

Fig. 4. Maps of the bouquet $B_4 = 14213243$ and its dual B_2^d in the torus.

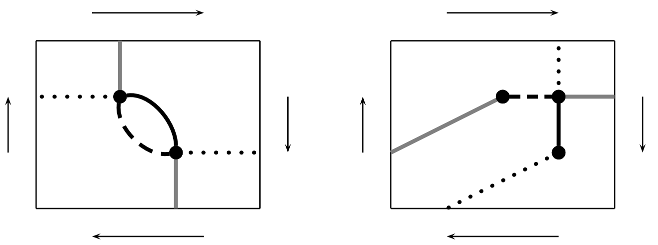

Fig. 5. Maps of the dipole D_4 and its dual B_2^d in the projective plane.

on either the vertex set or the edge set is not faithful. By the necklace problem, $D'(S_0C_n) = 2$ for $n > 5$, and $D'(S_0C_n) = 3$ for $n = 3, 4, 5$.

When n is even, $M = S_0C_n$ is again a cycle of length n in the sphere, but now $M^P = M$ and there are only three members of the triality class. Again, $M^* = S_0D_n$ is again a dipole in the sphere. The third member of the triality class is $M^{*P} = S_{1+(n-4)/2}D_n$, a self-dual map with two vertices and two faces in the orientable surface of genus $1+(n-4)/2$. Also when n is even, there is another map N_1C_n in the projective plane, which looks just like N_1C_n for odd n. Then $M^* = N_1B_n^A$ is a one-vertex map again in the projective plane, and its Petrie dual $M^{*P} = SB_n^A$ is a one-vertex map in an orientable surface with viewed again as a $2n$-gon with antipodal sides identified.

Example 2.6. Consider the map with the underlying graph the bouquet B_n with the cyclic rotation $1n213243\ldots n(n-1)$. Fig. 4 shows $S_1B_4, S_1B_4^*, S_1B_4^{*P}$. Note that the underlying graph for $S_1B_4^*$ has one vertex of valence 4 and two of valence 2, giving us B_2^d. The same graph underlies the dual of N_1D_4 (see Fig. 5).

Remark. For all of these maps, except for $S_0K_4^{P*}$, $S_2B_4^*$ and $N_1D_4^*$, if there are parallel edges, then the underlying graph is the dipole. If there are loops, the underlying graph is a bouquet. For S_0C_n, the automorphism group does not act faithfully on edges.

3. Edge stabilizers and vertex stabilizers

The distinguishing index of a map depends almost entirely on the way an edge stabilizer acts on incident vertices and faces, and the way a vertex stabilizer acts on incident edges. We first consider edge stabilizers. We recall at the outset that Aut(M) acts semi-regularly on flags, that is, the only automorphism fixing a flag is the identity. On the other hand, as we see from Example 2.2, it is possible for an edge-stabilizer to act trivially on edges (as well as vertices and faces).

Proposition 3.1. *Let e be an edge of the map M. Then the stabilizer* Stab(e) $\subset$ Aut(M) *acts faithfully on the four flags incident to e as a subgroup of the Klein group $V_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$. In particular, there is* *at most one element of* Stab(e) *that interchanges the edge-ends, but fixes the edge-sides; at most one that interchanges edge-sides, but fixes edge-ends; and at most one that interchanges both edge-ends and edge-sides. Each of these automorphisms can stabilize all other edges, so the action of* Stab(e) *on the edges of M need not be faithful.*

Next, we consider vertex stabilizers. Given a vertex v in a map M, choosing one of the two local orientations at v defines a cyclic order or *rotation* of the edge-ends incident to v; we use the term edge-ends because we allow loops in the graph underlying a map. The stabilizer of a vertex v of valence $n > 2$ induces a faithful action of a subgroup of the dihedral group Di_n on the edge-ends incident to v. On the other hand, since two ends can belong to the same edge, this action may not extend to a faithful action on all edges (as happens with the map with one vertex, one face, and two loops in the torus). Moreover, if v has valence 2, its stabilizer may not act faithfully on edge-ends: there can be a reflection fixing the two incident edges but interchanging the two incident faces.

Given the rotation $(1, 2, \ldots n)$ at vertex v, we define the *angle measure* of edge-ends i, j, denoted by $m(i, j)$, as the smaller of $|i-j|$ and $n - |i-j|$. If $m(i, j) = 1$, we call the pair of edge-ends i, j a *corner*. If n is even and $m(i, j) = n/2$, we call i, j *antipodal*. The following proposition summarizes what we need for the action of a vertex stabilizer on incident edges.

