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Proper distinguishing arc-colourings of symmetric digraphs

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A symmetric digraph $\overleftrightarrow{G}$ arises from a simple graph G by substituting each edge uv by a pair of opposite arcs $\overrightarrow{uv}, \overrightarrow{vu}$. An arc-colouring c of $\overleftrightarrow{G}$ is distinguishing if the only automorphism of $\overleftrightarrow{G}$ preserving c is the identity. We study four types of proper arc-colourings of $\overleftrightarrow{G}$ corresponding to four definitions of adjacency of arcs. For each type, we investigate the distinguishing chromatic index of $\overleftrightarrow{G}$, i.e. the least number of colours in a distinguishing proper colouring of $\overleftrightarrow{G}$. We also determine tight bounds for chromatic indices of $\overleftrightarrow{G}$, i.e. for the least numbers of colours in each type of proper colourings. Colourings of arcs of a symmetric digraph $\overleftrightarrow{G}$ are equivalent to colourings of halfedges of the graph G, which have applications in computer science.

1. Introduction

We use standard graph theory notation. By $[k]$ we denote the set $\{1, \ldots, k\}$ of k smallest positive integers. Throughout, we consider only connected graphs.

A colouring c of a graph is called *distinguishing* (or *asymmetric*) if the only automorphism preserving c is the identity. The first papers on distinguishing vertex-colourings were published by Babai [2] in 1977 and Cameron [4] in 1986. However, a special interest in these issues began with a seminal paper [1] of Albertson and Collins in 1996. They introduced the definition of the *distinguishing number* of a graph as the least number of colours in a general, i.e. not necessarily proper, distinguishing vertex-colouring. Their paper spawned much more than a hundred papers on symmetry breaking in graphs by various types of colourings. In 2006, Collins and Trenk in [5] initiated investigations of proper distinguishing vertex-colourings. They defined the *chromatic distinguishing number* $\chi_D(G)$ of a graph G as the least number of colours in a proper distinguishing vertex-colouring of G.

In 2015, Kalinowski and Pilśniak in [11] introduced the *distinguishing index* $D'(G)$ of a graph G as the least number of colours in a general distinguishing edge-colouring of G, not necessarily proper. In the same paper, its counterpart for proper colourings called the *chromatic distinguishing index*, denoted by $\chi'_D(G)$ was also defined. Clearly, both invariants are defined for any connected graph except for K_2.

Total distinguishing colourings, both general and proper, were later studied by Kalinowski, Pilśniak and Woźniak in [14]. In the present paper, we initiate investigations of proper distinguishing arc-colourings of symmetric digraphs. By $\overleftrightarrow{G}$ we denote a symmetric digraph obtained from a simple graph G by replacing each edge uv by a pair of opposite arcs $\overrightarrow{uv}, \overrightarrow{vu}$.

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A definition of proper arc-colourings of a digraph depends on a definition of adjacent arcs. There are four digraphs $A_i, i = 1, \dots, 4,$ with two arcs having at least one vertex in common:

- *2-cycle* A_1 with arcs $\overrightarrow{uv}, \overrightarrow{vu}$,
- *2-path* A_2 with arcs $\overrightarrow{uv}, \overrightarrow{vw}$,
- *source* A_3 with arcs $\overrightarrow{uv}, \overrightarrow{uw}$,
- *sink* A_4 with arcs $\overrightarrow{uv}, \overrightarrow{wv}$. Thus, there are 15 possible definitions of a proper colouring of a digraph since there are 15 possible definitions of adjacency of arcs corresponding to non-empty forbidden monochromatic subsets of the set of the four digraphs $A_i, i = 1, \dots, 4$. We denote by $\chi'_i(\overleftrightarrow{G})$ the *chromatic index* of a symmetric digraph $\overleftrightarrow{G}$, i.e. the least number of colours in a proper arc-colouring of $\overleftrightarrow{G}$, when "proper" means "without monochromatic digraph A_i". We also use the notation $\chi'_{i,j}(\overleftrightarrow{G}), \chi'_{i,j,k}(\overleftrightarrow{G}), \chi'_{i,j,k,l}(\overleftrightarrow{G})$ if more monochromatic two-arc digraphs are forbidden.

