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Bounds on the paired domination number of graphs with minimum degree at least three

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ABSTRACT

A set S of vertices in a graph G is a paired dominating set if every vertex of G is adjacent to a vertex in S and the subgraph induced by S contains a perfect matching (not necessarily as an induced subgraph). The minimum cardinality of a paired dominating set of G is the paired domination number $\gamma_{\rm pr}(G)$ of G. In this paper, we show that if G is a graph of order n and $\delta(G) \ge 3$, then $\gamma_{\rm pr}(G) \le \frac{19037}{30000} n < 0.634567\, n$.

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1. Introduction

A set S of vertices in a graph G is a *dominating set* if every vertex in $V(G) \setminus S$ is adjacent to a vertex of S. A *paired dominating set*, abbreviated PD-set, of G is a dominating set S of G such that the induced subgraph $G[S]$ contains a perfect matching M (not necessarily induced). Two vertices are *paired* in S if they form an edge of M. The *paired domination number*, $\gamma_{\rm pr}(G)$, of G is the minimum cardinality of a PD-set of G. A $\gamma_{\rm pr}$*-set* of G is a PD-set of G of minimum cardinality. Necessarily, the paired domination number of a graph is an even integer. Paired domination in graphs is well studied in the literature, and was first studied by Haynes and Slater [6,7] in 1995. A recent survey paper on paired domination in graphs can be found in [3,5]).

For graph theory notation and terminology, we follow [9]. The vertex set and edge set of G are denoted by $V(G)$ and $E(G)$, respectively. The order and size of G are given by $n(G) = |V(G)|$ and size $m(G) = |E(G)|$. If two vertices are adjacent, they are called *neighbors*. The set of neighbors of a vertex v in G is the set $N_G(v)$, called the *open neighborhood* of v. The degree, $d_G(v)$, of v is the number of neighbors of v in G. Moreover, if X is a set of vertices of G, then $d_X(v)$ is the number of neighbors of v in G that belong to the set X. In the special case when $X = V(G)$, we note that $d_X(v) = d_G(v)$. The minimum and maximum degree among the vertices of G is $\delta(G)$ and $\Delta(G)$, respectively. A graph is *k-regular* if every vertex has degree k. A 3-regular graph is commonly referred to a *cubic graph* in the literature. The set consisting of v and its neighbors is its *closed neighborhood* $N_G[v]$. For a set S of vertices, the *open* (resp., *closed*) *neighborhood* of S is union of the open (resp., closed) neighborhoods of vertices in S, denoted by $N_G(S)$ (resp., $N_G[S]$). For simplicity, we sometimes write $N(v)$ and $N[S]$ in place of $N_G(v)$ and $N_G[S]$, respectively.

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If S is a set of vertices in G, by $G - S$ we mean the graph obtained from G by removing the vertices (and their incident edges) from S. If $S = \{v\}$, then we simply write $G - v$ rather than $G - \{v\}$. The subgraph induced by the set S is given by $G[S]$. A path, cycle and complete graph on n vertices is given by P_n, C_n and K_n, respectively.

2. Known results

The paired domination number of a graph with minimum degree at least 2 is known to be at most two-thirds its order.

Theorem 1. *([7, 10]) If G is a connected graph of order $n \geq 6$ with $\delta(G) \geq 2$, then $\gamma_{\rm pr}(G) \leq \frac{2}{3}n$.*

The graphs achieving equality in Theorem 1 are characterized in [8]. Chen et al. [2] in 2007 established the best possible upper bound on the paired domination number of a cubic graph.

Theorem 2. *([2]) If G is a cubic graph of order n, then $\gamma_{\rm pr}(G) \leq \frac{3}{5}n$.*

Goddard and Henning [4] in 2009 showed that the only connected graph achieving equality in Theorem 2 is the Petersen graph, and conjectured that if we exclude this exceptional graph, then the bound can be improved to $\gamma_{\rm pr}(G) \leq \frac{4}{7}n$. This conjecture has yet to be resolved in general. However, Lu et al. [11] in 2019 proved the conjecture in the special case of claw-free graphs.

3. Main result

It remains an open problem to determine a best possible upper bound on the paired domination number of a connected graph with minimum degree at least 3 in terms of its order n. By Theorem 1, we have $\gamma_{\rm pr}(G) \leq \frac{2}{3}n$. However, this $\frac{2}{3}$-bound has yet to be improved. A best possible upper bound on the paired domination of a non-regular graph with minimum degree at least 3 is considerable more challenging to determine than in the case of 3-regular graphs. In this paper, we present such a bound that is an improvement of the $\frac{2}{3}$-upper bound, a proof of which is given in Section 4.

Theorem 3. *If G is a graph of order n with $\delta(G) \geq 3$, then $\gamma_{\rm pr}(G) \leq \frac{19{,}037}{30{,}000}n < 0.634567\,n$.*

4. Proof of Theorem 3

We define the *boundary* $\partial_G(D)$ of a set $D \subseteq V(G)$ in a graph G as all neighbors of vertices of D that belong outside the set D, that is, $\partial_G(D) = N_G[D] \setminus D$. We define the concept of a colored graph, which has a similar flavor to a residual graph defined by Bujtás [1].

Definition 1. Let G be a graph and let S be a set of vertices such that $G[S]$ contains a perfect matching. The *colored graph* G_S of G associated with the set S is the graph obtained from G as follow:

1. A vertex is colored **amber** if it has no neighbor in S.
2. A vertex is colored **beige** if it has a neighbor in S and a neighbor not dominated by S.
3. A vertex is colored **cyan** if it and all its neighbors are dominated by S.
4. All edges of G are removed from G_S, except for edges that join two amber vertices or an amber and a beige vertex.

Thus, each vertex in the colored graph G_S is colored amber, beige or cyan. In particular, a vertex in S is colored cyan. We let A, B, and C be the set of amber, beige, and cyan vertices, respectively, in G_S, and so (A, B, C) is a partition of $V(G)$. The *amber graph* is defined as the graph $G[A]$ induced by the set A of amber vertices. The number of amber and beige vertices adjacent to a vertex v in G_S is the *amber-degree* and *beige-degree*, respectively, of v, and is denoted by $d_A(v)$ and $d_B(v)$, respectively. The maximum amber-degree of a vertex in A (resp., B) is denoted by $\Delta_A(A)$ (resp., $\Delta_A(B)$). If v is an amber vertex, then its amber and beige neighbors are given by $N_A(v)$ and $N_B(v)$, respectively. We let $N_A[v] = N_A(v) \cup \{v\}$.

Throughout the proof we use the observation that an amber vertex has no cyan neighbor, and therefore its degree in G is the sum of its amber-degree and beige-degree in the colored graph G_S. Hence, the number of amber and beige neighbors of an amber vertex in G_S is precisely its degree in G, which is at least $\delta(G) \geq 3$. By construction of the colored graph, a beige vertex has at least one amber neighbor, but no beige or cyan neighbors in G_S. Moreover, if v is colored beige in G_S, then it has at least one neighbor in G that is colored cyan in G_S.

We are now in a position to prove Theorem 3. Recall its statement.

Theorem 3 *If G is a graph of order n with $\delta(G) \geq 3$, then $\gamma_{\rm pr}(G) \leq \frac{19{,}037}{30{,}000}n < 0.634567\,n$.*

Proof. Let G be a graph of order n with $\delta(G) \geq 3$. Removing edges from a graph in such a way that no isolated vertices are created, cannot decrease its paired-domination number. Hence, we may assume that the graph G is edge-minimal with respect to the condition that $\delta(G) \geq 3$, that is, if u and v are adjacent vertices, then at least one of u and v has degree 3 in G (or, equivalently, at most one of u and v has degree 4 or more in G). □

For a set S of vertices in G, we define a weak partition (where some of the sets may be empty) of the set of beige vertices in G_S by $B = (B_1, B_2, B_3, B_4)$, where B_i is the set of beige vertices having i amber neighbors in G_S for $i \in [3]$ and where B_4 is

Table 1
The weight $w(v)$ of a vertex v.

set containing v	A	B_4	B_3	B_2	B_1	C
w(v)	57,111	46,964	42,889	41,926	40,963	0

the remaining set of beige vertices. Thus, each vertex in B_4 has four or more amber neighbors. The *weight* $w(v)$ of a vertex v in the colored graph G_S is defined by the values given in Table 1.

The *weight* $w(G_S)$ of the colored graph G_S is the sum of the weights of the vertices, that is,

$$w(G_S) = \sum_{v \in V(G)} w(v) = 57,111|A| + 46,964|B_4| + 42,889|B_3| + 41,926|B_2| + 40,963|B_1|.$$

The set S is a PD-set in G if and only if all vertices are colored cyan in G_S, in which case $w(G_S) = 0$. Given a subset $S \subseteq V(G)$ such that $G[S]$ contains a perfect matching and given a subset $R \subseteq V(G) \setminus S$ where $G[R]$ contains a perfect matching, we define

$$\xi(R) = w(G_S) - w(G_{R \cup S}),$$

that is, $\xi(R)$ represented the total weight decrease when growing the set S to the set $R \cup S$. Such a set R is an *S-desirable set* if $G[R]$ contains a perfect matching and

$$\xi(R) \geq 90,000|R|.$$

Letting $S' = R \cup S$, we denote the resulting set of amber, beige and cyan vertices, respectively, in $G_{S'}$ by A', B' and C' respectively. Associated with the resulting set S', we define the weak partition $B' = (B'_1, B'_2, B'_3, B'_4)$ of the beige vertices in $G_{S'}$ in the natural way, where B'_i is the set of beige vertices having i amber neighbors in $G_{S'}$ for $i \in [3]$ and where B'_4 is the remaining set of beige vertices. Our key claim is that if the weight of the colored graph G_S is positive, then there exists an S-desirable set.