Proposition 3.2. *Let v be a vertex of valence $n > 2$, and suppose $\varphi \in$ Stab(v). Then*
(1) $m(\varphi(i), \varphi(j)) = m(i, j)$.
(2) If d, e are proper edges incident to v that are not antipodal, and φ fixes e and d, then φ is the identity.
(3) If $D'(M) > 2$ and φ stabilizes the proper edge e incident to v, then φ stabilizes every loop incident to v.
(4) If e is a loop, and its edge-ends are not antipodal, then $|\mathrm{Stab}(e)| \leq 2$. If $D'(M) > 2$, then Stab(e) $= 2$.

Proof. The first two statements follow from the faithful action of Stab(v) on edge-ends as a subgroup of Di_n. For (3), let d be any loop at v. If we color d, e black and all other edges white, the only non-identity color-preserving automorphism must stabilize d since it is the only black loop. Therefore it must fix v and e. The only possibility is that φ is the unique non-identity element of Stab(e) that fixes v.

For (4), for the dihedral action of Di_n on $(123 \ldots n)$, only the identity fixes the edge-ends i, j if they are not antipodal. Thus in the dihedral action $|\mathrm{Stab}(i, j)| \leq 2$. Since $n > 2$, the action of Stab(v) is faithful, so $|\mathrm{Stab}(e)| \leq 2$. When $D'(M) > 2$, coloring e black and all other edges white, we must have Stab(e) $= 2$. □

4. Loops and bouquets

Lemma 4.1. *Suppose $D'(M) > 2$, and M has a loop. Then the underlying graph is a bouquet.*

Proof. Suppose that M has more than one vertex, and that there is loop e at vertex v. Let G' be the graph obtained by removing all loops from the underlying graph G. Suppose that G' has a vertex of valence 3 or more. Let u be the closest such vertex to v, let R be the shortest path between v and u, and let d be the edge of R incident to u. Since u has valence at least 3 in G', there is a proper edge c that is not antipodal to d at u (see Fig. 6). Color the loop e, the path R and the edge c with black and all other edges with white. Any color-preserving automorphism φ must fix vertex v since the only black loop is at v. Therefore it also fixes u and hence the non-antipodal edge-ends d and c at u. By Proposition 3.2, φ is the identity. We conclude that $D'(M) \leq 2$.

Therefore all vertices of the graph G' have valence 1 or 2. Let d be a proper edge and e a loop incident to one of the endpoints of d. If we color d, e with black and all other edges with white, any color-preserving automorphism must fix d and its endpoints, and leave e invariant. Thus the automorphism fixing d and its endpoints must leave invariant all loops incident to either endpoint. Hence if we color with black one loop and the edges of G' and all other edges with white, the only color-preserving automorphism φ fixes all vertices of G', and leaves all loops invariant, making φ the identity on edges. It follows that if G has more than one vertex, then $D(G') \leq 2$. □

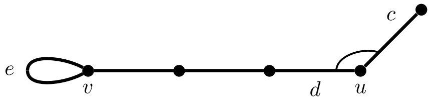

Fig. 6. Situation from Lemma 4.1.

We now find all bouquets M with $D'(M) > 2$. If e is a loop incident to v, its angle measure $m(e)$ is the angle measure of its edge-ends. We call a loop *antipodal* if its edge-ends are antipodal. We call a rotation *consecutive* if $m(e) = 1$ for every loop.

Lemma 4.2. *If M has one vertex v, and $D'(M) > 2$, then the number of loops n is at most 5 and the edge-ends of each loop have the same angle measure, which must be odd or antipodal. Thus the cyclic order is antipodal, consecutive, or 14213243 for $n = 4$, or 1521324354 for $n = 5$. The antipodal ones are in the triality class of S_0C_n for $n = 3, 5$, and N_1C_4. The consecutive ones are in the triality class of $S_0K_{1,n}$ for $n = 3, 4, 5$. The last two cyclic orders give S_1B_4 and S_2B_5.*

Proof. Suppose that the loop e is not antipodal. By item (4) of Proposition 3.2, there is exactly one non-identity automorphism φ stabilizing e. If φ also stabilizes all other loops, then the coloring: e with black and all other loops with white, is distinguishing. Therefore there is another loop d such that $\varphi(d) \neq d$. Coloring d, e with black and all others edges with white, we conclude there is an automorphism interchanging e and d, since there is no automorphism fixing both e and d. Thus $m(d) = m(e)$ by item (1) of Proposition 3.2. Let c be any other loop, and suppose that $m(c) \neq m(e)$. Then c must be stabilized by φ, and also by the non-identity element ψ of Stab(d). Since φ does not stabilize d, we have $\varphi \neq \psi$. So, $|\text{Stab}(c)| > 2$, contradicting item (4) of Proposition 3.2.