It has to be noted that so far, only two types of proper arc-colourings of digraphs were studied in literature. An arc-colouring of a digraph is called proper of type I if there are neither monochromatic 2-cycles nor 2-paths. Poljak and Rödl [18] proved a notable result stating that

$$\chi'_{1,2}(\overleftrightarrow{G}) = \min\left\{k \;:\; \chi(G) \le \binom{k}{\lfloor k/2 \rfloor}\right\}$$

for every graph G. An arc-coloring is proper of type II if there are neither monochromatic sources nor sinks. It is well known that $\chi'_{3,4}(\overleftrightarrow{G}) = \Delta(G)$ (cf. [15,20]).

Determining chromatic indices corresponding other types of proper arc-colourings is sometimes easy, but not always. For instance, it can be easily seen, using the correspondence between colourings of the symmetric digraph $\overleftrightarrow{G}$ and the subdivision $\widehat{G}$ described in the next section, that the chromatic index $\chi'_{1,2,3}(\overleftrightarrow{G})$ equals the incidence chromatic number of the graph G, introduced by Brualdi and Massey [3] in 1993. No general sharp upper bound for the incidence chromatic number is known. For the current state of art, see the homepage of Éric Sopena [19]. Similarly, $\chi'_{D_i}(\overleftrightarrow{G}), \chi'_{D_{i,j}}(\overleftrightarrow{G}), \chi'_{D_{i,j,k}}(\overleftrightarrow{G}), \chi'_{D_{i,j,k,l}}(\overleftrightarrow{G})$ stands for the *chromatic distinguishing index* of $\overleftrightarrow{G}$, i.e. the least number of colours in a distinguishing proper arc-colouring, where the indicated two-arc digraphs cannot be monochromatic.

In this paper, we determine the values of $\chi'_{D_1}(\overleftrightarrow{G}), \chi'_{D_3}(\overleftrightarrow{G}), \chi'_{D_4}(\overleftrightarrow{G}), \chi'_{D_{1,3,4}}(\overleftrightarrow{G})$. In each case, we also determine the corresponding chromatic index of $\overleftrightarrow{G}$. We settle the other types of proper arc-colourings in another papers [12,13].

2. Preliminaries

Let us first show a correspondence between arc-colourings of a symmetric digraph $\overleftrightarrow{G}$ and edge-colourings of a subdivision of the graph G. By $\widehat{G}$ we denote a *subdivision* of G, i.e. a graph obtained by replacing each edge uv of G by a path uxv of length two. A colouring of $\widehat{G}$ can be also described as a colouring of the two *halfedges* of each edge of G, and the problems we consider here have applications in computer science, where sometimes it is necessary to break all non-trivial symmetries of networks of anonymous ports or nodes by colouring endings of edges (cf. [6,7]).

Clearly, $\mathrm{Aut}(\overleftrightarrow{G}) = \mathrm{Aut}(G)$. If G is a cycle C_n, then $\widehat{G} = C_{2n}$. Hence, $\mathrm{Aut}(C_n)$ is the dihedral group of order $2n$, while $\mathrm{Aut}(\widehat{C_n})$ is a dihedral group of order $4n$, so $\mathrm{Aut}(C_n)$ is isomorphic to a proper subgroup of $\mathrm{Aut}(\widehat{C_n})$.

Proposition 1. *If a connected graph G is not a cycle, then*

$$\mathrm{Aut}(\overleftrightarrow{G}) \cong \mathrm{Aut}(\widehat{G}) \cong \mathrm{Aut}(G).$$

Proof. Clearly, $\mathrm{Aut}(G)$ is isomorphic to a subgroup of $\mathrm{Aut}(\widehat{G})$. Let $\varphi \in \mathrm{Aut}(\widehat{G})$. If G is not a cycle, then either G is a path or $\Delta(G) \ge 3$. Then $\widehat{G}$ contains a vertex v_0 of degree $d(v_0) \ne 2$. Hence, $\varphi(v_0) \in V(G)$. Obviously, a vertex $u \in V(\widehat{G})$ belongs to $V(G)$ if and only if the distance between u and v_0 is even. Every automorphism preserves the distance between vertices, so each vertex of G is mapped by φ into a vertex of G. That is, the restriction of φ to $V(G)$ is an automorphism of G. □