Claim 1. *If $w(G_S) > 0$, then there exists an S-desirable set in G_S.*

Proof. Let $w(G_S) > 0$, and suppose that there is no S-desirable set in G_S. We proceed with a series of claims describing some structural properties of G_S which culminate in the implication of its nonexistence. □

Claim 1.1. *The following hold in the amber graph $G[A]$.*

1. *There is no component of order 3 in $G[A]$.*
2. *Every component of order at least 4 in $G[A]$ has minimum degree at least 2.*

Proof. (a) Assume that C is a component of order 3 in $G[A]$. Thus, C contains a path P_3 given by $v_1v_2v_3$, where possibly v_1v_3 is an edge. Let u_2 be a beige neighbor of v_2, and let $R = \{u_2, v_2\}$. In the graph $G_{S'}$, the four vertices v_1, v_2, v_3, u_2 are all colored cyan, implying that $\xi(R) \geq 3 \times 57,111 + 1 \times 40,963 = 212,296 > 180,000 = 90,000|R|$, a contradiction. Hence, no component in $G[A]$ has order 3.

(b) Suppose that C is a component in the amber graph $G[A]$ such that $|V(C)| \geq 4$ and $\delta(C) = 1$. Let a_1 be a vertex of minimum (amber) degree in C, and so let a_2 be the only amber neighbor of a_1. Let a_3 be a neighbor of a_2 distinct from a_1 in C. Since C has order greater than 3, there is an (amber) vertex $p \notin \{a_1, a_2, a_3\}$ that is adjacent to at least one of a_2 or a_3. Letting $R = \{a_2, a_3\}$, the vertices a_1, a_2, a_3 are colored cyan in $G_{S'}$, while the vertex p is colored beige or cyan in $G_{S'}$. Each of a_1, a_2, a_3 therefore decreases the weight by 57,111, while the weight decrease of p is at least 10,147. Therefore, $\xi(R) \geq 57,111 \times 3 + 10,147 = 181,480 > 180,000 = 90,000|R|$, a contradiction. □

Claim 1.2. $\Delta_A(A) \leq 3$.

Proof. Suppose that $\Delta_A(A) \geq 4$. Let v be an amber vertex with $d_A(v) = \Delta_A(A) \geq 4$. By the edge-minimality of G, every neighbor of v has degree 3 in G. Let $X = \partial(N_A[v])$ be the boundary of the set $N_A[v]$ in the amber graph $G[A]$, and so X is the set of amber vertices that do not belong to $N_A[v]$ but have a neighbor in $N_A(v)$. Among all amber neighbors of v, let v' be chosen so that the number, $d_X(v')$, of neighbors of v' that belong to the set X is a maximum. Since the vertex v' has degree 3 in G, we have $d_X(v') \leq 2$. Let $R = \{v, v'\}$.

Suppose that $d_X(v') = 0$. Thus the vertex v and its amber neighbors are colored cyan in the colored graph $G_{S'}$, and therefore result in a weight decrease of $57,111 \times (1 + d_A(v)) \geq 57,111 \times 5 = 285,555$. Hence, $\xi(R) \geq 285,555 > 180,000 = 90,000|R|$. Thus, the set R is a S-desirable set, a contradiction. Therefore, $d_X(v') \geq 1$.

Suppose that $d_X(v') = 2$. Thus, v and v' are colored cyan in $G_{S'}$. Every amber neighbor u of v different from v' has degree 3 in G and therefore has degree at most 2 in $G_{S'}$, implying that $u \in B'_2 \cup B'_1 \cup C'$, resulting in a weight decrease of at least $57,111 - 41,926 = 15,185$. Further, the two amber neighbors of v' in X are colored beige or cyan in $G_{S'}$, and therefore their weight decreases by at least $57,111 - 46,964 = 10,147$. Hence, $\xi(R) \geq 57,111 \times 2 + 15,185 \times (d_A(v) - 1) + 10,147 \times d_{A'}(v') \geq 2 \times 57,111 + 3 \times 15,185 + 2 \times 10,147 = 180,071 > 180,000 = 90,000|R|$. Thus, the set R is a S-desirable set, a contradiction. Hence, $d_X(v')=1$. By our choice of the vertex v', every amber neighbor of v has at most one neighbor in X.

Suppose that some neighbor of v, say v'', has no amber neighbor in X, that is, $d_X(v'')=0$. Thus, the vertices v, v' and v'' are colored cyan. Moreover, the neighbors of v distinct from v' and v'' belong to the set $B'_1 \cup C'$. Hence, $\xi(R) \geq 57,111 \times 3 + 16,148 \times (d_A(v)-2) \geq 57,111 \times 3 + 16,148 \times 2 = 203,629 > 90,000|R|$, a contradiction. Thus, every amber neighbor of v has exactly one amber neighbor in X. Thus, for every amber neighbor u of v different from v' we have $u \in B'_1 \cup C'$, and therefore the vertex u decreases the weight by at least $57,111 - 40,963 = 16,148$. As before, the decrease of weight of the amber neighbor of v' outside the set $N[v]$ is at least 10,147. Hence if $d_A(v) \geq 5$, then $\xi(R) \geq 57,111 \times 2 + 16,148 \times 4 + 10,147 = 188,961 > 90,000|R|$, a contradiction. Therefore, $\Delta_A(A) = 4$. Let $N_A(v) = \{v_1, v_2, v_3, v_4\}$, where $v' = v_1$.

Suppose that two amber neighbors of v have a common amber neighbor in X. Without loss of generality, we may assume that v_1 and v_2 are two such neighbors of v. In this case, the vertices v, v_1 and v_2 are colored cyan in $G_{S'}$, implying that $\xi(R) \geq 57,111 \times 3 + 16,148 \times 2 + 10,147 = 213,776 > 90,000|R|$, a contradiction. Hence, no two amber neighbors of w have a common (amber) neighbor in X. Let x_i be the (amber) neighbor of v_i in X for $i \in [4]$. By our earlier observations, the vertices x_1, x_2, x_3, x_4 are distinct. Thus, $X = \{x_1, x_2, x_3, x_4\}$.

Let C_v be the component in the amber graph A that contains the vertex v. Thus, $N_A[v] \cup X \subseteq V(C_v)$, implying that the component C_v has order at least 9. Hence by Claim 1.1, we have $\delta(C_v) \geq 2$, and so every vertex in X has amber degree at least 2.

Suppose that $x_i x_j$ is an edge for some $i, j \in [4]$ where $i \neq j$. We may assume that $x_1 x_2$ is an edge. Let $R = \{v, v_3, x_1, x_2\}$ (with v and v_3 paired, and with x_1 and x_2 paired). All vertices in $\{v, v_1, v_2, v_3, x_1, x_2\}$ are colored cyan in $G_{S'}$, while the vertex $v_4 \in B'_1 \cup C'$ and the vertex x_3 is colored beige or cyan in $G_{S'}$. This implies that $\xi(R) \geq 6 \times 57,111 + 1 \times 16,148 + 1 \times 10,147 = 368,961 > 360,000 = 90,000|R|$. Thus, R is a S-desirable set, a contradiction. We deduce that X is an independent set in G.

Let Y be the set of all amber vertices that are not neighbors of v but have a neighbor in the set X. Each vertex in X has degree at least 2 and therefore has at least one (amber) neighbor in Y.

Suppose that two vertices in X, say x_1 and x_2, have a common amber neighbor, say y, in Y. Let $R = \{v, v_3, x_1, y\}$ (with v and v_3 paired, and with x_1 and y paired). All vertices in $\{v, v_1, v_2, v_3, x_1, y\}$ are colored cyan in $G_{S'}$. Further, the vertex $v_4 \in B'_1 \cup C'$, and the vertices x_2 and x_3 are recolored beige or cyan. This implies that $\xi(R) \geq 6 \times 57,111 + 1 \times 16,148 + 2 \times 10,147 = 368,961 > 360,000 = 90,000|R|$. Thus, R is a S-desirable set, a contradiction. We deduce that no two vertices in X have a common neighbor in Y.

Suppose that a vertex in X has only one (amber) neighbor in Y. We may assume that the vertex x_1 has a unique neighbor y_1 in Y. Thus, the vertex x_1 has amber-degree 2. Let z_1 be an amber neighbor of y_1 distinct from x_1. Thus, $z_1 \notin X$. Letting $R = \{v, v_2, y_1, z_1\}$ (with v and v_2 paired, and with y_1 and z_1 paired), the six vertices in $\{v, v_1, v_2, x_1, y_1, z_1\}$ are colored cyan in the colored graph $G_{S'}$, while the vertices $v_3, v_4 \in B'_1 \cup C'$ and the vertex x_2 is recolored beige or cyan, implying that $\xi(R) \geq 6 \times 57,111 + 2 \times 16,148 + 1 \times 10,147 = 385,109 > 90,000|R|$. Thus, R is a S-desirable set, a contradiction, implying that each vertex in X has at least two amber neighbors in Y.

Let y_i and y'_i be two distinct (amber) neighbors of x_i for $i \in [4]$. By our earlier observations, the vertices $y_1, y'_1, y_2, y'_2, y_3, y'_3, y_4, y'_4$ are distinct. Let $R = \{v_1, x_1, x_2, x_3, x_4, y_2, y_3, y_4\}$ (with v_1 and x_1 paired, and with x_i and y_i paired for $i \in \{2,3,4\}$). All 12 vertices in the set $N_A[v] \cup X \cup \{y_2, y_3, y_4\}$ are colored cyan in $G_{S'}$, and the vertices $y_1, y'_1, y'_2, y'_3, y'_4$ are recolored beige or cyan. This implies that $\xi(R) \geq 12 \times 57,111 + 5 \times 10,147 = 736,067 > 720,000 = 90,000|R|$, and so the set R is a S-desirable set, a contradiction. □

Claim 1.3. $\Delta_A(B) \leq 3$.