We conclude that all loops are antipodal, or have the same angle measure $m(e) = a < n$. Suppose that a is even, and loop e has edge-ends i, j. Then there is a loop d with edge-end k exactly half-way between i, j. The non-identity element φ of Stab(e) interchanges i, j, and fixes k, so it also stabilizes d, and therefore fixes the other end of d, which can only happen if d is antipodal, a contradiction.

We must show $n < 6$. Imagine a unit circle with edge-ends at angles π/n around the circle. Each edge-end pair $i\pi/n, (i+a)\pi/n$ determines a unique midpoint $(i+a/2)\pi/n$ on the circle. There are n of these, not necessarily equally spaced, and Aut(M) acts on them as a subgroup of the dihedral Di_{2n}, leaving them invariant. Suppose that $i < j$ are the closest midpoints, either π/n or $2\pi/n$ apart, since there are n midpoints. If the angle is $2\pi/n$, then the midpoints are equally spaced around the circle so the dihedral action on the midpoints is just the standard action of Di_n. By the necklace problem, we have $n < 6$. If the angle is π/n, let k be any other midpoint other than $i - \pi/n$ or $j + \pi/n$. Then the coloring: i, j, k with black and all other possible midpoints with white distinguishes the action of Di_{2n} on the set of $2n$ possible midpoints, and hence the action of Aut(M) on the midpoints.

The angle measure $a = 1$ (consecutive) occurs with $S_0K^*_{n,1}$ for $n = 3, 4, 5$. The antipodal case for $n = 3, 5$ occurs as $S_0C_n^{P*}$, and for $n = 4$ is $N_1C_4^*$. For $n = 3$, the only possible common angle measures are 1 (consecutive) or 3 (antipodal). For $n = 4, 5$ there is also the possibility that the common measure is $m(e) = 3$. The resulting cyclic orders must be 14213243 for $n = 4$, and 1521324354 for $n = 5$. $\square$

5. Parallel edges and dipoles

Next, we handle parallel edges. Just as the presence of a loop in a map with $D'(M) > 2$ forces M to be a bouquet, the presence of parallel edges forces a dipole, but this time with two exceptions.

Lemma 5.1. *Suppose $D'(M) > 2$ and M has parallel edges. Then the graph underlying M is either a dipole or the subdivided bouquet B_2^d or the doubled 3-cycle C_3^d.*

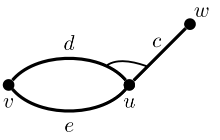

Fig. 7. Situation from Lemma 5.1.

Proof. Suppose that the underlying graph is not a dipole. Then there are vertices u, v with parallel edges d, e between u and v, and there is an edge c between u and a vertex $w \neq v$ such that cd is a corner (see Fig. 7). Color c, d, e with black and all other edges with white. Then there must be an automorphism fixing the edge c and interchanging d, e. Thus ce is also a corner. Now color just c, e with black and the other edges with white. Then there must be an automorphism φ fixing u and interchanging c, e, so $b = \varphi(d)$ is another edge between u and w. Applying the same argument to b, c, e, we have a corner be. The same argument for b, c, d gives a corner bd. There is no room for more edges at u so u has valence 4, with two edges to v and two edges to w. If v, w have valence 2, we have a single vertex of valence 4, two of valence 2 and all edges parallel, giving us B_2^d. There are only four maps with the underlying graph B_2^d. Two, namely $S_1B_2^d$ and $N_1B_2^d$, appear in our table. The map of B_2^d in the sphere has $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, and therefore has the distinguishing index 2. The same is true for the map with B_2^d in the Klein bottle, since any automorphism interchanging the two vertices of valence 2 would also interchange an orientation-preserving and orientation-reversing closed curve.