We apply the following one-to-one correspondence between arcs of $\overleftrightarrow{G}$ and edges of $\widehat{G}$. Let uv be an edge of G, and let $x \in V(\widehat{G}) \setminus V(G)$ be adjacent to u and v in $\widehat{G}$. Then the arc $\overrightarrow{uv}$ corresponds to the edge ux. Given an arc-colouring c of $\overleftrightarrow{G}$, we define the edge colouring $\widehat{c}$ of $\widehat{G}$ by setting $\widehat{c}(ux) = c(\overrightarrow{uv})$. In view of Proposition 1, the following lemma is obvious, and it clearly holds also for cycles.

Lemma 2. *An arc-colouring c of $\overleftrightarrow{G}$ is distinguishing if and only if the corresponding edge-colouring $\widehat{c}$ is a distinguishing colouring of $\widehat{G}$.* □

Given an arc-colouring c of $\overleftrightarrow{G}$, we say that a vertex $v \in V(G)$ is *fixed* if $\varphi(v) = v$ for every automorphism of $\overleftrightarrow{G}$ preserving c. The following observation is very useful in some cases of proper arc-colourings.

Lemma 3. *Let c be an arc-colouring of $\overleftrightarrow{G}$ without monochromatic sources or sinks. If there exists a fixed vertex, then c is a distinguishing colouring.*

Proof. Suppose that there are no monochromatic sources in a colouring c. Let v be a fixed vertex. Every arc outgoing from v has a distinct colour, and therefore each neighbour of v is also fixed. Then the claim follows by induction of the distance from v.

We argue analogously, if monochromatic sinks are forbidden. □

In the next section we use some known results concerning the distinguishing index of simple graphs.

Theorem 4. *([11]) If G is a connected graph of order $n \geq 3$, then*

$$D'(G) \leq \Delta(G)$$

unless G is a cycle of length at most five.

A tree is *symmetric* (resp. *bisymmetric*) if it has a central vertex v_c (resp. a central edge e_c), all leaves are of the same distance from v_c (resp. e_c) and every vertex that is not a leaf has the same degree.

Theorem 5. *([17]) Let G be a connected graph with $\Delta(G) \geq 3$. Then $D'(G) \leq \Delta(G) - 1$ unless G is either a symmetric or bisymmetric tree, or G is K_4 or $K_{3,3}$.*

Theorem 6. *([10]) If G is a connected graph without pendant edges, then*

$$D'(G) \leq \left\lceil \sqrt{\Delta(G)} \right\rceil + 1.$$

3. Proper arc-colourings

3.1. Forbidden monochromatic 2-cycles

Trivially, $\chi'_1(\overleftrightarrow{G}) = 2$ by putting the same pair of distinct colours on each 2-cycle. However, $\chi'_{D_1}(\overleftrightarrow{G})$ can be arbitrarily large.

Proposition 7. *For every symmetric tree T,*

$$\chi'_{D_1}(\overleftrightarrow{T}) = \min\{k : \Delta(T) \leq k(k-1)\}.$$

Proof. Let T be a symmetric tree with a central vertex v_0. By Theorem 5, $D'(T) = \Delta(T)$. Let $c : E(T) \to [\Delta(T)]$ be a distinguishing colouring. Denote $k_0 = \min\{k : \Delta(T) \leq k(k-1)\}$, and $K = [k_0]^2 \setminus \{(j, j) : j \in [k_0]\}$. Thus, there is an injection $\iota : [\Delta(T)] \to K$.

Let $e = uv$ be any edge of T such that the vertex u lies on the path between v_0 and v. If $\iota(c(uv)) = (i, j)$, then we put colour i on the arc $\overrightarrow{uv}$, and colour j on the opposite arc $\overrightarrow{vu}$ to obtain an arc-colouring c' of $\overleftrightarrow{T}$. The colouring c' is obviously distinguishing since c was a distinguishing edge-colouring of T. Hence, $\chi'_{D_1}(\overleftrightarrow{T}) \leq k_0$.