Proof. Suppose that $\Delta_A(B) \geq 4$. Let w be a beige vertex with $d_A(w) = \Delta_A(B)$. By the edge-minimality of G, every neighbor of w has degree 3 in G. Let $X = \partial(N_A[w])$ be the boundary of the set $N_A[w]$ in the amber graph $G[A]$, and so X is the set of amber vertices that do not belong to $N_A[v]$ but have a neighbor in $N_A(v)$. Among all amber neighbors of w, let w' be chosen so that $d_A(w')$ is a maximum. Since the vertex w' has degree 3 in G, we have $d_A(w') \leq 2$. Let $R = \{w, w'\}$.

If $d_X(w') = 0$, then the vertex w and all its amber neighbors in G are colored cyan in $G_{S'}$, implying that $\xi(R) \geq 46,964 + 4 \times 57,111 \geq 275,408 > 180,000 = 90,000|R|$, a contradiction. Hence, $d_X(w') \geq 1$.

Suppose that $d_X(w') = 2$. The vertices w and w' are colored cyan in $G_{S'}$. Every amber neighbor u of v distinct from v' has degree 3 in G and therefore has degree at most 2 in $G_{S'}$, and so $u \in B'_2 \cup B'_1 \cup C'$. Thus, the vertex u decreases the weight by at least 15,185. Further, every amber neighbor of w' in X is colored beige or cyan in $G_{S'}$ and, by Claim 1.2, has at most two amber neighbors in $G_{S'}$, implying that it belongs to the set $B'_2 \cup B'_1 \cup C'$ and therefore its weight decrease is also at least 15,185. Hence, $\xi(R) \geq 46,964 + 57,111 + 15,185 \times (d_A(w) - 1 + d_X(w')) \geq 46,964 + 57,111 + 15,185 \times 5 = 180,000 = 90,000|R|$. This contradicts our assumption, therefore, $d_X(w') = 1$.

Suppose that $\Delta_A(B) \geq 5$. The weight decrease of every amber neighbor of w different from w' is at least 15,185, as is the weight decrease of the amber neighbor of w' in X. Hence, $\xi(R) \geq 46,964 + 57,111 + 5 \times 15,185 = 90,000|R|$, a contradiction. Thus, $\Delta_A(B) \leq 4$. By supposition, $\Delta_A(B) \geq 4$. Consequently, $d_A(w) = \Delta_A(B) = 4$. Let w_1, w_2, w_3, w_4 be the amber neighbors of w.

By our earlier observations, $d_X(w') = 1$, implying that every amber neighbor of w has at most one neighbor in X. If an amber neighbor of w has no neighbor in X, then such a vertex is recolored cyan in $G_{S'}$, implying that $\xi(R) \geq 46,964 + 2 \times$

$57,111 + 3 \times 15,185 = 206,741 > 90,000|R|$, a contradiction. Hence, every amber neighbor of w has exactly one neighbor in X, that is, $d_X(w_i) = 1$ for all $i \in [4]$. We can therefore choose the vertex w' (which has the maximum number of neighbors in X) as an arbitrary vertex among w_1, w_2, w_3, w_4.

Suppose that two amber neighbors, say w_1 and w_2, of w have a common amber neighbor in X. We may assume that $w' = w_1$. The vertices w, w_1, w_2 are colored cyan in $G_{S'}$, implying once again that $\xi(R) \geq 46,964 + 2 \times 57,111 + 3 \times 15,185 > 90,000|R|$, a contradiction. Hence, no two amber neighbors of w have a common (amber) neighbor in X. Let x_i be the (amber) neighbor of w_i in X for $i \in [4]$. The vertices x_1, x_2, x_3, x_4 are distinct. Thus, $X = \{x_1, x_2, x_3, x_4\}$.

Suppose that x_ix_j is an edge for some $i, j \in [4]$ where $i \neq j$. We may assume that x_1x_2 is an edge. Let $R = \{w, w_3, x_1, x_2\}$ (with w and w_3 paired, and with x_1 and x_2 paired). All vertices in $\{w, w_1, w_2, w_3, x_1, x_2\}$ are colored cyan in $G_{S'}$, while the vertex $w_4 \in B'_1 \cup C'$ and the vertex $x_3 \in B'_2 \cup B'_1 \cup C'$. This implies that $\xi(R) \geq 46,964 + 5 \times 57,111 + 16,148 + 15,185 = 363,852 > 360,000 = 90,000|R|$. Thus, R is a S-desirable set, a contradiction. We deduce that $X = \{x_1, x_2, x_3, x_4\}$ is an independent set in G.

Let Y be the set of all amber vertices that are not neighbors of w but have a neighbor in the set X. Each vertex in X has amber degree at least 2, and therefore has at least one neighbor in Y. Suppose that two vertices in X, say x_1 and x_2, have a common amber neighbor, say y, in Y. Let $R = \{w, w_3, x_1, y\}$ (with w and w_3 paired, and with x_1 and y paired). The six vertices in $\{w, w_1, w_2, w_3, x_1, y\}$ are colored cyan in $G_{S'}$. Moreover, $w_4, x_2 \in B'_1 \cup C'$ and $x_3 \in B'_2 \cup B'_1 \cup C'$. This implies that $\xi(R) \geq 46,964 + 5 \times 57,111 + 2 \times 16,148 + 15,185 = 380,000 > 90,000|R|$. Thus, R is a S-desirable set, a contradiction. We deduce that for any two vertices $x_1, x_2 \in X$ there is no vertex $y \in Y$ that is a neighbor of both x_1 and x_2.

Let y_i be an (amber) neighbor of x_i that belongs to the set Y for $i \in [4]$. The vertices y_1, y_2, y_3, y_4 are distinct. Let $R = X \cup \{y_1, y_2, y_3, y_4\}$. The vertex w is colored cyan in $G_{S'}$, as are the vertices w_i, x_i, y_i for all $i \in [4]$, implying that $\xi(R) \geq 46,964 + 12 \times 57,111 = 732,296 > 720,000 = 90,000|R|$, and so the set R is a S-desirable set, a contradiction. $\square$

As a consequence of Claim 1.2, we have $\Delta_A(A) \leq 3$, and by Claim 1.3, we have $\Delta_A(B) \leq 3$. Thus, $B = B_1 \cup B_2 \cup B_3$, where we recall that if $w \in B_i$ for $i \in [3]$, then $d_A(w) = i$.

Claim 1.4. *There is no subgraph isomorphic to K_4 or $K_4 - e$ in $G[A]$.*

Proof. Assume that F is an (amber) subgraph in $G[A]$ isomorphic to K_4. Since $\Delta_A(A) \leq 3$, F is an (amber) component in $G[A]$. Let $V(F) = \{v_1, v_2, v_3, v_4\}$ and let $R = \{v_1, v_2\}$. All four (amber) vertices in F are colored cyan in $G_{S'}$, and so $\xi(R) \geq 4 \times 57,111 = 228,444 > 90,000|R|$, a contradiction. Hence, there is no subgraph isomorphic to K_4 in $G[A]$.

Assume that F is an (amber) subgraph in $G[A]$ isomorphic to $K_4 - e$. Let $V(F) = \{v_1, v_2, v_3, v_4\}$, where v_2 and v_3 have degree 3 in F. By Part (a), we note that v_1v_4 is not an edge. Letting $R = \{v_1, v_2\}$, the vertices v_1, v_2, v_3 are colored cyan in $G_{S'}$ and $v_4 \in B'_1 \cup C'$. Thus, $\xi(R) \geq 3 \times 57,111 + 16,148 = 187,481 > 90,000|R|$, a contradiction. $\square$

Claim 1.5. *The removal of an edge that joins a beige vertex v to an amber vertex results in a decrease in the weight of v by at least 963.*

Proof. Let e be an edge of G_S joining a beige vertex v to an amber vertex u. If the vertex u is added to the set S and the edge e is removed, then the vertex v has $i - 1$ amber neighbors, whence $i \in \{2, 3\}$ and $v \in B_{i-1}$ in G_S or $i = 1$ and v is recolored cyan in G_S. This implies (see, Table 1) that the removal of the edge e decreases the weight of v by 963 if $i \in \{2, 3\}$, and by 40963 if $i = 1$. $\square$

Since we frequently use Claim 1.5 in the remaining part of the proof, we often omit the reference to this claim when we apply it.

Claim 1.6. *There is no subgraph isomorphic to K_3 in $G[A]$.*

Proof. Assume that T is an (amber) triangle in $G[A]$, where $V(T) = \{v_1, v_2, v_3\}$. Let C be the (amber) component in $G[A]$ that contains the triangle T. By Claim 1.1(a), the subgraph T is not a component in $G[A]$, and so the component C has order at least 4. Thus by Claim 1.1(b), every vertex that belongs to the component C has degree at least 2. Let $X = \partial(V(T))$ be the boundary of the set $V(T)$ in the amber graph $G[A]$. Thus, X consists of all amber vertices not in T that have a neighbor in T. Since the component C has order at least 4, we note that $X \neq \emptyset$. We may assume that v_1 has an (amber) neighbor in X, say x_1.

Suppose that a vertex in T, say v_3, has no (amber) neighbor in X. In this case, we let $R = \{v_1, v_2\}$. All three vertices in T are colored cyan in $G_{S'}$, while the vertex $x_1 \in B'_2 \cup B'_1 \cup C$, implying that $\xi(R) \geq 3 \times 57,111 + 15,185 = 186,518 > 90,000|R|$, a contradiction. Hence, each vertex in the triangle T has an amber neighbor in X. Let x_i be the (amber) neighbor of v_i that belongs to X for $i \in [3]$. By Claim 1.4, the vertices x_1, x_2, x_3 are distinct. Hence, $X = \{x_1, x_2, x_3\}$.