If instead v, w are adjacent to vertices other than u, then as with u they must have valence 4 with another parallel edges. Continuing this way, we conclude that the graph G underlying M is a doubled cycle. Suppose the cycle has length at least 4. Let u, v, w, x be consecutive vertices on the cycle. Color both edges between u, v with black, one edge between v, w with black and the other with white, and one edge between w, x with black and other with white. Color all other edges with white. We observe that u has two incident parallel edges black, v has three black edges, w has two non-parallel black edges, and x has one black. Thus any color-preserving automorphism φ must fix u, v, w, x, and hence fix all vertices of the cycle. Since φ fixes w and its neighbors, but cannot fix all edges incident to w, it must interchange the parallel pair vw or the parallel pair wx, but these pairs are both colored with black–white. Therefore φ fixes all edges incident to w and hence fixes all corners, making φ the identity.

It follows that if G is not a dipole or B_2^d, then it is a doubled 3-cycle, where the cyclic order at any vertex v alternates between the two pairs of parallel edges 1212. The stabilizer of v must contain an automorphism interchanging 1 and 2. Since this holds for every pair of consecutive edges at every vertex, the map is edge-transitive. Since there are 6 edges, Aut(M) has order 6, 12, 24. Since $D'(M) > 2$, the only possibility is Aut$(M) = 24$ so M is a regular (reflexible) map. If we trace out the face with corner 12 at v, the corner involving 1 at vertex w, involves edges going to the third vertex z. Similarly, for the corners involving 2 at z. Thus the faces have size either 3 or 6. If the faces all have size 3, the dual is K_4, giving us $S_0K_4^{P*}$. If the faces have size 6, the dual has two vertices, and this map is handled in the next Lemma. □

We want to classify dipole maps M with $D'(M) > 2$.

Lemma 5.2. *Suppose that the graph underlying M is a dipole with n edges, and $D'(M) > 2$. Then $n \leq 5$. For $n = 3, 5$, the only possibilities are in the triality classes of S_0C_n. For $n = 4$, in addition to the class of S_0C_4, there is map N_1D_4.*

Proof. Suppose that $n > 5$. Look at e_1, e_2, e_4 in the rotation $e_1e_2e_3e_4\dots$ at vertex u. Color them with black and all other edges with white. Only the identity fixes u, and preserves colors, so any non-identity color-preserving automorphism φ must interchange u, v, fix e_4 (since it is the only one

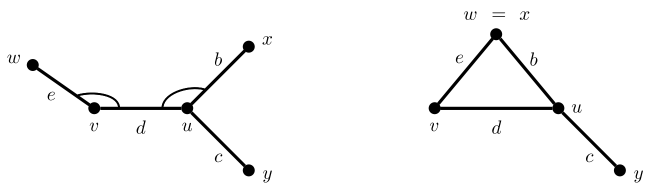

Fig. 8. Two situations from Lemma 6.1.

of the three not involved in a corner with one of the others), and either fix or interchange e_1, e_2. Since e_1, e_2 forms a corner at u, we find that e_1, e_2 also form a corner at v. Since e_1, e_2 are arbitrary, we conclude that the rotation at v is either the same, or the reverse of the rotation at u. But, if $n > 5$, then the reverse does not fix e_4, and leave e_1, e_2 invariant. Therefore φ preserves the rotation and, since it fixes e_1, e_2, e_4, it must fix all edges. Thus the only automorphism preserving the coloring of e_1, e_2, e_4 is the identity on the edge set, so $D'(M) = 2$.

We conclude that M has $n < 6$ edges. If $n = 5$, and there are two different face sizes, then the stabilizer of each vertex is at most $\mathbb{Z}_2$, since no cyclic symmetry around a vertex is possible. Thus $\text{Aut}(M)$ is at most $\mathbb{Z}_2 \times \mathbb{Z}_2$, and any such action on 5 edges has the distinguishing index 2. Therefore all faces have the same size. As the underlying graph is D_5, no face has odd size. We conclude that either all faces have size 2, or there is one face of size 10: both are in the class of S_0C_5. The same argument shows for C_3 that the only possible maps are again in the class of S_0C_3. For D_4, when all faces are the same size, we get maps in the class of S_0C_4. However, there is now the possibility of one face of size 4 and two of size 2 with $\mathbb{Z}_2 \times \mathbb{Z}_2$ stabilizing the vertices, allowing the possibility of full dihedral symmetry on the four edges of D_4. The resulting surface has Euler characteristic $2 - 4 + 3 = 1$, giving us N_1 and the map N_1D_4. □

6. Simple graphs

We now consider maps M with a simple underlying graph. If M has a vertex of valence 1 or 2, then M^* has a loop or parallel edges, and such maps with $D'(M) > 2$ have been classified.