On the other hand, suppose that there exists a distinguishing arc-colouring c' of $\overleftrightarrow{T}$ with less that k_0 colours. Every edge uv of T, such that u lies on a path from v_0 to v, can be coloured by a pair (i, j), where $i = c'(\overrightarrow{uv})$ and $j = c'(\overrightarrow{vu})$. This yields a distinguishing colouring of T with less than $\Delta(T)$ colours. This contradicts the fact that $D'(T) = \Delta(T)$. □

Here is an upper bound for $\chi'_{D_1}(\overleftrightarrow{G})$ with respect to the maximum degree $\Delta(G)$ of the underlying graph G.

Proposition 8. *For every connected graph G,*

$$\chi'_{D_1}(\overleftrightarrow{G}) \leq \lceil \sqrt{\Delta(G)} \rceil + 1.$$

Proof. Suppose first that the graph is a tree T. Then by Theorem 4, $D'(T) \leq \Delta(T)$, whence $D'(T) \leq k(k-1)$ for $k = \lceil \sqrt{\Delta(T)} \rceil + 1$. If the tree T has a central vertex, then, using the same method as in the proof of Proposition 7, we construct a distinguishing arc-colouring of $\overleftrightarrow{T}$ with at most $\lceil \sqrt{\Delta(T)} \rceil + 1$ colours. If the tree T has a central edge $e_0 = u_0v_0$, then we colour the pair of arcs between u_0 and v_0 with a pair of distinct colours, thus fixing both vertices u_0, v_0. Each of the two subtrees of $T - e$ has the distinguishing number less than $\Delta(T)$, and we can analogously construct a distinguishing arc-colouring of $\overleftrightarrow{T}$ with $\lceil \sqrt{\Delta(T)} \rceil + 1$ colours.

Now, consider any connected graph G with a cycle. Consider the maximal subgraph G' of G with $\delta(G') \geq 2$, i.e. the subgraph G' obtained by deleting all pendant subtrees of G. By Theorem 6, there exists a distinguishing colouring c of G' with $\lceil \sqrt{\Delta(G)} \rceil + 1$ colours. Since $\lceil \sqrt{\Delta(G)} \rceil + 1 \geq 3$, we have

$$\binom{\lceil \sqrt{\Delta(G)} \rceil + 1}{2} \geq \lceil \sqrt{\Delta(G)} \rceil + 1,$$

hence to each colour in c we can assign a distinct 2-element subset $\{i, j\}$ of the set of integers $[\lceil \sqrt{\Delta(G)} \rceil + 1]$. For each edge uv of G', we colour the pair of opposite arcs $\overrightarrow{uv}, \overrightarrow{vu}$ with two pairs $(i, j), (j, i)$, where $\{i, j\}$ is the set assigned to $c(uv)$. This way, we obtain an arc-colouring c' of G' such that $V(G')$ is fixed pointwise by every automorphism of G preserving c'.

To complete the proof, it suffices to extend the colouring c' to a distinguishing colouring for every maximal pendant subtree T of G. As we have already shown at the beginning of the proof, this can be done with $\lceil\sqrt{\Delta(G)}\rceil+1$ colours. □

The bound in Proposition 8 is tight for infinitely many values of $\Delta(G)$, since by Proposition 7, it is achieved by infinitely many symmetric trees with maximum degree Δ. Indeed, it is easy to check that

$$\min\{k:\Delta\le k(k-1)\}\in\{\lceil\sqrt{\Delta}\rceil,\lceil\sqrt{\Delta}\rceil+1\},$$

and both values are achieved with asymptotically equal frequency, as Δ tends to infinity. In view of Theorem 4, we suppose that the following generalization of Proposition 7 holds.