Suppose that x_ix_j is an edge for some $i, j \in [3]$ where $i \neq j$. We may assume that x_1x_2 is an edge. We now consider the set $R = \{v_1, x_1\}$. The vertices v_1, v_2, x_1 are colored cyan in $G_{S'}$, while $v_3, x_2 \in B'_1 \cup C'$, implying that $\xi(R) \geq 3 \times 57,111 + 2 \times 16,148 > 90,000|R|$, a contradiction. The set X is therefore an independent set.

Let Y be the set of all amber vertices that do not belong to the triangle T but have a neighbor in the set X. Every vertex in X has at least two amber neighbors, and therefore has either one or two neighbors in Y.

Suppose that two vertices, x_1 and x_2, in X have a common amber neighbor, say y, in Y. If x_1 has amber degree 2, then letting $R = \{v_3, x_2, x_3, y\}$ (with x_2 and y_1 paired, and v_3 and x_3 paired), the seven vertices in $V(T) \cup X \cup \{y\}$ are colored cyan in the colored graph $G_{S'}$, implying that $\xi(R) \ge 7 \times 57,111 = 399,777 > 360,000 = 90,000|R|$, a contradiction. Hence, x_1 has amber degree 3. Let y_1 be the amber neighbor of x_1 different from v_1 and y. Letting $R = \{x_1, y, v_3, x_3\}$ (with x_1 and y paired, and v_3 and x_3 paired), the six vertices $v_1, v_2, v_3, x_1, x_3, y$ are all colored cyan in $G_{S'}$, while $x_2 \in B'_1 \cup C'$ and $y_1 \in B'_2 \cup B'_1 \cup C'$, implying that $\xi(R) \ge 6 \times 57,111 + 16,148 + 15,185 = 373,999 > 360,000 = 90,000|R|$, a contradiction. Hence, no two vertices in X have a common neighbor in Y. Let y_i be an (amber) neighbor of x_i for $i \in [3]$. By our earlier observations, the vertices y_1, y_2, y_3 are distinct.

Suppose that y_iy_j is an edge for some $i, j \in [3]$ where $i \neq j$. We may assume that y_1y_2 is an edge. Let $R = \{v_3, x_3, y_1, y_2\}$ (with v_3 and x_3 paired, and y_1 and y_2 paired). In the colored graph $G_{S'}$, the six vertices $v_1, v_2, v_3, x_3, y_1, y_2$ are colored cyan, while the vertices $x_1, x_2 \in B'_1 \cup C'$, implying that $\xi(R) \ge 6 \times 57,111 + 2 \times 16,148 = 374,962 > 90,000|R|$, a contradiction. Hence, the set $\{y_1, y_2, y_3\}$ is an independent set in G.

Suppose that a vertex in X has at least two (amber) neighbors in Y. We may assume that x_1 has two neighbors in Y. Let y'_1 be a neighbor of x_1 in Y different from y_1. Interchanging the roles of y_1 and y'_1, the vertex y'_1 is not adjacent to y_2 or y_3. Let z_2 be an (amber) neighbor of y_2 different from x_2. By our earlier observations, $z_2 \notin X \cup \{y_1, y'_1, y_3\}$. Letting $R = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ (with x_i and y_i paired for $i \in [3]$), all nine vertices in the set $V(T) \cup X \cup \{y_1, y_2, y_3\}$ are colored cyan in $G_{S'}$, while the vertices $y'_1, z_2 \in B'_2 \cup B'_1 \cup C'$, implying that $\xi(R) \ge 9 \times 57,111 + 2 \times 15,185 = 544,369 > 540,000 = 90,000|R|$. Thus, R is a S-desirable set, a contradiction. Every vertex in X therefore has exactly one (amber) neighbor in Y, implying by our earlier observations that $N_A(x_i) = \{v_i, y_i\}$ for $i \in [3]$.

Let Z be the set of amber vertices not in X but having a neighbor in the set Y. Every vertex in Y has at least two amber neighbors, and therefore has either one or two neighbors in Z. Letting $R = X \cup Y$ (with x_i and y_i paired for $i \in [3]$), the vertices in $V(T) \cup X \cup Y$ are all colored cyan in $G_{S'}$, while the vertices in Z belong to the set $B'_2 \cup B'_1 \cup C'$. If $|Z| \ge 2$, then $\xi(R) \ge 9 \times 57,111 + 2 \times 15,185 = 544,369 > 540,000 = 90,000|R|$, a contradiction. Hence, $|Z| = 1$. Let $Z = \{z\}$. By our earlier observations, every vertex in Y has a neighbor in Z, and so z is adjacent to all three vertices in Y. Thus, $z \in C'$ and $\xi(R) \ge 10 \times 57,111 = 571,110 > 90,000|R|$, a contradiction. □

Claim 1.7. *There is no 4-cycle in $G[A]$.*

Proof. Suppose that $C : v_1v_2v_3v_4v_1$ is an amber 4-cycle in $G[A]$. Let $X = \partial(V(C))$ be the boundary of the set $V(C)$ in the amber graph $G[A]$. Thus, X consists of all amber vertices not in C that have a neighbor in C. By Claim 1.6, there is no amber triangle, implying that the cycle C is an induced cycle in $G[A]$. By Claim 1.1, every vertex in the (amber) component in $G[A]$ containing the cycle C has degree at least 2.

Suppose that a vertex in C, say v_1, has no (amber) neighbor in X. In this case, we let $R = \{v_2, v_3\}$. All three vertices v_1, v_2, v_3 are colored cyan in $G_{S'}$, while the vertex $v_4 \in B'_1 \cup C$, implying that $\xi(R) \ge 3 \times 57,111 + 16,148 = 187,481 > 90,000|R|$, a contradiction. Hence, each vertex in the cycle C has an amber neighbor in X. Let x_i be the (amber) neighbor of v_i that belongs to X for $i \in [4]$.

Suppose that $x_i = x_j$ for some i and j where $1 \le i < j \le 4$. Since there is no (amber) triangle in $G[A]$, we have $j = i + 2$. For notational simplicity, we may assume that $x_1 = x_3$. Let $R = \{x_1, v_3\}$ (with x_1 and v_3 paired). All three vertices v_1, v_3, x_1 are colored cyan in $G_{S'}$, while the vertices $v_2, v_4 \in B'_1 \cup C'$, implying that $\xi(R) \ge 3 \times 57,111 + 2 \times 16,148 = 203,629 > 90,000|R|$, a contradiction. Hence, the vertices x_1, x_2, x_3, x_4 are distinct. Thus, $X = \{x_1, x_2, x_3, x_4\}$.

Suppose that x_ix_{i+1} is an edge for some $i \in [4]$ (where addition is taken modulo 4). By symmetry, we may assume that x_1x_2 is an edge. Letting $R = \{x_1, x_2, v_3, v_4\}$ (with x_1 and x_2 paired, and v_3 and v_4 paired), all six vertices in $V(C) \cup \{x_1, x_2\}$ are colored cyan in $G_{S'}$, while the vertices $x_3, x_4 \in B'_2 \cup B'_1 \cup C'$, implying that $\xi(R) \ge 6 \times 57,111 + 2 \times 15,185 = 373,036 > 360,000 = 90,000|R|$, a contradiction. Hence, x_ix_{i+1} is not an edge for all $i \in [4]$.

Suppose that x_ix_{i+2} is an edge for some $i \in [4]$ (where addition is taken modulo 4). By symmetry, we may assume that x_1x_3 is an edge. If x_2x_4 is an edge, then letting $R = X$ (with x_1 and x_3 paired, and x_2 and x_4 paired), all eight vertices in $V(C) \cup X$ are colored cyan in $G_{S'}$, implying that $\xi(R) \ge 8 \times 57,111 = 456,888 > 90,000|R|$, a contradiction. Hence, x_2x_4 is not an edge, implying by our earlier observations that neither x_2 nor x_4 has a neighbor in X.

Let y_2 and y_4 be amber neighbors of x_2 and x_4, respectively, that lie outside the cycle C. Further, let y_2 and y_4 be chosen, if possible, to be distinct. If $y_2 \neq y_4$, then letting $R = X \cup \{y_2, y_4\}$ (with x_1 and x_3 paired, x_2 and y_2 paired, and x_4 and y_4 paired), all 10 vertices in $V(C) \cup X \cup \{y_2, y_4\}$ are colored cyan in $G_{S'}$, implying that $\xi(R) \ge 10 \times 57,111 = 571,110 > 540,000 = 90,000|R|$, a contradiction. Hence, $y_2 = y_4$, implying by our choice of y_2 and y_4 that the vertices x_2 and x_4 both have (amber) degree 2 in $G[A]$. In this case, letting $R = \{v_1, v_4, x_4, y_4\}$ (with v_1 and v_4 paired, and x_4 and y_4 paired), all six vertices in $\{v_1, v_2, v_4, x_2, x_4, y_4\}$ are colored cyan in $G_{S'}$, while the vertex $v_3 \in B'_1 \cup C'$ and the vertex $x_1 \in B'_2 \cup B'_1 \cup C'$, implying that $\xi(R) \ge 6 \times 57,111 + 16,148 + 15,185 = 373,999 > 360,000 = 90,000|R|$, a contradiction. Hence, x_ix_{i+2} is not an edge for all $i \in [4]$, implying by our earlier observations that X is an independent set.