Lemma 6.1. *Suppose M has a simple underlying graph with no vertices of valence 1 or 2. If $D'(M) > 2$, then $M = S_0K_4$ or $M = S_0K_4^P$.*

Proof. Let $d = uv$ be any edge, and let $xu = b$ and $yu = c$ form the two corners at u involving d, and let $e = vw$ form a corner at v with uv (see Fig. 8). Suppose all five vertices are distinct. Then since v is the only vertex of valence two in the subgraph H consisting of edges b, c, d, e, any automorphism φ leaving H invariant fixes v and also fixes w, as it is the only neighbor of u of valence one in H. Therefore φ fixes the corner cd, and must be the identity.

We conclude that the five vertices are not distinct. Since M has no parallel edges or loops, the only possibility is that $w = x$ or $w = y$. Suppose that $w = x$. Then the graph H consists of the triangle uwv with an extra edge uy (see Fig. 8). The only non-identity symmetry of H is one interchanging $x = w$ and v, and fixing edge c. This means xuy is also a corner, so u has valence 3. The same argument applies if $w = y$ with the automorphism fixing edge b. If we apply the same argument beginning with b or c playing the role of d, we find that there is another automorphism fixing a second edge at u. Since u was arbitrary, we conclude that all vertices have valence 3, and that the stabilizer of each vertex acts transitively on its three incident edges. Since we also found a triangle incident to v, we conclude that the underlying graph is K_4. The only edge-transitive maps with the underlying graph K_4 are the tetrahedron and its Petrie dual; note there is no such edge-transitive map in the torus or in the Klein bottle since with only two faces some edges appear on two faces and some do not. □

7. Alternative approaches

When we first considered this problem, we thought of using the medial map M, denoted by Med(M), since the action of Aut(M) on the edges of M is the same as the action of Aut(Med(M)) = Aut(M) on the vertices of Med M (cf. [2,15]). Thus $D'(M) = D(\text{Med}(M))$, so we can use the classification of maps M with $D(M) > 2$ given in [19]. This approach does work, but there are various difficulties with parallel edges and loops which can arise in Med(M), and [18,19] consider only maps with simple underlying graphs. Moreover, this approach would mean that the classification of maps with $D'(M) > 2$ would depend on [19], which is far, far more complicated than the self-contained proof we have given here.

Another approach is to focus more on vertices of valence 1 or 2. Our proof concentrates on loops and parallel edges. We did not have to worry about vertices of valence 1 or 2, because they lead to loops or parallel edges in the dual graph. But we could have started by trying to get a direct proof for maps with simple underlying graphs, which entail handling vertices of valence 1 (for $S_0K_{1,n}$) or valence 2 (for S_0C_n). Unfortunately the difficulty of bouquets cannot be avoided because the maps S_1B_4 and S_2B_5 have no simple graphs in their triality class.

Finally, we could use the Motion Lemma [16]. For a group A acting on Ω, the *motion*, denoted by $m(A)$, is the least number of points moved by any element that does not fix all points of Ω (it is also called the *minimal degree* of a permutation group). Then $m(A) > 2\log_2 |A|$ implies $D(A)$. This often leads to "all-but-finitely" many examples. For example, when Ω is a group G and $A = \text{Aut}(G)$, then it is easily shown [6] that $D(\text{Aut}(G), G) \leq 2$ when $|G| > 257$, since the fixed points of Aut(G) form a proper subgroup of G, forcing $m(\text{Aut}(G)) \geq |G|/2$. One then uses a computer to consider all $|G| \leq 256$. For maps, the fact that an automorphism cannot fix the edge-ends of a corner creates large motion. In [19], it is shown that $D(M) \leq 2$, if the number of vertices is at least 87. But this all-but-finitely result is not strong enough to allow machine computation, since a single graph, even with 10 vertices, has many possible embeddings.

For edges, the motion is at least $E/2$, where E is the number of edges: adding up the number of corners over all vertices gives the number $2E$ of edge-ends, and at least one edge-end must be moved at every corner, so at least E edge-ends are moved (and every edge has two edge-ends). Since edge stabilizers have order at most 4, we get $|\text{Aut}(M)| \leq 4E$. By the Motion Lemma, we have $D'(M) \leq 2$ when $E/2 > 2\log_2(4E)$. Thus $E \geq 28$ suffices. Then $|\text{Aut}(M)| \leq 4E = 107$; this looks feasible. For example, if one can establish that $D'(M) > 2$ implies M is edge transitive, one could use the census in [13].

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