Conjecture 9. *For every connected graph G,*

$$\chi'_{D_1}(\overleftrightarrow{G})\le\min\left\{k:D'(G)\le k(k-1)\right\}.$$

For $D'(G)=2$, this conjecture, if true, would imply that every graph with $D'(G)\le 2$ has an asymmetric orientation, that is, one can substitute each edge of G with an arc such that the resulting oriented graph had trivial automorphism group. Indeed, we would then have $\chi'_{D_1}(\overleftrightarrow{G})=2$, and the arcs in one colour would create an asymmetric orientation of G. Let us mention that the problem of the existence of asymmetric orientations for some classes of graphs was investigated by Harary et al. [8,9] and by Meslem and Sopena [16]. Conjecture 9 is true for bipartite graphs and for traceable graphs. Recall that Pilśniak [17] proved that $D'(G)\le 2$ for every traceable graph G of order $n\ge 7$, and $D'(G)\le 3$ for graphs of smaller orders.

Proposition 10. *If G is a connected bipartite graph, then*

$$\chi'_{D_1}(\overleftrightarrow{G})\le\min\left\{k:D'(G)\le k(k-1)\right\}.$$

Proof. Let A and B be two independent sets of a bipartition of the graph G, and let c be a distinguishing edge-colouring of G with the set $[D'(G)]$ of colours. Denote $k_0=\min\{k:D'(G)\le k(k-1)\}$. Let

$$\iota:[D'(G)]\longrightarrow[k_0]^2\setminus\{(j,j):j\in[k_0]\}$$

be an injection.

For every edge uv of G with $u\in A$ and $v\in B$, we colour the two arcs between u and v as follows. If $\iota(c(uv))=(i,j)$, then we colour the arc $\overrightarrow{uv}$ with i and the arc $\overrightarrow{vu}$ with j. Let φ be any automorphism of $\overleftrightarrow{G}$ preserving our colouring. Then φ also preserves the colouring c of G, unless there exist two edges $uv,u'v'$ such that $\iota(c(uv))=(i,j)$ and $\iota(c(u'v'))=(j,i)$, $\varphi(u)=v'$ and $\varphi(v)=u'$. However, since G is bipartite and connected, then either all vertices of A are mapped by φ into A or all of them are mapped into B. Therefore, the above situation is not possible. Hence, φ is the identity automorphism. Consequently, $\chi'_{D_1}(\overleftrightarrow{G})\le k_0(k_0-1)$. □

Proposition 11. *If G is a traceable graph, then*

$$\chi'_{D_1}(\overleftrightarrow{G})\le 2.$$

Proof. Let $P=v_1\cdots v_n$ be a Hamilton path of G. For $i<j$, we colour each arc $\overrightarrow{v_iv_j}$ of $\overleftrightarrow{G}$ with colour 1, and each arc $\overrightarrow{v_jv_i}$ with colour 2. Then v_1 is the only vertex with all outgoing arcs coloured with 1, so it is fixed. Moreover, P is the only directed Hamilton path of $\overleftrightarrow{G}$ coloured with 1, so each vertex of P, and thus each of $\overleftrightarrow{G}$, is also fixed. □

3.2. Forbidden monochromatic either sources or sinks

We begin with determining the chromatic distinguishing index of $\overleftrightarrow{G}$ in each of the two cases.

Proposition 12. *If G is a connected graph, then*

$$\chi'_3(\overleftrightarrow{G})=\chi'_4(\overleftrightarrow{G})=\Delta(G).$$

Proof. The subdivision $\widehat{G}$ is a bipartite graph, hence it admits a proper edge-colouring $\hat{c}$ with $\Delta(\widehat{G})=\Delta(G)$ colours, by Kőnig's theorem. The corresponding arc-colouring c of $\overleftrightarrow{G}$ does not have monochromatic sources. Hence, $\chi'_3(\overleftrightarrow{G})=\Delta(G)$.

If monochromatic sinks are not allowed, then we reverse the arcs in $\overleftrightarrow{G}$ coloured by c to obtain an arc-colouring without monochromatic sinks. Thus, $\chi'_4(\overleftrightarrow{G})=\Delta(G)$. □

Now we show that every symmetric digraph $\overleftrightarrow{G}$ admits a proper distinguishing arc-colouring with the minimum number of colours, whenever G is connected.