Let Y be the set of all amber vertices that do not belong to the 4-cycle C but have a neighbor in the set X. Every vertex in X has either one or two (amber) neighbors in Y. Suppose that x_i and x_{i+1} have a common (amber) neighbor, say y, in Y for some $i \in [4]$ (where addition is taken modulo 4). We may assume that x_1 and x_2 have a common amber neighbor, say y_1, in Y. Letting $R = \{x_1, y_1, v_3, v_4\}$ (with x_1 and y_1 paired, and v_3 and v_4 paired), the six vertices $v_1, v_2, v_3, v_4, x_1, y$ are all colored cyan, while the vertex $x_2 \in B'_1 \cup C'$ and the vertices $x_3, x_4 \in B'_2 \cup B'_1 \cup C'$, implying that $\xi(R) \ge 6 \times 57,111 + 16,148 + 2 \times 15,185 > 90,000|R|$, a contradiction. Hence, x_i and x_{i+1} have no common neighbor in Y for $i \in [4]$.

Let y_i be an (amber) neighbor of x_i for $i \in [4]$, where the vertices y_1, y_2, y_3, y_4 are chosen so that the set $\cup_{i=1}^{4}\{y_i\}$ is as large as possible. By our earlier observations, $y_i \neq y_{i+1}$ for $i \in [4]$ (where addition is taken modulo 4). Suppose that x_i

and x_{i+2} have a common (amber) neighbor in Y for some $i \in [4]$. Renaming vertices if necessary, we may assume that x_1 and x_3 have a common amber neighbor in Y. Thus, $y_1 = y_3$. If x_1 or x_3 has amber degree 3, then we could have chosen y_1 and y_3 to be distinct, a contradiction to our choice of the vertices y_1, y_2, y_3, y_4. Hence, both x_1 and x_3 have amber degree 2. Letting $R = \{v_4, x_2, x_3, x_4, y_1, y_2\}$ (with v_4 and x_4 paired, x_2 and y_2 paired, and x_3 and y_1 paired), the ten vertices in $V(C) \cup X \cup \{y_1, y_2\}$ are all colored cyan, implying that $\xi(R) \geq 10 \times 57,111 > 90,000|R|$, a contradiction. We deduce that no two vertices in X have a common neighbor in Y. Thus, the vertices y_1, y_2, y_3, y_4 are distinct.

Suppose that $y_i y_{i+1}$ is an edge for some $i \in [4]$ (where addition is taken modulo 4). By symmetry, we may assume that $y_1 y_2$ is an edge. Letting $R = \{v_3, v_4, y_1, y_2\}$ (with v_3 and v_4 paired, and y_1 and y_2 paired), the vertices $v_1, v_2, v_3, v_4, y_1, y_2$ are colored cyan in $G_{S'}$, while the vertices $x_1, x_2 \in B'_1 \cup C'$ (and the vertices $x_3, x_4 \in B'_2 \cup B'_1 \cup C'$), implying that $\xi(R) > 6 \times 57,111 + 2 \times 16,148 > 90,000|R|$, a contradiction. Hence, $y_i y_{i+1}$ is not an edge for all $i \in [4]$.

Suppose that $y_i y_{i+2}$ is an edge for some $i \in [4]$ (where addition is taken modulo 4). By symmetry, we may assume that $y_1 y_3$ is an edge. Letting $R = \{v_2, x_2, x_4, y_1, y_3, y_4\}$ (with v_2 and x_2 paired, x_4 and y_4 paired, and y_1 and y_3 paired), the nine vertices $v_1, v_2, v_3, v_4, x_2, x_4, y_1, y_3, y_4$ are colored cyan in the colored graph $G_{S'}$, while the vertices $x_1, x_3 \in B'_1 \cup C'$ and the vertex $y_2 \in B'_2 \cup B'_1 \cup C'$, implying that $\xi(R) > 9 \times 57,111 + 2 \times 16,148 + 15,185 > 90,000|R|$, a contradiction. Hence, $y_i y_{i+2}$ is not an edge for all $i \in [4]$. Hence, $Y' = \{y_1, y_2, y_3, y_4\}$ is an independent set in $G[A]$.

We show that Y is an independent set of amber vertices. Suppose, to the contrary, that the vertex y_1 is adjacent to some other vertex, y'_1 say, in Y. Let x_i be the neighbor of y'_1 that belongs to the set X for some $i \in [4]$. Since there is no amber triangle, we note that $i \neq 1$. We may assume that $y'_1 = y_i$. This contradicts our earlier observation that the set Y' is an independent set of amber vertices. Hence, Y is an independent set of amber vertices.

Let Z be the set of amber vertices outside the set X that have a neighbor in the set Y. Since every amber vertex has at least two amber neighbors, the vertex y_i has at least one neighbor in Z for $i \in [4]$. Since the maximum amber degree is 3, this implies that $|Z| \geq 2$. Let $R = X \cup Y$ (with x_i and y_i paired for $i \in [4]$).

If $|Z| \geq 3$, then the 12 vertices $V(C) \cup X \cup Y'$ are colored cyan in $G_{S'}$, while the vertices in Z belong to the set $B'_2 \cup B'_1 \cup C'$, implying that $\xi(R) \geq 12 \times 57,111 + 3 \times 15,185 = 730,887 > 720,000 = 90,000|R|$, a contradiction. Hence, $|Z| = 2$. Let $Z = \{z_1, z_2\}$. If a vertex $z \in Z$ belongs to the set C' in $G_{S'}$, then the 13 vertices $V(C) \cup X \cup Y' \cup \{z\}$ are colored cyan in $G_{S'}$, implying that $\xi(R) \geq 13 \times 57,111 = 742,443 > 720,000 = 90,000|R|$, a contradiction. Hence, $z_1, z_2 \in B'_2 \cup B'_1$. In particular, this implies that no vertex in Z has three (amber) neighbors in Y. Thus, both z_1 and z_2 have exactly two (amber) neighbors in Y' and $z_1, z_2 \in B'_1$. Moreover, neither z_1 nor z_2 has an (amber) neighbor in X. Since Y is an independent set, each vertex y_i has exactly two amber neighbors, namely v_i and one of z_1 and z_2, for $i \in [4]$.

Recall that $Y' = \{y_1, y_2, y_3, y_4\}$ and $Y' \subseteq Y$. If $Y' \subset Y$, then by our earlier observations this implies that one of z_1 and z_2 has three neighbors in Y, and therefore belongs to the set C', a contradiction. Hence, $Y' = Y$, implying that each vertex x_i has exactly two amber neighbors, namely v_i and y_i, for $i \in [4]$. Each vertex x_i is therefore adjacent to at least one beige vertex for $i \in [4]$. By our earlier observations, each vertex y_i is adjacent to at least one beige vertex for $i \in [4]$. This yields the existence of at least eight edges joining amber and beige vertices that get deleted, which contribute a weight decrease of at least 8×963, implying that $\xi(R) \geq 12 \times 57,111 + 2 \times 16,148 + 8 \times 963 = 725,332 > 720,000 = 90,000|R|$, a contradiction. □

Claim 1.8. *No component in the amber graph $G[A]$ is a cycle.*

Proof. Suppose that the amber graph $G[A]$ contains a component C that is a cycle. Let C be the cycle $v_1 v_2 \dots v_q$ for some $q \geq 3$ in $G[A]$. By Claim 1.6, there is no amber 3-cycle, and by Claim 1.7, there is no amber 4-cycle. Hence, $q \geq 5$. We note that every vertex in the component C has at least one beige neighbor. Let w_i be a beige neighbor of v_i for $i \in [q]$.

Suppose that $q = 5$. If $w_i = w_{i+2}$ for all $i \in [5]$ (with addition taken modulo 5), then this implies that $w_i = w_j$ for all $i, j \in [5]$. But then w_1 is adjacent to all vertices on the cycle C, and would therefore have amber degree at least 5, a contradiction. Hence, we may assume that w_1 and w_3 are distinct. In this case, we let $R = \{v_1, v_3, w_1, w_3\}$ with v_1 and w_1 paired, and v_3 and w_3 paired. The vertices in $V(C) \cup \{w_1, w_3\}$ are all colored cyan in $G_{S'}$, implying that $\xi(R) \geq 5 \times 57,111 + 2 \times 40,963 = 367,481 > 360,000 = 90,000|R|$, a contradiction. Hence, $q \neq 5$, implying that $q \geq 6$.

We now let $R = \{v_2, v_5, v_6, w_2\}$ (with v_2 and w_2 paired, and v_5 and v_6 paired). If $q = 6$, then all six vertices of C are colored cyan in the graph $G_{S'}$, implying that $\xi(R) \geq 6 \times 57,111 + 40,963 = 383,629 > 360,000 = 90,000|R|$, a contradiction. Hence, $q \geq 7$. In this case, the six vertices $w_2, v_2, v_3, v_4, v_5, v_6$ are all colored cyan in the graph $G_{S'}$, while the vertices v_1 and v_7 belong to the set $B'_1 \cup C'$. The vertex w_2 is adjacent to at most three amber vertices, implying that at least four edges joining (amber) vertices in $\{v_1, v_3, v_4, v_5, v_6, v_7\}$ to beige vertices distinct from w_1 are deleted when constructing $G_{S'}$, yielding an additional decrease in weight of at least 4×963. Therefore, $\xi(R) \geq 5 \times 57,111 + 40,963 + 2 \times 16,148 + 4 \times 963 = 362,666 > 90,000|R|$, a contradiction. Hence, there is no (amber) cycle component in $G_S[A]$. □

By Claim 1.6, there is no amber 3-cycle and by Claim 1.7, there is no amber 4-cycle. Hence, every amber cycle has length at least 5. By Claim 1.8, there is no amber component that is a cycle.

Claim 1.9. *The amber graph $G[A]$ contain no adjacent vertices of degree 2.*

Proof. Suppose that the amber graph $G[A]$ has a component C that contains two adjacent (amber) vertices of degree 2. By Claim 1.8, the component C is not a cycle, implying that C contains a vertex of (amber) degree 3. Hence, there must exist a vertex v of (amber) degree 3 in the component C with an (amber) neighbor v_1 of degree 2 that is adjacent to an (amber) neighbor x_1 of degree 2 in $G[A]$. Let v_2 and v_3 be the other two amber neighbors of v, and so $N_A(v) = \{v_1, v_2, v_3\}$. Since there is no amber 3-cycle, the set $N_A(v)$ is an independent set.

Let $X = \partial(N_A[v])$ be the boundary of the set $N_A[v]$ in the amber graph $G[A]$, and so X is the set of amber vertices outside $N_A[v]$ that have a neighbor in $N_A(v)$. By Claim 1.1, every vertex in C has amber degree at least 2. Since there is no amber cycle of length 4, no two vertices in $N_A(v)$ have a common neighbor in X. Let x_i be an (amber) neighbor of v_i that belongs to X for $i \in [3]$. By our earlier observation, the vertices x_1, x_2 and x_3 are distinct. By supposition, the vertex x_1 has (amber) degree 2 in $G[A]$, and is the unique (amber) neighbor of v_1 in X.

Suppose that x_1x_i is an edge for some $i \in \{2, 3\}$. We may assume that x_1x_2 is an edge. If v_2 has amber degree 2, then letting $R = \{x_1, x_2, x_3, v_3\}$ (with x_1 and x_2 paired, and v_3 and x_3 paired), the seven vertices in $N_A[v] \cup \{x_1, x_2, x_3\}$ are colored cyan, implying that $\xi(R) \ge 7 \times 57,111 = 399,777 > 360,000 = 90,000|R|$, a contradiction. Hence, v_2 has amber degree 3. Let x_2' be the neighbor of v_2 in X different from x_2. Letting $R = \{v, v_2, v_3, x_2\}$ (with v and v_3 paired, and x_2 and v_2 paired), the six vertices in $N_A[v] \cup \{x_1, x_2\}$ are colored cyan, while the vertices $x_2', x_3 \in B_2' \cup B_1' \cup C'$, implying that $\xi(R) \ge 6 \times 57,111 + 2 \times 15,185 = 373,036 > 360,000 = 90,000|R|$, a contradiction. Hence, x_1x_i is not an edge for $i \in \{2, 3\}$, implying that x_1 is not adjacent to any vertex of X. Let y_1 be the (amber) neighbor of x_1 different to v_1. Thus, $y_1 \notin X$.

If y_1 is adjacent to both x_2 and x_3, then letting $R = \{v, v_2, x_2, y_1\}$ (with v and v_2 paired, and x_2 and y_1 paired), the six vertices in $\{v, v_1, v_2, x_1, x_2, y_1\}$ are colored cyan, while the vertices $v_3, x_3 \in B_2' \cup B_1' \cup C'$, implying that $\xi(R) \ge 6 \times 57,111 + 2 \times 15,185 > 90,000|R|$, a contradiction. Hence, we may assume that y_1 is not adjacent to x_2. Let y_2 be a neighbor of y_1 different from x_1. Thus, $y_2 \ne x_2$ but possibly $x_3 = y_2$. Letting $R = \{v, v_2, y_1, y_2\}$ (with v and v_2 paired, and y_1 and y_2 paired), the six vertices in $\{v, v_1, v_2, x_1, y_1, y_2\}$ are colored cyan, while the vertices $v_3, x_2 \in B_2' \cup B_1' \cup C'$, implying that $\xi(R) \ge 6 \times 57,111 + 2 \times 15,185 > 90,000|R|$, a contradiction. □

Recall that every amber cycle has length at least 5. By Claim 1.9, a vertex of degree 2 in $G[A]$ has both its neighbors of degree 3 in $G[A]$.

Claim 1.10. *A vertex of degree 2 in $G[A]$ does not belong to an amber 5- or 6-cycle.*

Proof. Suppose that $G[A]$ contains a vertex v_1 of degree 2 that belongs to an amber 5- or 6-cycle C_v. Let v and x_1 be the neighbors of v_1 in the cycle C_v. By Claim 1.9, both v and x_1 have degree 3 in $G[A]$. Let v_2 and v_3 be the two amber neighbors of v different from v_1, where v_2 belongs to the cycle C_v. Since there is no amber 3-cycle, the set $N_A(v) = \{v_1, v_2, v_3\}$ is an independent set. Let $X = \partial(N_A[v])$ be the boundary of the set $N_A[v]$ in the amber graph $G[A]$. Let x_i be an (amber) neighbor of v_i that belongs to X for $i \in \{2, 3\}$. Since there is no amber 4-cycle, no two neighbors of v in $G[A]$ have a common neighbor in X. In particular, x_1, x_2, x_3 are distinct. □

Claim 1.10.1. *The cycle C_v is not a 5-cycle.*

Proof. Suppose that C_v is a 5-cycle. We may assume that x_1x_2 is an edge of the cycle C_v. Thus, C_v is the cycle $vv_1x_1x_2v_2v$. Let $R = \{x_1, x_2, x_3, v_3\}$ (with x_1 and x_2 paired, and v_3 and x_3 paired). The six vertices in $\{v, v_1, v_3, x_1, x_2, x_3\}$ are colored cyan in $G_{S'}$. If v_2 has amber degree 2, then the vertex v_2 is colored cyan in $G_{S'}$, implying that $\xi(R) \ge 7 \times 57,111 = 399,777 > 360,000 = 90,000|R|$, a contradiction. Hence, v_2 has amber degree 3. Let x_2' be the amber neighbor of v_2 in X different from x_2. We note that the vertex $v_2 \in B_1'$. Analogously, if x_1x_3 is an edge, then v_3 has amber degree 3.

Let y_1 be the (amber) neighbor of x_1 different from v_1 and x_2. Since there is no amber 4-cycle, we note that $y_3 \ne x_2'$. Suppose that $y_1 = x_3$, that is, x_1x_3 is an edge, implying by our earlier observations that v_3 has amber degree 3. Let x_3' be the amber neighbor of v_3 in X different from x_3. As observed earlier, no two neighbors of v in $G[A]$ have a common neighbor in X, and so the vertices $x_1, x_2, x_2', x_3, x_3'$ are distinct. Recall that $R = \{x_1, x_2, x_3, v_3\}$ and $v_2 \in B_1'$. The vertex $x_3' \in B_2' \cup B_1' \cup C'$, implying that $\xi(R) \ge 6 \times 57,111 + 16,148 + 15,185 = 373,999 > 360,000 = 90,000|R|$, a contradiction. Hence, $y_1 \ne x_3$. In this case, the vertex $y_1 \in B_2' \cup B_1' \cup C'$, implying once again that $\xi(R) \ge 6 \times 57,111 + 16,148 + 15,185 > 90,000|R|$, a contradiction. □

By Claim 1.10.1, the cycle C_v is a 6-cycle. More generally, no vertex of degree 2 in $G[A]$ belongs to an amber 5-cycle. Thus, x_1 is adjacent to no vertex in X. Let y_1 and y_2 be the two amber neighbors of x_1 different from v_1. Thus, $y_1, y_2 \notin X$. Recall that v_1, v_2 and v_3 are the three amber neighbors of v, and that x_i is an amber neighbor of v_i that belongs to X for $i \in [3]$. We may assume that the cycle C_v is the cycle $vv_1x_1y_2x_2v_2v$.

Suppose x_2 has a common neighbor with v_3. In this case, we can choose x_3 so that x_2x_3 is an edge. Letting $R = \{v, v_3, x_1, y_2\}$ (with x_1 and y_2 paired, and v and v_3 paired), the six vertices in $\{v, v_1, v_3, x_1, x_2, y_2\}$ are colored cyan, while the vertices $v_2, x_3 \in B_1' \cup C'$ and the vertex $y_1 \in B_2' \cup B_1' \cup C'$, implying that $\xi(R) \ge 6 \times 57,111 + 2 \times 16,148 + 15,185 = 390,147 > 360,000 = 90,000|R|$, a contradiction. Hence, x_2 has no neighbor in X.

If x_2 has only v_2 and y_2 as its amber neighbors, then letting $R = \{v, v_3, x_1, y_2\}$, the six vertices in $\{v, v_1, v_3, x_1, x_2, y_2\}$ are colored cyan, while the vertex $v_2 \in B_1' \cup C'$ and the vertices $x_3, y_1 \in B_2' \cup B_1' \cup C'$, implying that $\xi(R) \ge 6 \times 57,111 + 16,148 + 2 \times 15,185 = 389,184 > 360,000 = 90,000|R|$, a contradiction. Hence, v_2 has amber degree 3. Let y_3 be the third amber neighbor of x_2, and so $N_A(x_2) = \{v_2, y_2, y_3\}$. Since there is no amber 4-cycle, we note that $y_1 \ne y_3$.

If v_2 has amber degree 2, then letting $R = \{v, v_3, x_1, y_2\}$, the six vertices in $\{v, v_1, v_2, v_3, x_1, y_2\}$ are colored cyan, while the vertex $x_2 \in B_1' \cup C'$ and the vertices $x_3, y_1 \in B_2' \cup B_1' \cup C'$, implying that $\xi(R) \ge 6 \times 57,111 + 16,148 + 2 \times 15,185 > 90,000|R|$, a contradiction. Hence, v_2 has amber degree 3. Let x_2' be the amber neighbor of v_2 in X different from x_2, and let y_4 be an amber neighbor of x'_2 different from v_2. Further, we choose y_4, if possible, to be distinct from y_1. Since there is no amber 4-cycle, we note that $y_4 \neq y_2$ and $y_4 \neq y_3$.