Theorem 13. *For every connected graph G,*

$$\chi'_{D_3}(\overleftrightarrow{G})=\chi'_{D_4}(\overleftrightarrow{G})=\Delta(G).$$

Proof. Suppose first that sources cannot be monochromatic. By Kőnig's theorem, the subdivision $\widehat{G}$, being a bipartite graph, admits a proper edge-colouring $\hat{c}$ with $\Delta(G)$ colours. Clearly, $\hat{c}$ need not be distinguishing. Observe that, for every vertex $x \in V(\widehat{G}) \setminus V(G)$, both edges of $\widehat{G}$ incident to x have distinct colours. However, this is not required in this type of proper colouring, and we shall use this fact to obtain a distinguishing colouring by exchanging colours of some edges.

We consider Kempe chains with respect to $\hat{c}$, that is, connected components of subgraphs induced by edges coloured by $\hat{c}$ with two given colours. Suppose first that there exists a Kempe chain which is a cycle C. Clearly, C is of length at least six. We exchange colours of two adjacent edges xv, vy in C, where $v \in V(G)$, thus obtaining a colouring c. Now, v is the only vertex adjacent in $\overleftrightarrow{G}$ to two monochromatic 2-cycles. Hence, v is fixed by every automorphism of $\overleftrightarrow{G}$ preserving the corresponding colouring c which is still proper. By Lemma 3, c is distinguishing.

Suppose then that every Kempe chain is a path. There always exists in $\widehat{G}$ a Kempe path of length at least three, in particular the one containing a vertex of maximum degree, since all colours of $\hat{c}$ appear on its incident edges. Let P be such a path with an end-vertex $u \in V(G)$, and let v be a vertex of distance two on P. We exchange the colours of the two edges of P incident to v. Thus v becomes a unique vertex of $\overleftrightarrow{G}$ incident to arcs in both colours of P, one of which on a monochromatic 2-cycle. Thus v is fixed. If P has both end-vertices in $V(\widehat{G}) \setminus V(G)$ and is of length at least six, then we exchange the colours of the last two edges of P with a common vertex $v \in V(G)$. For the same reason as in the previous case, v is fixed. In both cases, we obtain a proper distinguishing colouring of $\overleftrightarrow{G}$, by Lemma 3.

Thus we are left with a situation, when all Kempe paths of $\hat{c}$ in $\widehat{G}$ are of length four and have both ends in $V(\widehat{G}) \setminus V(G)$. Let $P = xvywz$ be such a Kempe path and let u be the other neighbour of x in $\widehat{G}$. If there is no edge incident to u coloured with $\hat{c}(vy)$, then we recolour the edge xu with $\hat{c}(vy)$. Otherwise, if there is such an edge, say ut, then we exchange colours of xu and ut. In both cases, we obtain a longer Kempe path, and we proceed as previously. Note that in the latter case, the exchange of colours may create a monochromatic path of length two incident to u, but it does not matter for the next procedure since this would be a colour different from those of P.

If the monochromatic sinks are forbidden, then we take a distinguishing colouring of $\overleftrightarrow{G}$ constructed above and reverse the arcs. □

3.3. Forbidden monochromatic 2-cycles, sources and sinks

In this subsection, we determine both invariants $\chi'_{1,3,4}(\overleftrightarrow{G})$ and $\chi'_{D_{1,3,4}}(\overleftrightarrow{G})$ for connected graphs G.

Theorem 14. *For every connected graph G,*

$$\Delta(G) \leq \chi'_{1,3,4}(\overleftrightarrow{G}) \leq \Delta(G) + 1.$$

Moreover, $\chi'_{1,3,4}(\overleftrightarrow{G}) = \Delta(G)$ whenever $\Delta(G)$ is even.

Proof. We add a matching M between vertices of odd degrees, thus obtaining an Eulerian multigraph $G + M$. We take an Eulerian tour W and fix its orientation. Let $V(G) = \{v_1, \ldots, v_n\}$. We define a bipartite multigraph $G' = (X, Y, E)$ as follows: $X = V(G), Y = \{v'_1, \ldots, v'_n\}$ and $v_i v'_j \in E$ if $\overrightarrow{v_i v_j}$ is an oriented edge of W. By Kőnig's theorem, which is also valid for multigraphs, G' has a proper edge-colouring with $\Delta(G') = \Delta(G+M)/2 = \lceil \Delta(G)/2 \rceil$ colours. The k-th colour class corresponds in G to a subgraph H_k with maximum degree at most two, for $k \in [\lceil \Delta(G)/2 \rceil]$.