Suppose that $y_4 = x_3$, that is, x'_2x_3 is an edge. Since no vertex of degree 2 in $G[A]$ belong to an amber 5-cycle, we note that v_3 has amber degree 3. Let x'_3 be the amber neighbor of v_3 in X different from x_3. Letting $R = \{v_3, x_1, x'_2, x_3, x'_3, y_2\}$ (with v_3 and x'_3 paired, x_1 and y_2 paired, and x'_2 and x_3 paired), the nine vertices in $N_A[v] \cup \{x_1, x'_2, x_3, x'_3, y_2\}$ are colored cyan in $G_{S'}$, while the vertices $x_2 \in B'_1 \cup C'$ and $y_1 \in B'_2 \cup B'_1 \cup C'$, implying that $\xi(R) \geq 9 \times 57,111 + 15,185 + 16,148 = 545,332 > 540,000 = 90,000|R|$, a contradiction. Hence, $y_4 \neq x_3$. This implies that X is an independent set.

Suppose that $y_1 \neq y_4$. Letting $R = \{v_3, x_1, x'_2, x_3, y_2, y_4\}$ (with v_3 and x_3 paired, x_1 and y_2 paired, and x'_2 and y_4 paired), the nine vertices in $N_A[v] \cup \{x_1, x'_2, x_3, y_2, y_4\}$ are colored cyan in $G_{S'}$, while the vertex $x_2 \in B'_1 \cup C'$ and the vertex $y_1 \in B'_2 \cup B'_1 \cup C'$, implying once again that $\xi(R) \geq 9 \times 57,111 + 15,185 + 16,148 > 90,000|R|$, a contradiction. Hence, $y_1 = y_4$, implying that the vertex x'_2 has amber degree 2, where v_2 and y_1 are the two (amber) neighbors of x'_2. However, interchanging the roles of x_2 and x'_2, and taking the 6-cycle C_v to be the cycle $vv_1x_1y_1x'_2v_2v$, the vertex x'_2 has amber degree 3, a contradiction. (□)

Claim 1.11. *A vertex of degree 3 in $G[A]$ has at most one neighbor of degree 2 in $G[A]$.*

Proof. Suppose that v is a vertex of degree 3 in the induced graph $G[A]$. Suppose, to the contrary, that at least two neighbors of v have degree 2 in $G[A]$. Let v_1, v_2, v_3 be the three amber neighbors of v, where v_1 and v_2 have degree 2 in $G[A]$. Since there is no amber 3-cycle, the set $N_A(v) = \{v_1, v_2, v_3\}$ is an independent set. Let $X = \partial(N_A[v])$ be the boundary of the set $N_A[v]$ in the amber graph $G[A]$. Let x_i be an (amber) neighbor of v_i that belongs to X for $i \in [3]$. Since there is no amber 4-cycle, no two neighbors of v in $G[A]$ have a common neighbor in X. Thus, x_1, x_2, x_3 are distinct. By supposition, the vertices x_1 and x_2 are the unique neighbors of v_1 and v_2, respectively, in X. By Claim 1.10, and since there is no amber 5-cycle with an amber vertex of amber degree 2, the set X is an independent set.

By Claim 1.9, the vertices x_1 and x_2 have degree 3 in $G[A]$. Let y_i and y'_i be the two amber neighbors of x_i different from v_i for $i \in [2]$. By Claim 1.10, the vertices y_1, y'_1, y_2, y'_2 are distinct. Letting $R = \{v_3, x_1, x_2, x_3, y_1, y_2\}$ (with v and v_3 paired, and x_i and y_i paired for $i \in [2]$), the nine vertices in $N_A[v] \cup \{x_1, x_2, x_3, y_1, y_2\}$ are colored cyan, while the vertices $y'_1, y'_2 \in B'_2 \cup B'_1 \cup C'$, implying that $\xi(R) \geq 9 \times 57,111 + 2 \times 15,185 = 544,369 > 90,000|R|$, a contradiction. □

Claim 1.12. *If a component in the amber graph $G[A]$ is different from a path P_1 or P_2, then it has minimum degree 2 and maximum degree 3.*

Proof. Suppose that C is a component in the amber graph $G[A]$ distinct from a path P_1 or P_2. By Claim 1.1, the component C has order at least 4 and minimum degree at least 2. By Claim 1.9, no two adjacent vertices in C both have (amber) degree 2. At least one vertex of C therefore has degree 3. Suppose that C is 3-regular. In this case, we choose the set R to be a minimum PD-set in $G[A]$. By Theorem 2, we have $|R| \leq \frac{3}{5}|A|$. In the colored graph $G_{S'}$, all vertices are colored cyan. In particular, all vertices of A are colored cyan in $G_{S'}$, implying that $\xi(R) \geq 57,111 \times |A| \geq 57,111 \times \frac{5}{3}|R| = 95185|R| > 90,000|R|$, a contradiction. Hence, at least one vertex in the component C has degree 2. □

Claim 1.13. *Every component in the amber graph $G[A]$ is a path P_1 or P_2.*

Proof. Let C be a component in the amber graph $G[A]$, and suppose that $C \neq P_1$ and $C \neq P_2$. By Claim 1.12, the amber component C has minimum degree 2 and maximum degree 3. Hence, there exists a vertex v of (amber) degree 3 in C that has a neighbor of (amber) degree 2 in C. Let v_1, v_2, v_3 be the three amber neighbors of v, where v_1 has degree 2 in C. By Claim 1.11, both v_2 and v_3 have degree 3 in C. Recall that there is no amber 3- or 4-cycle in G. In particular, every two amber vertices have at most one amber vertex in common.

Let $X = \{x_1, x_2, \ldots, x_5\}$ be the set of amber vertices different from v that have a neighbor in $N_A(v)$, where x_1 is the neighbor of v_1 in X, and x_{2i} and x_{2i+1} are the two neighbors of v_i in X for $i \in \{2, 3\}$. By Claim 1.10, the vertex v_1 does not belong to an amber 5-cycle, and so the vertex x_1 has no neighbor in X. By Claim 1.9, the vertex x_1 has degree 3 in C. Let y_1 and y_2 be the two neighbors of x_1 in C different from v_1. By Claim 1.11, the vertices y_1 and y_2 have degree 3 in C. Let z_{2i-1} and z_{2i} are the two neighbors of y_i different from x_i for $i \in [2]$. Since there is no amber 3- or 4-cycle, the vertices $y_1, y_2, z_1, z_2, z_3, z_4$ are distinct. Let $Z = \{z_1, z_2, z_3, z_4\}$. By Claim 1.10, the vertex v_1 does not belong to a 5- or 6-cycle in the component C, implying that $X \cap (\{y_1, y_2\} \cup Z) = \emptyset$.

Since there is no amber 3- or 4-cycle in G, there are at most two edges in $G[X]$. We may assume that the only possibly edges joining vertices of X are x_2x_5 and x_3x_4. If x_2x_5 is an edge, then since no vertex of amber degree 2 belongs to an amber 5-cycle, both x_2 and x_5 have amber degree 3. Similarly, if x_3x_4 is an edge, then both x_3 and x_4 have amber degree 3. Let w_i be an amber neighbor of x_i that does not belong to the set $N_A(v) \cap X$ for $i \in \{2, 3, 4, 5\}$. Recall that no vertex of amber degree 2 belongs to an amber 6-cycle. We can therefore choose the vertices w_2, w_3, w_4, w_5 so that $w_2 \neq w_5$ and $w_3 \neq w_4$. Let $R = \{w_2, w_5\} \cup \{w_3, w_4\}$. Possibly, $A \subseteq Z$.

Suppose that at least one vertex of A does not belong to the set $\{z_1, z_2\}$. We may assume that $w_3 \notin \{z_1, z_2\}$. Let $R = \{v_2, v_3, x_1, x_3, x_4, y_1\}$ (with x_1 and y_1 paired, v_2 and x_3 paired, and v_3 and x_4 paired). The eight vertices in $N_A[v] \cup \{x_1, x_3, x_4, y_1\}$ are colored cyan in $G_{S'}$, while the six vertices $x_2, x_5, y_2, w_3, z_1, z_2 \in B'_2 \cup B'_1 \cup C'$, implying that $\xi(R) \geq 8 \times 57,111 + 6 \times 15,185 = 547,998 > 540,000 = 90,000|R|$, a contradiction. Hence, $A \subseteq \{z_1, z_2\}$. However letting $R = \{v_2, v_3, x_1, x_3, x_4, y_2\}$ (where now x_1 and y_2 are paired), analogous arguments show that $A \subseteq \{z_3, z_4\}$. Since $\{z_1, z_2\} \cap \{z_3, z_4\} = \emptyset$, this gives a contradiction. □

By Claim 1.13, every amber component is either a path P_1 or a path P_2.

Claim 1.14. *No amber component is a path P_2.*

Proof. Suppose that there exists an amber component C that is a path P_2. Let $V(C) = \{v_1, v_2\}$. The number of amber and beige neighbors of an amber vertex in G_S is precisely its degree in G, which is at least 3. Hence, each vertex of C has at least two beige neighbors in G_S.

Suppose that there exists a beige vertex w that is adjacent to no amber vertex different from v_1 and v_2. Necessarily, w is adjacent to at least one of v_1 and v_2. We may assume that v_2w is an edge. Let w_2 be a beige neighbor of v_2 different from w, and let $R = \{v_2, w_2\}$. The vertices v_1, v_2, w_1, w_2 are all colored cyan in $G_{S'}$, implying that $\xi(R) \geq 2 \times 57,111 + 2 \times 40,963 = 196,148 > 90,000|R|$, a contradiction. Hence, every beige vertex that has a neighbor in an amber P_2-component has a neighbor in an amber component different from that component.