For each k, we fix an orientation of each component of H_k (which is a cycle or a path) and colour the arcs of $\overleftrightarrow{H_k}$ according to the fixed orientation with colour $2k-1$, while the opposite arcs get colour $2k$. Thus we obtain a proper colouring of $\overleftrightarrow{G}$ with $2\lceil \Delta(G)/2 \rceil \leq \Delta(G) + 1$ colours. □

For odd Δ, there are infinitely many graphs G with $\Delta(G) = \Delta$ and $\chi'_{1,3,4}(\overleftrightarrow{G}) = \Delta + 1$. This is the case, for instance, for a Δ-regular graph G having a bridge xy (clearly, such graphs exist for every odd Δ). Indeed, in any colouring of arcs of $\overleftrightarrow{G}$ with Δ colours without monochromatic sources and sinks, each arc of $\overleftrightarrow{G}$ belongs to a monochromatic directed cycle. The only possible monochromatic cycle for arcs between x and y is the 2-cycle between them. Thus, we need an extra colour to avoid monochromatic 2-cycles.

Theorem 15. *If G is a connected graph, then*

$$\Delta(G) \leq \chi'_{D_{1,3,4}}(\overleftrightarrow{G}) \leq \Delta(G) + 1.$$

Proof. Clearly, $\Delta(G) \leq \chi'_{1,3,4}(\overleftrightarrow{G}) \leq \chi'_{D_{1,3,4}}(\overleftrightarrow{G})$. If $\Delta(G)$ is even, then $\overleftrightarrow{G}$ admits a proper colouring with $\Delta(G)$ colours. We simply put an extra colour on a certain arc. Hence, both ends of this arc are fixed. Then each vertex of $\overleftrightarrow{G}$ is fixed, by Lemma 3.

Let $\Delta(G)$ be odd. We select an edge $uv \in E(G)$ which lies on a cycle C, and we join u and v by additional edge uv. When G does not contain a cycle, i.e. G is a tree, then we earlier add an edge between two pendant vertices of G to create a cycle. This way we get a multigraph G_1 with a double edge uv. Next, we add a matching M between vertices of odd degrees in G_1 to obtain an Eulerian multigraph $G_1 + M$. We pick an oriented Euler tour W of $G_1 + M$ such that both edges between u

and v are traversed in the same direction. To see that such a tour W exists, start it with one edge from u to v, and traverse the cycle C until we again reach v through the other edge uv. Then we continue our tour as long as possible. If we do not complete an Euler tour this way, then we break our tour in a suitable vertex (perhaps several times) and complete it to an Euler tour without changing the direction of edges of previous parts.

Next, we proceed as in the proof of Theorem 14. The bipartite multigraph G' with a double edge uv' admits a proper edge-colourings with $\lceil (\Delta(G)+1)/2 \rceil$ colours. Thus the double edge uv' gets two distinct colours, say 1 and 2. We consider the corresponding colouring of the graph G. Without loss of generality, we may assume that the edge uv gets colour 1. Again, for $k = 1, \ldots, \lceil \Delta(G)/2 \rceil$, we fix an orientation of each component of the colour class H_k and colour its arcs with colour $2k-1$. The opposite arcs of $\overleftrightarrow{H_k}$ we colour with $2k$. Thus we obtain a proper arc-colouring of $\overleftrightarrow{G}$. Now, we take the arc between u and v which is coloured with 2, and recolour it with 3. The colouring remains proper since the double edge uv' of G' got colours 1 and 2. Consequently, the pair $\overrightarrow{uv}, \overrightarrow{vu}$ creates the only 2-cycle coloured with the pair 1,3, therefore both u and v are fixed. By Lemma 3, this colouring is distinguishing. □

The left-hand side equality is achieved by trees, and the right-hand side by cycles.

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