Suppose that v_1 and v_2 have a common (beige) neighbor w. The vertex w has a neighbor, say v, in an amber component, C' say, different from C. We note that $w \in B_3$. Let $R = \{v, w\}$. The vertices v_1, v_2, v, w are all colored cyan in $G_{S'}$, implying that $\xi(R) \geq 3 \times 57,111 + 42,889 = 214,222 > 90,000|R|$, a contradiction. Hence, there is no beige vertex adjacent to both (amber) vertices belonging to the same P_2-component in the amber graph. Let u_i and w_i be two beige neighbors of v_i for $i \in [2]$. The vertices u_1, u_2, w_1, w_2 are distinct.

Suppose that a beige neighbor of a vertex in C is adjacent to a vertex, say v, from an amber P_1-component. We may assume that w_1 is such a vertex. Letting $R = \{v_1, w_1\}$, the vertices v, v_1, v_2, w_1 are colored cyan in $G_{S'}$, implying that $\xi(R) \geq 3 \times 57,111 + 41,926 = 213,259 > 90,000|R|$, a contradiction. Hence, every beige neighbor of an amber P_2-component has all its amber neighbors belonging to amber P_2-components.

Suppose that a beige neighbor of v_1 and a beige neighbor of v_2 have neighbors in different amber P_2-components. We may assume that w_i has a neighbor in an amber P_2-component C^i for $i \in [2]$, where the components C, C^1, C^2 are distinct. Let x_i be the neighbor of w_i that belongs to the P_2-component C^i for $i \in [2]$, and let $R = \{w_1, w_2, x_1, x_2\}$ (with w_i and x_i paired for $i \in [2]$). In the graph $G_{S'}$, all six (amber) vertices in the components C, C^1, C^2 are colored cyan, as are the (beige) vertices w_1 and w_2, implying that $\xi(R) \geq 6 \times 57,111 + 2 \times 41,926 = 426,518 > 360,000 = 90,000|R|$, a contradiction.

We deduce, therefore, that every amber neighbor of a beige vertex from the set $\{u_1, u_2, w_1, w_2\}$ outside the component C belongs to the same amber P_2-component, say C'. Let $V(C') = \{v'_1, v'_2\}$. Thus, $V(C) \cap V(C') = \emptyset$, and every beige neighbor of a vertex in C or C' belongs to the set $\{u_1, u_2, w_1, w_2\}$. Further, every amber neighbor of a vertex in the set $\{u_1, u_2, w_1, w_2\}$ belongs to C or C'. We now let $R = \{v_1, v_2, v'_1, v'_2\}$. In the colored graph $G_{S'}$, all vertices in $V(C) \cup V(C') \cup \{u_1, u_2, w_1, w_2\}$ are colored cyan, implying that $\xi(R) \geq 4 \times 57,111 + 4 \times 41,926 = 396,148 > 360,000 = 90,000|R|$, a contradiction. □

By Claims 1.13 and 1.14, every amber component is a path P_1. Every amber vertex therefore has three or more beige neighbors.

Claim 1.15. *Every vertex in B has exactly two amber neighbors.*

Proof. Suppose a vertex $w \in B$ has three or more amber neighbors. Since $\Delta_A(B) \leq 3$, we have $d_A(w) = \Delta_A(B) = 3$. Let v_1, v_2, v_3 be the three amber neighbors of w in G_S. Letting $R = \{w, v_1\}$, the vertex w and its three amber neighbors are colored cyan in $G_{S'}$, implying that $\xi(R) \geq 3 \times 57,111 + 42,889 = 214,222 > 90,000|R|$, a contradiction. Hence, every vertex in B has at most two amber neighbors. Recall that every beige vertex has at least one amber neighbor.

Suppose a beige vertex w_1 has exactly one amber neighbor, say v_1. If all (beige) neighbors of v_1 have only v_1 as their only amber neighbor, then let $R = \{v_1, w_1\}$. In this case, v_1 and all its beige neighbors are colored cyan in $G_{S'}$, implying that $\xi(R) \geq 57,111 + 3 \times 40,963 = 180,000 = 90,000|R|$, a contradiction. Hence, there is a beige neighbor of v_1, say w, that has two amber neighbors. Let v_2 be the amber neighbor of w distinct from v_1. Letting $R = \{v_1, w\}$, the four vertices v_1, v_2, w, w_1 are colored cyan in $G_{S'}$, implying that $\xi(R) \geq 2 \times 57,111 + 40,963 + 41,926 = 197,111 > 90,000|R|$, a contradiction. □

By Claim 1.15, if a vertex belongs to B, then it has exactly two amber neighbors.

Claim 1.16. *Every two amber vertices have at most one common (beige) neighbor in G_S.*

Proof. Suppose that two amber vertices v_1 and v_2 have two common (beige) neighbors, say w_1 and w_2. Letting $R = \{v_1, w_1\}$, the vertices v_1, v_2, w_1, w_2 are colored cyan in $G_{S'}$, implying that $\xi(R) \geq 2 \times 57,111 + 2 \times 41,926 = 198,074 > 90,000|R|$, a contradiction. □

We now return to the proof of Claim 1 one final time. By our earlier observations, every beige vertex has exactly two amber neighbors, while every amber vertex has at least three beige neighbors. By Claim 1.16, there is no 4-cycle containing two amber vertices. Let $w_1 \in B$ and let v_1 and v_2 be the two amber neighbors of w_1. Let w_2 be a beige neighbor of v_2 different from w_1. Let v_3 be the amber neighbor of w_2 different from v_2. By Claim 1.16, we note that $v_1 \neq v_3$. Let w_3 and w'_3 be two beige neighbors of v_3 different from w_2. By Claim 1.16, at most one neighbor of v_3 is adjacent to v_1. Let w_3 be a neighbor of v_3 different from w_2 that is not adjacent to v_1. Let v_4 be the amber neighbor of w_3 different from v_3. By our earlier observations, the vertices $v_1, v_2, v_3, v_4, w_1, w_2, w_3$ are distinct and $P : v_1w_1v_2w_2v_3w_3v_4$ is an induced path on seven vertices in G_S starting and ending at amber vertices, and alternating between amber and beige vertices.

Let $R = \{v_2, v_4, w_1, w_3\}$ (with v_2 and w_1 paired, and v_4 and w_3 paired). All vertices on the path P are colored cyan in $G_{S'}$, as are the three beige vertices w_1, w_2, w_3. Moreover, since P is an induced path and every amber vertex has at least three beige neighbors, at least six edges join vertices on the path P to beige vertices that do not belong to the set $\{w_1, w_2, w_3\}$. These edges when removed decrease the weight by at least 6×963 when constructing $G_{S'}$ from G_S. Therefore, $\xi(R) \geq 4 \times 57,111 + 3 \times 41,926 + 6 \times 963 = 360,000 = 90,000|R|$, a contradiction. This completes the proof of Claim 1. (□)

We now return to the proof of Theorem 3. By Claim 1, if $\mathrm{w}(G_S) > 0$, then there is a S-desirable set in the graph G. Let $S_0 = \emptyset$ and let $G_0 = G_{S_0}$, and so G_0 is the graph G with all vertices colored amber. We note that $V(G_0) = A$ and $\mathrm{w}(G_0) = 57,111n$. By Claim 1, there exists a S_0-desirable set R_1, and so letting $S_1 = S_0 \cup R_1 = R_1$ and $G_1 = G_{S_1}$, we have $\mathrm{w}(G_0) - \mathrm{w}(G_1) \geq 90,000|R_1|$, that is,

$$\mathrm{w}(G_1) \leq \mathrm{w}(G_0) - 90,000|R_1|.$$

If $\mathrm{w}(G_1) > 0$, then there is a S_1-desirable set R_2 by Claim 1, and so letting $S_2 = R_1 \cup R_2$ and $G_2 = G_{S_2}$, we have $\mathrm{w}(G_1) - \mathrm{w}(G_2) \geq 90,000|R_2|$, that is,

$$\mathrm{w}(G_2) \leq \mathrm{w}(G_1) - 90,000|R_2|.$$

If $\mathrm{w}(G_2) > 0$, then we repeat the process, thereby obtaining a sequence of colored graphs $G_0, G_1, \ldots, G_k$ and a PD-set $S = R_1 \cup \cdots \cup R_k$ of G such that

$$\begin{aligned} 0 = \mathrm{w}(G_k) &\leq \mathrm{w}(G_{k-1}) - 90,000|R_k| \\ &\leq \mathrm{w}(G_0) - 90,000\sum_{i=1}^{k}|R_i| \\ &= 57111n - 90,000|S|. \end{aligned}$$

Consequently,

$$\gamma_{\mathrm{pr}}(G) \leq |S| \leq \frac{57,111}{90,000}n = \frac{19,037}{30,000}n < 0.634567\,n. \qquad \square$$

5. Concluding remarks

If G is a 3-regular graph of order n, then by the result (see Theorem 2) of Chen et al. [2] in 2007, we have $\gamma_{\mathrm{pr}}(G) \leq \frac{3}{5}n$. This bound is best possible, and is achieved by the Petersen graph. Given the considerable work to date on the problem, it is evident that determining a tight upper bound on the paired domination of a non-regular graph with minimum degree at least 3 is more challenging than in the regular case. In this paper, we prove that in this non-regular case we have $\gamma_{\mathrm{pr}}(G) \leq \frac{19037}{30000}n < 0.634567\,n$. However, it is unlikely that this bound is achievable. It would be interesting to close the gap between this current best known bound of $\gamma_{\mathrm{pr}}(G) \leq \frac{19037}{30000}n$ and the best possible general upper bound we can hope for, namely $\gamma_{\mathrm{pr}}(G) \leq 0.6\,n$ (which is achieved by the Petersen graph).

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